A result on singularly perturbed elliptic problems

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Abstract : We consider a class of equations of the form

$$-\varepsilon^2 \Delta u + V(x)u = f(u), \quad u \in H^1(\mathbf{R}^N).$$

For a local minimum x_0 of the potential $V(x)$, we show that there exists a sequence $\varepsilon_n \to 0$, for which corresponding solutions $u_n(x) \in H^1(\mathbf{R}^N)$ concentrate at x_0. Our assumptions on $f(\xi)$ are mainly the ones under which the associated autonomous problem

$$-\Delta v + V(x_0)v = f(v), \quad v \in H^1(\mathbf{R}^N),$$

admits a non trivial solution.

0. Introduction

In this paper we study the existence of positive solutions for the equation

$$-\varepsilon^2 \Delta u + V(x)u = f(u), \quad u \in H^1(\mathbf{R}^N). \tag{0.1}$$

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We assume $V(x)$ locally Hölder continuous and that there exists $V_0 > 0$ such that

$$V(x) \geq V_0 > 0 \quad \text{for all } x \in \mathbf{R}^N . \tag{0.2}$$

A basic motivation to study (0.1) comes from the nonlinear Schrödinger equation

$$i\hbar \frac{\partial \Phi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Phi + W(x)\Phi - g(|\Phi|)\Phi. \tag{0.3}$$

We are interested in standing wave solutions, namely solutions of the form $\Phi(x,t) = u(x)e^{-\frac{iEt}{\hbar}}$ and it is easily observed that a $\Phi(x,t)$ of this form satisfies (0.3) if and only if $u(x)$ is a solution of (0.1) with $V(x) = W(x) - E$, $\varepsilon^2 = \frac{\hbar^2}{2m}$ and $f(u) = g(u)u$.

An interesting class of solutions of (0.1), sometimes called semi-classical states, are families of solutions $u_\varepsilon(x)$ which concentrate and develop a spike shape around one, or more, special points in $\mathbf{R}^N$, while vanishing elsewhere as $\varepsilon \to 0$.

The existence of single and multiple spike solutions was first studied by Floer and Weinstein [**FW**]. In the one dimensional case and for $f(u) = u^3$ they construct a single spike solution concentrating around any given non-degenerate critical point of the potential $V(x)$. Oh [**O1**, **O2**] extended this result in higher dimension and for $f(u) = |u|^{p-1}u$ $(1 < p < \frac{N+2}{N-2})$. The arguments in [**FW**, **O1**, **O2**] are based on a Lyapunov-Schmidt reduction and rely on the uniqueness and non-degeneracy of the ground state solutions of the autonomous problems :

$$-\Delta v + V(x_0)v = f(v) \quad \text{in } H^1(\mathbf{R}^N) \quad (x_0 \in \mathbf{R}^N). \tag{0.4}$$

We remark that if we introduce a rescaled (around $x_0 \in \mathbf{R}^N$) function $v(y) = u(\varepsilon y + x_0)$, (0.1) becomes $-\Delta v + V(x_0 + \varepsilon y)v = f(v)$ and (0.4) appears as a limit as $\varepsilon \to 0$.

Subsequently reduction methods were also found suitable to find solutions of (0.1) concentrating around possibly degenerate critical points of $V(x)$, when the ground state solutions of the limit problems (0.4) are unique and non-degenerate. In [**ABC**] Ambrosetti, Badiale and Cingolani consider concentration phenomena at isolated local minima and maxima with polynomial degeneracy and in [**YYL**] Y. Li deals with C^1-stable critical points of V. See also [**AMS**, **CN**, **Gr**, **KW**, **P**, **S**] for related results.

We remark that the uniqueness and non-degeneracy of the ground state solutions of (0.4) are, in general, rather difficult to prove. They are known so far only for a rather restricted class of nonlinearities $f(\xi)$. To attack the existence of positive solutions of (0.1) without these assumptions, the variational approach, initiated by Rabinowitz [**R**], proved to be successful. In [**R**] he proves, by a mountain pass argument, the existence of positive solutions of (0.1), for $\varepsilon > 0$ small, whenever

$$\liminf_{|x|\to\infty} V(x) > \inf_{x\in\mathbf{R}^N} V(x).$$

Later Wang [**W**] showed that these solutions concentrate at global minimum points of $V(x)$.

In 1996, del Pino and Felmer [**DF1**] by introducing a penalization approach, so called *local mountain pass*, managed to handle the case of a, possibly degenerate, local minimum of $V(x)$. They assume that an open bounded set $\Lambda \subset \mathbf{R}^N$ satisfies

$$\inf_{x\in\Lambda} V(x) < \min_{x\in\partial\Lambda} V(x) \tag{0.5}$$

and they show the existence of a single spike solution concentrating around minimizer of $V(x)$ in Λ. Later, under stronger assumptions on $f(\xi)$, they extended their result to the existence of multiple spike solutions in a, possibly degenerate, saddle point setting [**DF4**]. As results in between [**DF1**] and [**DF4**] we mention [**DF2**, **DF3**, **Gu**].

In a recent paper Jeanjean and Tanaka [**JT3**] extend the result of [**DF1**] to a wider class of nonlinearities. In particular in [**JT3**] the monotonicity of the function $\xi \mapsto \frac{f(\xi)}{\xi}$ is not required and asymptotically linear as well as superlinear nonlinearities are deal with.

In the present paper we pursue the weakening of the conditions on $f(\xi)$. Our main result is the following :

Theorem 0.1. *Suppose $N \geq 3$ and let $\Lambda \subset \mathbf{R}^N$ be a bounded open set satisfying* (0.5). *We assume on $f(\xi)$,*

- (f0) $f(\xi) \in C^1(\mathbf{R}, \mathbf{R})$.
- (f1) $f(\xi) = o(\xi)$ *as* $\xi \sim 0$.
- (f2) *For some* $s \in (1, \frac{N+2}{N-2})$

$$\frac{f(\xi)}{\xi^s} \to 0 \quad \text{as } \xi \to \infty.$$

- (f3) *There exists* $\xi_0 > 0$ *such that*

$$F(\xi_0) - \frac{1}{2}(\inf_{x\in\Lambda} V(x))\xi_0^2 > 0,$$

where

$$F(\xi) = \int_0^\xi f(\tau)\, d\tau.$$

Then there exists a sequence (ε_n) decreasing to 0 such that, for any $n \in \mathbf{N}$, (0.1) *has a solution $u_{\varepsilon_n}(x)$ satisfying* *i)* $u_{\varepsilon_n}(x)$ *has unique local maximum (hence global maximum) in* $\mathbf{R}^N$ *at* $x_{\varepsilon_n} \in \Lambda$.

ii) $V(x_{\varepsilon_n}) \to \inf_{x\in\Lambda} V(x)$.

iii) There exist constants C_1, $C_2 > 0$ *such that*

$$u_{\varepsilon_n}(x) \le C_1 \exp\left(-C_2 \frac{|x - x_{\varepsilon_n}|}{\varepsilon_n}\right) \quad \textit{for } x \in \mathbf{R}^N .$$

We prove Theorem 0.1 under assumptions on $f(\xi)$ that we believe to be almost necessary. In particular no control on $f(\xi)$ between 0 and ∞ is required and it is a consequence of Pohozaev's identity that (f3) is necessary for the associated autonomous problem

$$-\Delta v + V(x_0)v = f(v), \quad v \in H^1(\mathbf{R}^N)$$

to have a non trivial solution (see [**BL**]). In turn this condition is required for the existence of solutions when $\varepsilon > 0$ is small. However this generalization is at the expense of a weakening of our knowledge on the concentration phenomena. We are only able to prove that it occurs on a sequence (ε_n) and not, as in [**JT3**] or [**DF1**], that i)-iii) in the statement of Theorem 0.1 hold for any $\varepsilon > 0$ sufficiently small.

Our solutions are obtained as critical points of penalized functionals

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbf{R}^N} \varepsilon^2|\nabla u|^2 + V(x)u^2\, dx - \int_{\mathbf{R}^N} G(x,u)\, dx$$

(see Section 4 for a precise definition). The purpose of introducing these functionals is to obtain solutions of (0.1) which are *localized* inside Λ.

We shall prove that $I_\varepsilon(u)$ has a mountain pass geometry when $\varepsilon > 0$ is small enough, and get our solutions at the mountain pass levels.

To reach the conclusion of Theorem 0.1 it is necessary to show that such solutions $u_\varepsilon(x)$ exists but also that they are sufficiently small. Indeed, as it is known from [**DF1**], a key point is to show that

$$\int_{\mathbf{R}^N} \varepsilon^2|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\, dx \le C\varepsilon^N \tag{0.6}$$

for a constant $C > 0$ independent of $\varepsilon > 0$. In [**DF1**] this was achieved by requiring the so called *global Ambrosetti-Rabinowitz's condition.* This requirement was weakened in [**JT3**] but still a global control on $f(\xi)$ was required. Here we manage to obtain (0.6) on a sequence, by using techniques which were developped by the second author in [**J**] (see also [**GJ**]) to deal with a bifurcation phenomena from the essential spectrum. Whether or not a solution exists and (0.6) holds for any $\varepsilon > 0$ small is an open question.

The proof of Theorem 0.1 consists of several steps. In Section 1, influenced by the work of del Pino-Felmer [**DF1**], we introduce the penalized problems. In Section 2, following [**JT3**] we define re-scaled functionals and recall some concentration-compactness type arguments. In Section 3 we state results on autonomous problems which were already at the heart of [**JT3**], in particular for proving that solutions of the penalized problems are also solutions of (0.1). Roughly speaking these results say that for (0.4) the mountain pass level corresponds to the ground state level. In Section 4, we prove that the functionals $I_\varepsilon(u)$ have a mountain pass geometry and derive estimates on the mountain pass levels. In Section 5, we prove that for almost every $\varepsilon > 0$ sufficiently small the penalized problems have a solution. Finally in Section 6 we prove the existence of a special sequence (ε_n) decreasing to 0 for which the corresponding critical points satisfies (0.6). Then we show, following [**JT3**], that for this sequence the critical points of the modified functionals satisfies the original problem (0.1).

1. Modification of the nonlinearity $f(\xi)$

In this section and the next two, we give some preliminaries for the proof of Theorem 0.1. Since we seek positive solutions, we can assume that $f(\xi) = 0$ for all $\xi \leq 0$. Also, under (f0)–(f2), for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$|f(\xi)| \leq \delta|\xi| + C_\delta|\xi|^s \quad \text{for all } \xi \in \mathbf{R}\,. \tag{1.1}$$

To find a solution $u_\varepsilon(x)$ concentrating in a given set Λ, we modify the nonlinearity $f(\xi)$. Here our approach is closely related to the one of del Pino-Felmer [**DF1**] (see also [**JT3**]).

Let $f(\xi)$ be a function satisfying (f0)–(f2). We choose a number $\nu \in (0, \frac{V_0}{2})$ and we set

$$\underline{f}(\xi) = \begin{cases} \min\{f(\xi), \nu\xi\} & \text{for } \xi \geq 0, \\ 0 & \text{for } \xi < 0. \end{cases} \tag{1.2}$$

By (f1) we see that there exists a small $r_\nu > 0$ such that

$$\underline{f}(\xi) = f(\xi) \quad \text{for } |\xi| \leq r_\nu.$$

Moreover it holds that

$$\underline{f}(\xi) = \begin{cases} \nu\xi & \text{for large } \xi \geq 0, \\ 0 & \text{for } \xi \leq 0. \end{cases} \tag{1.3}$$

Next, let $\Lambda \subset \mathbf{R}^N$ be a bounded open set satisfying (0.5). We may assume that the boundary $\partial\Lambda$ is smooth. We choose an open subset $\Lambda' \subset \Lambda$ with a smooth boundary $\partial\Lambda'$ and a function $\chi(x) \in C^\infty(\mathbf{R}^N, \mathbf{R})$ such that

$$\begin{aligned}
&\inf_{x\in\Lambda\setminus\Lambda'} V(x) > \inf_{x\in\Lambda} V(x), \\
&\min_{x\in\partial\Lambda'} V(x) > \inf_{x\in\Lambda'} V(x) = \inf_{x\in\Lambda} V(x), \\
&\chi(x) = 1 \qquad \text{for } x \in \Lambda', \\
&\chi(x) \in (0,1) \quad \text{for } x \in \Lambda \setminus \overline{\Lambda'}, \\
&\chi(x) = 0 \qquad \text{for } x \in \mathbf{R}^N \setminus \Lambda.
\end{aligned}$$

In what follows we assume, without loss of generality, that

$$0 \in \Lambda' \quad \text{and} \quad V(0) = \inf_{x\in\Lambda} V(x). \tag{1.4}$$

Finally we define

$$g(x,\xi) = \chi(x) f(\xi) + (1-\chi(x))\underline{f}(\xi) \quad \text{for } (x,\xi) \in \mathbf{R}^N \times \mathbf{R} \tag{1.5}$$

and we write $\underline{F}(\xi) = \int_0^\xi \underline{f}(\tau)\, d\tau$, $G(x,\xi) = \int_0^\xi g(x,\tau)\, d\tau = \chi(x)F(\xi) + (1-\chi(x))\underline{F}(\xi)$.

From now on we try to find a solution of the following problem :

$$-\varepsilon^2 \Delta u + V(x) u = g(x,u) \quad \text{in } \mathbf{R}^N. \tag{1.6}$$

We will find a solution $u_\varepsilon(x)$ of (1.6) via a mountain pass argument and besides other properties we will show that $u_\varepsilon(x)$ satisfies for small $\varepsilon > 0$

$$|u_\varepsilon(x)| \le r_\nu \quad \text{for } x \in \mathbf{R}^N \setminus \Lambda', \tag{1.7}$$

that is, $u_\varepsilon(x)$ also solves the original problem (0.1).

We give some properties of $\underline{f}(\xi)$.

Lemma 1.2. *(i)* $\underline{f}(\xi) = 0$, $\underline{F}(\xi) = 0$ *for all* $\xi \le 0$.
(ii) $\underline{f}(\xi) \le \nu\xi$, $\underline{F}(\xi) \le F(\xi)$ *for* $\xi \ge 0$.
(iii) $\underline{f}(\xi) \le f(\xi)$ *for* $\xi \ge 0$.

The proofs are direct from the definition of $\underline{f}(\xi)$. Also we clearly have

Corollary 1.3. *(i)* $g(x,\xi) \le f(\xi)$, $G(x,\xi) \le F(\xi)$ *for all* $(x,\xi) \in \mathbf{R}^N \times \mathbf{R}$.
(ii) $g(x,\xi) = f(\xi)$ *if* $|\xi| < r_\nu$.
(iii) For any $\delta > 0$ *there exists* $C_\delta > 0$ *such that*

$$|g(x,\xi)| \le \delta|\xi| + C_\delta|\xi|^s \quad \textit{for all } (x,\xi) \in \mathbf{R}^N \times \mathbf{R}\,. \tag{1.8}$$

2. Modified functionals and concentration-compactness type arguments

Introducing the re-scaled function $v(y) = u(\varepsilon y)$ we can rewrite (1.6) as

$$-\Delta v + V(\varepsilon y)v = g(\varepsilon y, v) \quad \text{in } \mathbf{R}^N\,. \tag{2.1}$$

The functional corresponding to (2.1) is

$$J_\varepsilon(v) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla v|^2 + V(\varepsilon y)v^2\,dy - \int_{\mathbf{R}^N} G(\varepsilon y, v)\,dy.$$

We consider $J_\varepsilon(v)$ on the following function space :

$$H_\varepsilon = \{v \in H^1(\mathbf{R}^N);\ \int_{\mathbf{R}^N} V(\varepsilon y)v^2\,dy < \infty\}$$

equipped with norm

$$\|v\|^2_{H_\varepsilon} = \int_{\mathbf{R}^N} |\nabla v|^2 + V(\varepsilon y)v^2\,dy.$$

Note that, because of (0.2), $H_\varepsilon \subset H^1(\mathbf{R}^N)$. In Proposition 3.2 of [**JT3**] we obtained a description of the sequences of points $(v_{\varepsilon_n}) \subset H_{\varepsilon_n}$ which satisfies, when $\varepsilon_n \to 0$,

$$J_{\varepsilon_n}(v_{\varepsilon_n}) \to c \in \mathbf{R}, \tag{2.2}$$

$$J'_{\varepsilon_n}(v_{\varepsilon_n}) = 0, \tag{2.3}$$

$$||v_{\varepsilon_n}||_{H_{\varepsilon_n}} \le m, \tag{2.4}$$

where the constants c, m are independent of ε.

To state this result, we need some definitions. For $x_0 \in \mathbf{R}^N$, let $\Phi_{x_0} : H^1(\mathbf{R}^N) \to \mathbf{R}$ be given by

$$\Phi_{x_0}(v) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla v|^2 + V(x_0)v^2\,dy - \int_{\mathbf{R}^N} G(x_0, v)\,dy.$$

We choose a function $\psi(y) \in C_0^\infty(\mathbf{R}^N, \mathbf{R})$ such that

$$\psi(y) = 1 \quad \text{for } y \in \Lambda,$$
$$\psi(y) \in [0,1] \quad \text{for all } y \in \mathbf{R}^N\,.$$

We also define $\psi_\varepsilon(y) = \psi(\varepsilon y)$. Finally we set

$$H(x,\xi) = -\frac{1}{2}V(x)\xi^2 + \chi(x)F(\xi) + (1-\chi(x))\underline{F}(\xi)$$

and

$$\Omega = \{x \in \mathbf{R}^N;\ \sup_{\xi>0} H(x,\xi) > 0\}.$$

Remark 2.1. (i) $\Omega \subset \Lambda$ and $0 \in \{x \in \Lambda';\ V(x) = \inf_{x\in\Lambda} V(x)\} \subset \Omega$.
(ii) $\Phi_{x_0}(v)$ has non-zero critical points if and only if $x_0 \in \Omega$. Indeed applying Proposition 3.1 with $H(\xi) = H(x_0,\xi) = -\frac{1}{2}V(x_0)\xi^2 + G(x_0,\xi)$, we can see that (h3) of Proposition 3.1 holds if and only if $x_0 \in \Omega$.

Now Proposition 3.2 of [**JT3**] is

Proposition 2.2. *Assume that* $f(\xi)$ *satisfies* (f0)–(f2) *and* $(v_{\varepsilon_n}) \subset H_{\varepsilon_n}$ *satisfies* (2.2)–(2.4). *Then there exists a subsequence, still denoted* $\varepsilon_n \to 0$, $\ell \in \mathbf{N}\cup\{0\}$, *sequences* $(y^k_{\varepsilon_n}) \subset \mathbf{R}^N$, $x^k \in \Omega$, $\omega^k \in H^1(\mathbf{R}^N)\setminus\{0\}$ $(k = 1,2,\cdots,\ell)$ *such that*

(i) $|y^k_{\varepsilon_n} - y^{k'}_{\varepsilon_n}| \to \infty$ *as* $j \to \infty$ *for* $k \neq k'$. (2.5)

(ii) $\varepsilon_n y^k_{\varepsilon_n} \to x^k \in \Omega$ *as* $j \to \infty$. (2.6)

(iii) $\omega^k \not\equiv 0$ *and* $\Phi'_{x^k}(\omega^k) = 0$. (2.7)

(iv) $\left\| v_{\varepsilon_n} - \psi_{\varepsilon_n}\left(\sum_{k=1}^{\ell} \omega^k(y - y^k_{\varepsilon_n})\right)\right\|_{H_{\varepsilon_n}} \to 0$ *as* $j \to \infty$. (2.8)

(v) $J_{\varepsilon_n}(v_{\varepsilon_n}) \to \sum_{k=1}^{\ell} \Phi_{x^k}(\omega^k)$. (2.9)

Remark 2.3. (i) When $\ell = 0$ in the statement of Proposition 2.2, it means that

$$\|v_{\varepsilon_n}\|_{H_{\varepsilon_n}} \to 0 \quad \text{and} \quad J_{\varepsilon_n}(v_{\varepsilon_n}) \to 0.$$

(ii) Since we do not assume any growth condition on $V(x)$, in general $\omega \notin H_\varepsilon$ for a critical point $\omega(y)$ of $\Phi_{x_0}(v)$ and $\varepsilon > 0$. This motivates the introduction of a cut-off function $\psi_\varepsilon(y)$ in (iv) of Proposition 2.2.

3. Some results on autonomous problems

In this section we deal with the the limit functionals $\Phi_{x_0}(v)$ for $x_0 \in \mathbf{R}^N$. The following result is due to Berestycki-Lions [**BL**].

Proposition 3.1. ([**BL**]). *Assume that* $N \geq 3$ *and that* $h(\xi)$ *satisfies*

(h0) $h(\xi) \in C(\mathbf{R}, \mathbf{R})$ *is continuous and odd.*

(h1) $-\infty < \liminf_{\xi \to 0} \frac{h(\xi)}{\xi} \leq \limsup_{\xi \to 0} \frac{h(\xi)}{\xi} < 0.$

(h2) $\displaystyle \lim_{\xi \to \infty} \frac{h(\xi)}{\xi^{\frac{2N}{N-2}}} = 0.$

Then the problem

$$-\Delta u = h(u) \quad \text{in } \mathbf{R}^N, \quad u(x) \in H^1(\mathbf{R}^N) \tag{3.1}$$

has a non-zero solution if and only if the following condition is satisfied.

(h3) *There exists* $\xi_0 > 0$ *such that* $H(\xi_0) > 0$, *where* $H(\xi) = \int_0^\xi h(\tau)\, d\tau$.

Moreover under (h0)–(h3), (3.1) *has a least energy solution* $u(x)$ *which satisfies* $u(x) > 0$ *and is radially symmetric in* $\mathbf{R}^N$. ∎

By a *least energy solution* we mean a solution $\omega(x)$ which satisfies $\tilde{I}(\omega) = m$, where

$$m = \inf\{\tilde{I}(u);\, u \in H^1(\mathbf{R}^N) \setminus \{0\} \text{ is a solution of (4.1)}\}, \tag{3.2}$$

$$\tilde{I}(u) = \int_{\mathbf{R}^N} \frac{1}{2}|\nabla u|^2 - H(u)\, dy.$$

It is also shown that $m > 0$.

In our recent work [**JT2**], we have revisited (3.1) and enlighten a mountain pass characterization of least energy solutions.

Proposition 3.2. *([**JT2**]). Assume that* (h0)–(h3) *hold. Then* $\tilde{I}(u)$ *has a mountain pass geometry and there holds that*

$$b = m, \tag{3.3}$$

where m *is defined in* (3.2) *and* b *is the mountain pass value for* $\tilde{I}(u)$*;*

$$b = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \tilde{I}(\gamma(t)),$$

$$\Gamma = \{\gamma(t) \in C([0,1], H^1(\mathbf{R}^N));\, \gamma(0) = 0, \tilde{I}(\gamma(1)) < 0\}.$$

Moreover for any least energy solution $\omega(x)$ *of* (3.1) *there exists a path* $\gamma(t) \in \Gamma$ *such that*

$$\tilde{I}(\gamma(t)) \leq m = \tilde{I}(\omega) \quad \text{for all } t \in [0,1], \tag{3.4}$$

$$\omega \in \gamma([0,1]). \tag{3.5}$$

∎

Remark 3.3. Both Propositions 3.1 and 3.2 are stated for odd nonlinearities $h(\xi)$. Since we just consider positive solutions, extending the nonlinearity $f(\xi)$ to an odd function on $\mathbf{R}$, we can apply Propositions 3.1 and 3.2 to our setting (see [**JT1**] for more details).

For $x \in \mathbf{R}^N$ we set

$$m(x) = \begin{cases} \text{least energy level of } \Phi_x(v) & \text{if } x \in \Omega, \\ \infty & \text{if } x \in \mathbf{R}^N \setminus \Omega. \end{cases}$$

By Proposition 3.2, $m(x)$ is equal to the mountain pass value for $\Phi_x(v)$ if $x \in \Omega$. We have the following.

Proposition 3.4. *$m(x_0) = \inf_{x\in\mathbf{R}^N} m(x)$ if and only if $x_0 \in \Lambda$ and $V(x_0) = \inf_{x\in\Lambda} V(x)$. In particular, $m(0) = \inf_{x\in\mathbf{R}^N} m(x)$.*

Proof. Suppose that $x_0 \in \Lambda$ satisfies $V(x_0) = \inf_{x\in\Lambda} V(x)$. By our choice of Λ' and χ, we have $x_0 \in \Lambda'$ and $\chi(x_0) = 1$. We also have $x_0 \in \Omega$ by Remark 2.1. Using $V(x) \ge V(x_0)$ in Λ, $G(x,\xi) \le F(\xi)$ for all (x,ξ), we have for any $x \in \Omega$,

$$\begin{aligned} \Phi_x(v) &= \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}V(x)\|v\|_2^2 - \int_{\mathbf{R}^N} G(x,v)\,dy \\ &\ge \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}V(x_0)\|v\|_2^2 - \int_{\mathbf{R}^N} F(v)\,dy \\ &= \Phi_{x_0}(v) \quad \text{for all } v \in H^1(\mathbf{R}^N). \end{aligned}$$

(We remark that this inequality is strict if $V(x) > V(x_0)$ and $v \not\equiv 0$). Thus $m(x_0) \le m(x)$ for all $x \in \mathbf{R}^N$.

Next suppose that $x' \in \Lambda$ satisfies $V(x') > V(x_0)$. We take a path $\gamma \in \Gamma$ such that (3.4)–(3.5) are satisfied for $\tilde{I}(v) = \Phi_{x'}(v)$. Then

$$m(x_0) \le \max_{t\in[0,1]} \Phi_{x_0}(\gamma(t)) < \max_{t\in[0,1]} \Phi_{x'}(\gamma(t)) = m(x').$$

Therefore Proposition 3.4 holds. ∎

4. Mountain pass geometry for I_ε and estimates

After these preliminaries we now turn back to equation (1.6). Associated to (1.6) is the *energy* functional

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbf{R}^N} \varepsilon^2|\nabla u|^2 + V(x)u^2\,dx - \int_{\mathbf{R}^N} G(x,u)\,dx$$

which is well defined for $u \in H$ where

$$H = \{u \in H^1(\mathbf{R}^N, \mathbf{R}) : \int_{\mathbf{R}^N} V(x)u^2(x)dx < \infty\}.$$

H becomes a Hilbert space, continuously embedded in $H^1(\mathbf{R}^N)$ when equipped with the inner product

$$< u, v >= \int_{\mathbf{R}^N} \nabla u \nabla v + V(x)uv\,dx$$

whose associated norm we denote by $||\cdot||$. Let us show that I_ε has a mountain pass geometry in H.

Proposition 4.1. *$I_\varepsilon(u) \in C^1(H, \mathbf{R})$ and it has a Mountain Pass Geometry that is uniform with respect to $\varepsilon \in (0,1]$ in the following sense :*

1° $I_\varepsilon(0) = 0$.

2° *There are constants $\rho_\varepsilon > 0$ and $\delta_\varepsilon > 0$ such that*

$$I_\varepsilon(u) \geq \delta_\varepsilon \quad \text{for all } \|u\|_{H^1(\mathbf{R}^N)} = \rho_\varepsilon$$

and

$$I_\varepsilon(u) > 0 \quad \text{for all } 0 < \|u\|_{H^1(\mathbf{R}^N)} \leq \rho_\varepsilon.$$

3° *There is a $u_0(x) \in C_0^\infty(\mathbf{R}^N)$ and $\varepsilon_0 > 0$ such that*

$$I_\varepsilon(u_0) < 0 \quad \text{for all } \varepsilon \in (0, \varepsilon_0].$$

Proof. From (1.8) it is clear that $I_\varepsilon(u) \in C^1(H, \mathbf{R})$. 1° is also trivial. To show 2°, we use (0.2) and (1.8) with $s = \frac{N+2}{N-2} = 2^* - 1$. We get, for $\varepsilon \in (0,1]$,

$$\begin{aligned} I_\varepsilon(v) &\geq \frac{\varepsilon^2}{2}||u||^2 - \frac{\varepsilon^2}{4}V_0 \int_{\mathbf{R}^N} u^2\,dx - C(\varepsilon)\int_{\mathbf{R}^N} |u|^{2^*}\,dx, \\ &\geq \frac{\varepsilon^2}{4}||u||^2 - C(\varepsilon)\int_{\mathbf{R}^N} |u|^{2^*}\,dx \end{aligned} \tag{4.1}$$

for some positive constant $C(\varepsilon)$. Thus, by the Sobolev's embeddings, there are constants $\rho_\varepsilon > 0$ and $\delta_\varepsilon > 0$ such that the statement 2° holds. To show 3°, we choose $v_0 \in C_0^\infty(\mathbf{R}^N)$ such that

$$\frac{1}{2}\int_{\mathbf{R}^N} |\nabla v_0|^2 + V(0)v_0^2\,dy - \int_{\mathbf{R}^N} F(v_0)\,dy < 0.$$

Because of (f3) the existence of such $v_0 \in C_0^\infty(\mathbf{R}^N)$ follows from Proposition 3.2. Since we are assuming $0 \in \Lambda'$, we observe that

$$J_\varepsilon(v_0) \to \frac{1}{2}\int_{\mathbf{R}^N} |\nabla v_0|^2 + V(0)v_0^2\,dy - \int_{\mathbf{R}^N} F(v_0)\,dy < 0 \quad \text{as } \varepsilon \to 0.$$

Now setting $u_0(x) = v_0(\frac{x}{\varepsilon})$ we have that $I_\varepsilon(u_0) = \varepsilon^N J_\varepsilon(v_0)$ and thus we get 3° for a $\varepsilon_0 > 0$ small enough. ∎

By Proposition 4.1, we can define the mountain pass value. For $\varepsilon \in (0, \varepsilon_0]$ we set

$$c_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} I_\varepsilon(\gamma(t)), \tag{4.2}$$

$$\Gamma_\varepsilon = \{\gamma \in C([0,1], H);\, \gamma(0) = 0, I_\varepsilon(\gamma(1)) < 0\}. \tag{4.3}$$

Next we derive estimates on the behavior of c_ε as $\varepsilon \to 0$.

Proposition 4.2. *Let $(c_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ be the mountain pass value of $I_\varepsilon(v)$ defined in (4.2)–(4.3). Then*

$$c_\varepsilon \le \varepsilon^N(m(0)+o(1)) \tag{4.4}$$

as $\varepsilon\to 0$, where $m(0)$ is the mountain pass value of the functional Φ_0. Also we have that

$$c_\varepsilon \ge C\varepsilon^N \tag{4.5}$$

for a $C>0$ independent of $\varepsilon>0$.

Proof. By Proposition 3.2 there exists a path $\gamma\in C([0,1],H)$ such that

$$\begin{aligned}
&\gamma(0)=0,\ \ \Phi_0(\gamma(1))<0,\\
&\Phi_0(\gamma(t))\le m(0) \quad \text{for all } t\in[0,1],\\
&\max_{t\in[0,1]}\Phi_0(\gamma(t))=m(0).
\end{aligned}$$

Let $\varphi(y)\in C_0^\infty(\mathbf{R}^N)$ be such that $\varphi(0)=1$ and $\varphi\ge 0$. Setting

$$\gamma_R(t)(y)=\varphi(y/R)\gamma(t)(y),$$

we have $\gamma_R(t)\in C([0,1],H)$, $\gamma_R(0)=0$ and $\Phi_0(\gamma_R(1))<0$ for sufficiently large $R>1$. Also for any fixed $R>0$,

$$J_\varepsilon(\gamma_R(t))\to\Phi_0(\gamma_R(t)) \quad \text{as } \varepsilon\to 0 \text{ uniformly in } t\in[0,1]$$

and thus

$$\max_{t\in[0,1]}J_\varepsilon(\gamma_R(t))\to\max_{t\in[0,1]}\Phi_0(\gamma_R(t)) \quad \text{as } \varepsilon\to 0. \tag{4.6}$$

Also we have

$$\max_{t\in[0,1]}\Phi_0(\gamma_R(t))\to m(0) \quad \text{as } R\to\infty. \tag{4.7}$$

Now defining $\tilde\gamma_R(t)$ by $\tilde\gamma_R(t)(x)=\gamma_R(t)(\frac{x}{\varepsilon})$ we see immediately that $\tilde\gamma_R(t)\in\Gamma_\varepsilon$ for sufficiently large $R>1$ and thus

$$c_\varepsilon\le\max_{t\in[0,1]}I_\varepsilon(\tilde\gamma_R(t))=\varepsilon^N\max_{t\in[0,1]}J_\varepsilon(\gamma_R(t)). \tag{4.8}$$

Then from (4.6)-(4.8) we get that

$$\limsup_{\varepsilon\to 0}c_\varepsilon\le\varepsilon^N m(0).$$

Finally we see that (4.5) hold from (4.1) noting that we can take ρ_ε to be $C\varepsilon^{N-2}$ with $C > 0$ sufficiently small. ∎

5. Solutions for the modified problems

In this section we establish the existence of a sequence (ε_n) decreasing to 0 for which the corresponding critical points $u_{\varepsilon_n}(x)$ of $I_{\varepsilon_n}(u)$ satisfies

$$\int_{\mathbf{R}^N} \varepsilon_n^2 |\nabla u_{\varepsilon_n}|^2 + V(x) u_{\varepsilon_n}^2 \, dx \leq C \varepsilon_n^N \tag{5.1}$$

for a $C > 0$ independent of $n \in \mathbf{N}$.

Observe that if $\varepsilon_1 < \varepsilon_2$ we have that $I_{\varepsilon_1}(u) \leq I_{\varepsilon_2}(u)$ for all $u \in H$. Thus, $\Gamma_{\varepsilon_2} \subset \Gamma_{\varepsilon_1}$ and we see that c_ε is a nondecreasing function of ε. Consequently c'_ε, the derivative of c_ε, exists almost everywhere.

From now one we make the change of variables $\lambda := \varepsilon^2$ to simplify the calculations. We also denote $I_\lambda = I_\varepsilon$, $\Gamma_\lambda = \Gamma_\varepsilon$ and by c'_λ the derivative of c_λ with respect to λ.

We claim that for any $\lambda > 0$ small enough where c'_λ exists, there is a sequence of paths $(\gamma_m) \subset \Gamma_\lambda$ with

$$\max_{t \in [0,1]} I_\lambda(\gamma_m(t)) \to c_\lambda$$

having "nice" localization properties. Namely, starting from a level strictly below c_λ, the "top" of each path is contained in a same ball centered at the origin whose radius $\beta(\lambda) > 0$ is sufficiently small as $\lambda \to 0$. To see this let $\lambda \in (0, \sqrt{\varepsilon_0}]$ be an arbitrary but fixed value where c'_λ exists. Let (λ_m) be a strictly decreasing sequence to λ. Our claim is a direct consequence of the following result.

Proposition 5.1. *For any $\delta > 0$ there exists a sequence of paths $(\gamma_m) \subset \Gamma_\lambda$ such that for m large enough we have*

i) $\|\nabla \gamma_m(t)\|_2^2 \leq 2c'_\lambda + 5\delta$ when

$$I_{\lambda_m}(\gamma_m(t)) \geq c_{\lambda_m} - \delta(\lambda_m - \lambda) \tag{5.2}$$

ii)

$$\max_{t \in [0,1]} I_\lambda(\gamma_m(t)) \leq c_\lambda + (c'_\lambda + 2\delta)(\lambda_m - \lambda). \tag{5.3}$$

iii) Making the choice $\delta = c_\lambda$ we have when (5.2) hold that

$$\int_{\mathbf{R}^N} \lambda |\nabla \gamma_m(t)|^2 + V(x)\gamma_m^2(t) \, dx \leq C\Big(\lambda(2c'_\lambda + 5c_\lambda) + (c_\lambda + (2c'_\lambda + 5c_\lambda)^{\frac{2*}{2}}\Big) := \beta(\lambda). \tag{5.4}$$

Proof. Let $(\gamma_m) \subset \Gamma_\lambda$ be an arbitrary sequence such that

$$\max_{t\in[0,1]} I_{\lambda_m}(\gamma_m(t)) \le c_{\lambda_m} + \delta(\lambda_m - \lambda). \tag{5.5}$$

Note that such sequence exists since $\Gamma_{\lambda_m} \subset \Gamma_\lambda$ for all $m \in \mathbf{N}$. From the definitions of $I_\lambda(u)$ and $I_{\lambda_m}(u)$ we obtain

$$\|\nabla\gamma_m(t)\|_2^2 = 2\,\frac{I_{\lambda_m}(\gamma_m(t)) - I_\lambda(\gamma_m(t))}{\lambda_m - \lambda}.$$

Thus if $\gamma_m(t)$ satisfies (5.2), by (5.5) we obtain, for m large enough, that

$$\|\nabla\gamma_m(t)\|_2^2 \le 2\,\frac{c_{\lambda_m} - c_\lambda}{\lambda_m - \lambda} + 4\delta \le 2c'_\lambda + 5\delta.$$

This proves (i). For m large enough we have

$$c_{\lambda_m} \le c_\lambda + (c'_\lambda + \delta)(\lambda_m - \lambda). \tag{5.6}$$

Thus since $I_{\lambda_m}(v) \ge I_\lambda(v)$ for all $v \in H$, we get from (5.5), (5.6), and for any $t \in [0,1]$

$$I_\lambda(\gamma_m(t)) \le I_{\lambda_m}(\gamma_m(t)) \le c_\lambda + (c'_\lambda + 2\delta)(\lambda_m - \lambda). \tag{5.7}$$

To prove (iii) we observe that

$$\int_{\mathbf{R}^N} V(x)\gamma_m^2(t)\,dx \le 2\Big(I_\lambda(\gamma_m(t)) + \int_{\mathbf{R}^N} G(x,\gamma_m(t))\,dx\Big).$$

Applying (1.8) with $s = \frac{N+2}{N-2} = 2^* - 1$ we can write

$$\int_{\mathbf{R}^N} G(x,\gamma_m(t))dx \le \rho\|\gamma_m(t)\|_2^2 + C_\rho\|\gamma_m(t)\|_{2^*}^{2^*}, \tag{5.8}$$

for any $\rho > 0$. Also, choosing m large enough such that $\delta(\lambda_m - \lambda) \le c_\lambda$ and $c'_\lambda(\lambda_m - \lambda) \le c_\lambda$ we obtain from (5.7)

$$I_\lambda(\gamma_m(t)) \le 4c_\lambda. \tag{5.9}$$

Gathering (5.8) and (5.9) it follows that

$$\int_{\mathbf{R}^N} V(x)\gamma_m^2(t)\,dx \le 2\Big(4c_\lambda + \rho\|\gamma_m(t)\|_2^2 + C_\rho\|\gamma_m(t)\|_{2^*}^{2^*}\Big).$$

Now using the Sobolev embedding $||u||_{2^*} \le C||\nabla u||_2, \forall u \in H$, and choosing $\rho < \frac{V_0}{4}$ we get

$$\int_{\mathbf{R}^N} V(x)\gamma_m^2(t)\,dx \le C\Big(c_\lambda + (2c'_\lambda + 5c_\lambda)^{\frac{2^*}{2}}\Big) \tag{5.10}$$

for some $C > 0$. Now we get (iii) from Point i) and (5.10). ∎

We shall now prove that when c'_λ exists the functional $I_\lambda(u)$ has a critical point at the mountain pass level c_λ which is contained in the set

$$C = \{u \in H; \int_{\mathbf{R}^N} \lambda|\nabla u|^2 + V(x)u^2\, dx \leq 2\beta(\lambda)\}$$

where $\beta(\lambda)$ is defined in Proposition 5.1. When $\lambda > 0$ is fixed, the norm

$$||u||_\lambda^2 = \int_{\mathbf{R}^N} \lambda|\nabla u|^2 + V(x)u^2\, dx$$

is equivalent to the norm $||\cdot||$. For $a > 0$ we define the set

$$F_a = C \cap I_\lambda^{-1}([c_\lambda - a, c_\lambda + a]).$$

Proposition 5.2. *For all $a > 0$,*

$$\inf_{u \in F_a} \|I'_\lambda(u)\| = 0. \tag{5.11}$$

Proof. Seeking a contradiction we assume that (5.11) does not hold. Then, because of the mountain pass geometry (see Proposition 4.1), $a > 0$ can be choosen such that for any $u \in F_a$, $||I'_\lambda(u)|| \geq a$ and $0 < a < \frac{1}{2}c_\lambda$. Using a deformation argument, there exist $\mu \in (0, a)$ and a homeomorphism $\eta : H \to H$ such that

i) $\eta(u) = u$ outside $I_\lambda^{-1}([c_\lambda - a, c_\lambda + a])$ and

$$I_\lambda(\eta(u)) \leq I_\lambda(u), \quad \text{for all} \;\; u \in H, \tag{5.12}$$

ii) for $||u||_\lambda^2 \leq \beta(\lambda)$ such that $I_\lambda(u) \leq c_\lambda + \mu$,

$$I_\lambda(\eta(u)) \leq c_\lambda - \mu. \tag{5.13}$$

Let $(\gamma_m) \subset \Gamma_\lambda$ be the sequence obtained in Proposition 5.1 where the choice $\delta = \frac{2}{5}c_\lambda$ is made. By Proposition 5.1 (ii) we can select a $k \in \mathbf{N}$ sufficiently large so that

$$\max_{t \in [0,1]} I_\lambda(\gamma_k(t)) \leq c_\lambda + \mu. \tag{5.14}$$

Clearly by i), $\eta \circ \gamma_k \in \Gamma_\lambda$. Now if $u = \gamma_k(t)$ with $I_\lambda(u) \leq c_\lambda - c_\lambda(\lambda_k - \lambda)$ then (5.12) implies that

$$I_\lambda(\eta(u)) \leq c_\lambda - c_\lambda(\lambda_k - \lambda). \tag{5.15}$$

On the other hand if $u = \gamma_k(t)$ with $I_\lambda(u) > c_\lambda - c_\lambda(\lambda_k - \lambda)$ then Proposition 5.1 and (5.14) imply that u is such that $||u||_\lambda^2 \leq \beta(\lambda)$ with $I_\lambda(u) \leq c_\lambda + \mu$. Now (5.13) gives that

$$I_\lambda(\eta(u)) \leq c_\lambda - \mu \tag{5.16}$$

which, combined with (5.15), yields

$$\max_{t \in [0,1]} I_\lambda(\eta \circ \gamma_k(t)) < c_\lambda.$$

This contradicts the variational characterization of c_λ and proves the proposition. ∎

Lemma 5.3. *Let λ be small enough and such that c'_λ exists. Then there exists a critical point of* (1.6), $u_\lambda \in H$ *satisfying* $I_\lambda(u_\lambda) = c_\lambda$ *and such that*

$$\int_{\mathbf{R}^N} \lambda |\nabla u_\lambda|^2 + V(x) u_\lambda^2 \, dx \leq 2\beta(\lambda).$$

Proof. From Proposition 5.2 when c'_λ exists, $I_\lambda(u)$ has a Palais-Smale sequence $(u_n) \subset H$ at the level c_λ which satisfies $||u_n||_\lambda^2 \leq 2\beta(\lambda)$. Since (u_n) is bounded in H, after extracting a subsequence if necessary, we may assume that $u_n \rightharpoonup u_\lambda$ in H. To show that the convergence is actually strong we adapt an argument of [**DF1**] who observe that it suffices to show that for any given $\delta > 0$ there exists $R > 0$ such that

$$\limsup_{j \to \infty} \int_{|y| \geq R} \lambda |\nabla u_n|^2 + V(x) u_n^2 \, dx < \delta. \tag{5.17}$$

Let $\eta_R \in C^\infty(\mathbf{R}^N, [0,1])$ such that $\eta_R(x) = 1$ for $|x| > R$, $\eta_R(x) = 0$ for $|x| \leq \frac{R}{2}$ and $|\nabla \eta_R(x)| \leq \frac{C}{R}$ in $\mathbf{R}^N$ for some positive constant $C > 0$.

Since $I'_\lambda(u_n)(\eta_R u_n) = o(1)$, we obtain that sufficiently large $R > 0$,

$$\begin{aligned}\int_{\mathbf{R}^N} (\lambda |\nabla u_n|^2 + V(x) u_n^2) \eta_R + \lambda u_n \nabla u_n \nabla \eta_R \, dx &= \int_{\mathbf{R}^N} \underline{f}(u_n) u_n \eta_R \, dx + o(1) \\ &\leq \nu \int_{\mathbf{R}^N} |u_n|^2 \eta_R \, dx + o(1).\end{aligned}$$

Therefore

$$\frac{1}{2} \int_{|y| \geq R} \lambda |\nabla u_n|^2 + V(x) u_n^2 \, dx \leq \frac{\lambda C}{R} \|u_n\|_2 \|\nabla u_n\|_2 + o(1)$$

and (5.17) clearly follows. ∎

6. End of the proof of Theorem 0.1

In this final section we end the proof of Theorem 0.1. First we show that there exist a special sequence (ε_n) decreasing to 0 on which the corresponding solutions $u_{\varepsilon_n}(x)$ satisfies the estimate (5.1). Having derive this bound the rest of the proof follows closely [**JT3**].

Lemma 6.1. *There exists a strictly decreasing sequence $\lambda_n \to 0$ such that*

$$c'_{\lambda_n} \leq C \lambda_n^{\frac{N}{2} - 1}$$

for some constant $C > 0$ independent of $n \in \mathbf{N}$. In particular, for the corresponding critical points $u_{\varepsilon_n}(x)$ of $I_{\varepsilon_n}(u)$ obtained in Lemma 5.3 we have

$$\int_{\mathbf{R}^N} \varepsilon_n^2 |\nabla u_{\varepsilon_n}|^2 + V(x) u_{\varepsilon_n}^2 \, dx \leq C \varepsilon_n^{\frac{N}{2}}. \tag{6.1}$$

Proof. Assume it is not true. We then have for any fixed $C > 0$,

$$\liminf_{\lambda \to 0} \frac{c'_\lambda}{\lambda^{\frac{N}{2}-1}} \geq C.$$

Thus for any λ small enough,

$$c_\lambda \geq c_\lambda - \lim_{h\to 0} c_h \geq \lim_{h\to 0} \int_h^\lambda c'_t dt \geq \lim_{h\to 0} \int_h^\lambda \frac{C}{2} t^{\frac{N}{2}-1} dt = \frac{C}{N} \lambda^{\frac{N}{2}}$$

and we conclude that, for λ small enough, $\frac{c_\lambda}{\lambda^{\frac{N}{2}}} \geq \frac{C}{N}$. This contradicts (4.5). ∎

Proposition 6.2. *Let (ε_n) be the sequence obtained in Lemma 6.1 and let $v_{\varepsilon_n}(x)$ denote the critical points of $J_{\varepsilon_n}(v)$ defined by $v_{\varepsilon_n}(x) = u_{\varepsilon_n}(\varepsilon_n x)$. Then for any subsequence of (ε_n) there exist a subsequence — denoted by ε_j — and (y_{ε_j}), x^1, ω^1 such that*

(i) $\varepsilon_j y_{\varepsilon_j} \to x^1$. (6.2)

(ii) $x^1 \in \Lambda'$ *satisfies* $V(x^1) = \inf_{x\in\Lambda} V(x)$. (6.3)

(iii) $\omega^1(y)$ *is a least energy solution of* $\Phi'_{x^1}(v) = 0$. (6.4)

(iv) $\left\| v_{\varepsilon_j} - \psi_{\varepsilon_j} \omega^1(y - y_{\varepsilon_j}) \right\|_{H_{\varepsilon_j}} \to 0$. (6.5)

(v) $J_{\varepsilon_j}(v_{\varepsilon_j}) \to m(x^1) = m(0)$. (6.6)

Proof. Taking into account Proposition 3.4, Proposition 6.2 hold true if $\ell = 1$ in Proposition 2.2. To see this let us apply Proposition 2.2 to the sequence (v_{ε_n}). By Proposition 2.2 there exist a subsequence denoted (ε_j), $\ell \in \mathbf{N} \cup \{0\}$, $(y^k_{\varepsilon_j})$, x^k, ω^k $(k = 1, 2, \cdots, \ell)$ satisfying (2.5)–(2.9). If we assume that $\ell = 0$, then (2.9) implies that $b_{\varepsilon_j} = J_{\varepsilon_j}(v_{\varepsilon_j}) \to 0$ in contradiction with (4.5) (see Remark 2.3 (i)). Thus $\ell \geq 1$ and again from (2.9) it follows that

$$\lim_{j\to\infty} b_{\varepsilon_j} = \sum_{k=1}^{\ell} \Phi_{x^k}(\omega^k) \geq \sum_{k=1}^{\ell} m(x^k) \geq \ell m(0) \geq m(0). \tag{6.7}$$

Combining (4.4) and (6.7), we deduce that $l = 1$. ∎

Now we are ready to give the proof of Theorem 0.1. Here we follow closely [**JT3**].

Proof of Theorem 0.1. To prove that the sequence $u_{\varepsilon_n}(x)$ obtained in Lemma 6.1 has the desired properties we shall work on the associated sequence $v_{\varepsilon_n}(x)$ of solutions of (1.6). Because of (6.1) we have that $||v_{\varepsilon_n}||_{H_{\varepsilon_n}} \leq C$ for a $C > 0$ and in particular (v_{ε_n}) is bounded in $H^1(\mathbf{R}^N)$.

Let us show that for any subsequence of (ε_n) there exists a subsequence — denoted by ε_j — such that for large j, v_{ε_j} takes a unique local maximum at $\bar{x}_{\varepsilon_j} \in \Lambda/\varepsilon_j$ with $V(\varepsilon_j \bar{x}_{\varepsilon_j}) \to \inf_{x\in\Lambda} V(x)$ and decreases sufficiently fast away from $\bar{x}_{\varepsilon_j}$. If this is the case it readily implies Theorem 0.1 by a contradiction argument.

We shall proceed in several steps. Let $\varepsilon_j \to 0$ be an arbitrary fixed sequence. Applying Proposition 6.2 we can assume that there exists (y_{ε_j}), x^1, ω^1 such that (6.2)–(6.6) hold. Moreover, by the maximum principle, $v_{\varepsilon_j}(y) \geq 0$ for all $y \in \mathbf{R}^N$.

Step 1 : *If a sequence $(z_{\varepsilon_j}) \subset \mathbf{R}^N$ satisfies*

$$\liminf_{j\to\infty} \int_{B_1(z_{\varepsilon_j})} |v_{\varepsilon_j}|^2\, dy > 0,$$

then $\limsup_{j\to\infty} |z_{\varepsilon_j} - y_{\varepsilon_j}| < \infty$. *In particular we have* $\lim_{j\to\infty} |\varepsilon_j z_{\varepsilon_j} - x^1| = 0$. *Conversely if (z_{ε_j}) satisfies $|z_{\varepsilon_j} - y_{\varepsilon_j}| \to \infty$, we have* $\int_{B_1(z_{\varepsilon_j})} |v_{\varepsilon_j}|^2\, dy \to 0$.

This clearly follows from (6.2), (6.5). ∎

Step 2 : $\sup_{z\in(\overline{\Lambda}\setminus\Lambda')/\varepsilon_j} |v_{\varepsilon_j}(z)| \to 0$ *as* $j \to \infty$. (6.8)

It follows from Step 1 that

$$\sup_{z\in(\overline{\Lambda}\setminus\Lambda')/\varepsilon_j} \int_{B_1(z)} |v_{\varepsilon_j}|^2\, dy \to 0 \quad \text{as } j \to \infty.$$

It also follows from the boundedness of (v_{ε_j}) in $H^1(\mathbf{R}^N)$ that

$$\|v_{\varepsilon_j}\|_{L^{s+1}(B_1(z))} \to 0 \quad \text{uniformly in } z \in (\overline{\Lambda} \setminus \Lambda')/\varepsilon_j. \tag{6.9}$$

We remark that $V(\varepsilon_j y)$, $\chi(\varepsilon_j y)$ stay bounded uniformly in $(\overline{\Lambda} \setminus \Lambda')/\varepsilon_j$ as $j \to \infty$. Thus since $v_{\varepsilon_j}(y)$ is a solution of

$$-\Delta v + V(\varepsilon_j y)v = g(\varepsilon_j y, v) \quad \text{in } B_1(z).$$

By standard regularity arguments we have $v_{\varepsilon_j}(y) \in C(B_1(z))$, and (6.8) implies

$$\|v_{\varepsilon_j}\|_{L^\infty(B_1(z))} \to 0 \quad \text{as } j \to \infty$$

uniformly in $z \in (\overline{\Lambda} \setminus \Lambda')/\varepsilon_j$. ∎

Step 3 : *For the constant $r_\nu > 0$ given in Section 1, there holds*

$$v_{\varepsilon_j}(y) \leq r_\nu \quad \text{in } \mathbf{R}^N \setminus (\Lambda'/\varepsilon_j). \tag{6.10}$$

By Step 2, $\sup_{z\in(\overline{\Lambda}\setminus\Lambda')/\varepsilon_j} |v_{\varepsilon_j}(y)| \leq \frac{r_\nu}{2}$ for small ε_j. Since $(v_{\varepsilon_j}(y) - r_\nu)_+ \,|_{\mathbf{R}^N\setminus(\Lambda'/\varepsilon_j)} \in H_\varepsilon$ it follows from $J'_\varepsilon(v_{\varepsilon_j})\left((v_{\varepsilon_j}(y) - r_\nu)_+ \,|_{\mathbf{R}^N\setminus(\Lambda'/\varepsilon_j)}\right) = 0$ that

$$\int_{\mathbf{R}^N\setminus(\Lambda'/\varepsilon_j)} |\nabla(v_{\varepsilon_j} - r_\nu)_+|^2 + V(\varepsilon_j y)v_{\varepsilon_j}(v_{\varepsilon_j} - r_\nu)_+ - \underline{f}(v_{\varepsilon_j})(v_{\varepsilon_j} - r_\nu)_+\, dy = 0.$$

By Lemma 1.2 (ii),

$$\int_{\mathbf{R}^N\setminus(\Lambda'/\varepsilon_j)} |\nabla(v_{\varepsilon_j}-r_\nu)_+|^2+(V_0-\nu)v_{\varepsilon_j}(v_{\varepsilon_j}-r_\nu)_+\,dy\le 0.$$

Thus $(v_{\varepsilon_j}-r_\nu)_+\equiv 0$ in $\mathbf{R}^N\setminus(\Lambda'/\varepsilon_j)$. That is, (6.10) holds. ∎

By Step 3 we see that $v_{\varepsilon_j}(y)$ is a solution of the rescaled original problem :

$$-\Delta v+V(\varepsilon_j y)v=f(v)\quad\text{in }\mathbf{R}^N$$

for sufficiently small $\varepsilon_j>0$. Since $f(\xi)\in C^1(\mathbf{R}^N,\mathbf{R})$, we have $v_{\varepsilon_j}(y)\in C^2(\mathbf{R}^N)$ from a standard regularity argument. From the boundedness of $\|v_{\varepsilon_j}\|_{H_\varepsilon}$ we can see also that $\|v_{\varepsilon_j}\|_{C^2(K/\varepsilon_j)}$ is bounded on any compact set $K\subset\mathbf{R}^N$ as $j\to\infty$. We remark that $V(\varepsilon_j y)$ and $\chi(\varepsilon_j y)$ stay bounded uniformly in K/ε_j as $j\to\infty$.

<u>Step 4 :</u> *Suppose that $v_{\varepsilon_j}(y)$ takes a local maximum at z_{ε_j}. Then (z_{ε_j}) satisfies*

$$\limsup_{j\to\infty}|z_{\varepsilon_j}-y_{\varepsilon_j}|<\infty\quad\textit{and}\quad \varepsilon_j z_{\varepsilon_j}\to x^1.$$

By the maximum principle, we see that $v_{\varepsilon_j}(z_{\varepsilon_j})\ge r_\nu$. Since $v_{\varepsilon_j}(y)$ is bounded in C^2_{loc}, we can also get $\liminf_{j\to\infty}\int_{B_1(z_{\varepsilon_j})}|v_{\varepsilon_j}|^2\,dy>0$. We conclude by Step 1. ∎

<u>Step 5 :</u> *v_{ε_j} has only one local maximum for ε_j small.*

Assume that $v_{\varepsilon_j}(y)$ takes a local maximum at $y=z_{\varepsilon_j}$. By the maximum principle, $v_{\varepsilon_j}(z_{\varepsilon_j})\ge r_\nu$. Since v_{ε_j} is bounded in $H^1(\mathbf{R}^N)$ and $C^2_{loc}(\mathbf{R}^N)$, after extracting a subsequence, we may assume $v_{\varepsilon_j}(y+z_{\varepsilon_j})\to\omega(y)$ weakly in $H^1(\mathbf{R}^N)$ and strongly in C^2_{loc} with $\omega(y)$ satisfying

$$-\Delta\omega+V(x^1)\omega=f(\omega)\quad\text{in }\mathbf{R}^N$$

and having a local maximum at $y=0$. Thus by the result of [**GNN**], $\omega(y)$ is radially symmetric with respect to 0 and strictly decreasing with respect to $r=|y|$. Thus if $v_{\varepsilon_j}(y)$ takes two local maxima at $y=z_{\varepsilon_j}$ and $y=z'_{\varepsilon_j}$, then necessarily $|z_{\varepsilon_j}-z'_{\varepsilon_j}|\to\infty$. However Step 4 implies $\limsup|z_{\varepsilon_j}-z'_{\varepsilon_j}|\le\limsup|z_{\varepsilon_j}-y_{\varepsilon_j}|+\limsup|z'_{\varepsilon_j}-y_{\varepsilon_j}|<\infty$. This contradiction shows that $v_{\varepsilon_j}(y)$ takes only one local maximum. ∎

<u>Step 6 :</u> *There exists $\ell_0>0$ such that for small $\varepsilon_j>0$*

$$|v_{\varepsilon_j}(y)|<r_\nu\quad\textit{for all}\quad |y-\bar{x}_{\varepsilon_j}|\ge\ell_0,$$

where $\bar{x}_{\varepsilon_j}$ is the unique local maximum of $v_{\varepsilon_j}(y)$.

Indeed, if z_{ε_j} satisfies $v_{\varepsilon_j}(z_{\varepsilon_j}) \geq r_\nu$, then we have $\liminf_{j\to\infty} \int_{B_1(z_{\varepsilon_j})} |v_{\varepsilon_j}|^2\, dy > 0$ and Steps 1,4 implies that $\limsup |z_{\varepsilon_j} - \bar{x}_{\varepsilon_j}| \leq \limsup |z_{\varepsilon_j} - y_{\varepsilon_j}| + \limsup |y_{\varepsilon_j} - \bar{x}_{\varepsilon_j}| < \infty$. Thus there is no sequence (z_{ε_j}) satisfying $|z_{\varepsilon_j} - \bar{x}_{\varepsilon_j}| \to \infty$ and $v_{\varepsilon_j}(z_{\varepsilon_j}) \geq r_\nu$. Step 6 follows. ∎

Step 7 : *Conclusion.*

Consider the unique solution $\eta(y) \in H^1(|y| \geq \ell_0)$ of the following problem :

$$\begin{aligned} -\Delta\eta + \frac{V_0}{2}\eta = 0 \quad & \text{in } |y| \geq \ell_0, \\ \eta(y) = r_\nu \quad & \text{on } |y| = \ell_0. \end{aligned}$$

It is easily seen that $\eta(y)$ has an exponential decay and since $\frac{f(v_{\varepsilon_j}(y))}{v_{\varepsilon_j}(y)} \leq \frac{V_0}{2}$ when $|y| \geq \ell_0$, we have, by the maximum principle that $v_{\varepsilon_j}(y + \bar{x}_{\varepsilon_j}) \leq \eta(y)$ for $|y| \geq \ell_0$. Thus $v_{\varepsilon_j}(y)$ also has an exponential decay.

At this point it is clear that $u_{\varepsilon_j}(x) = v_{\varepsilon_j}(x/\varepsilon_j)$ has the desired properties. This concludes the proof of Theorem 0.1. ∎

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