Path integral in Quantum Theory

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1. Outline:
   - Path integral and Partition Functions
   - Wick Rotation
   - Quantum Field Theory
   - Bethe Ansatz
   - 1+1D QFT

Schrdinger's picture v.s. Heisenberg's picture

S picture

quantum state: $|\psi\rangle$

Observer: A

Dynamics:$|\psi(t)\rangle = U(t)|\psi\rangle$

$i\hbar\frac{\partial}{\partial t}|(\psi(t))\rangle = H|\psi(t)\rangle$

H Picture

$|\psi;t\rangle = U(-t)|\psi\rangle$

$A(t) = U(-t)AU(t)$

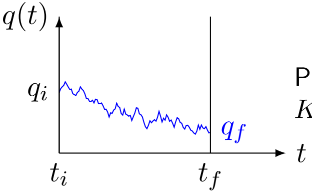
$|\psi(t);t\rangle = U(t)U(-t)|\psi\rangle = |\psi\rangle$

$i\hbar\frac{\partial}{\partial t}A(t) = -[H, A(t)]$

Path integral

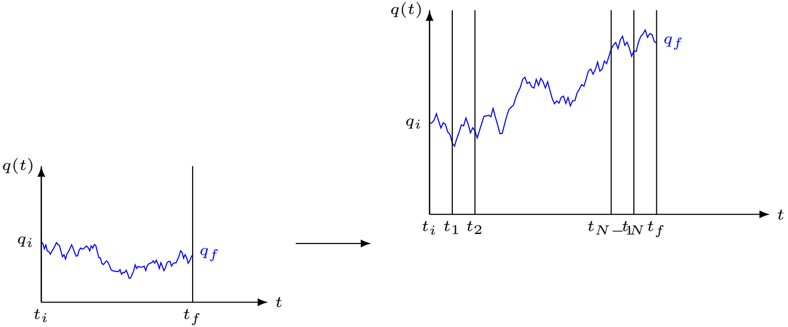
- 1D non-relativistic particle

$$|\psi(t)\rangle = U(t)\,|\psi\rangle \text{ Where } U(t) = e^{-\frac{it}{\hbar}H}$$



Probability Amplitude:
$K(q_f, t_f; q_i, t_i) = \langle q_f|U(t_f - t_i)\,|q_i\rangle$

Path Integral



Probability Amplitude:

$$
\begin{aligned}
\mathrm{K}(q_f, t_f; q_i, t_i) &= \langle q_f | \, U(t_f - t_i) \, | q_i \rangle \\
&= \langle q_f | \, e^{-\frac{i(t_f - t_i)}{\hbar} H} \, | q_i \rangle \\
&= \langle q_f | \, e^{-\frac{i\delta}{\hbar} H} e^{-\frac{i\delta}{\hbar} H} \ldots e^{-\frac{i\delta}{\hbar} H} \, | q_i \rangle
\end{aligned}
$$

where $\delta = \frac{t_f - t_i}{N}$

Path Integral

use identity: $\int |q\rangle \langle q| \, \mathrm{d}q = 1$ We have:

$$\langle q_f| e^{-\frac{i(t_f - t_i)}{\hbar} H} |q_i\rangle =$$

$$(\prod_{j=1}^{N-1} \int dq_j) \langle q_f| e^{-\frac{i\delta}{\hbar} H} |q_{N-1}\rangle \langle q_{N-1}| e^{-\frac{i\delta}{\hbar} H} |q_{N-2}\rangle \ldots \langle q_2| e^{-\frac{i\delta}{\hbar} H} |q_1\rangle \langle q_1| e^{-\frac{i\delta}{\hbar} H} |q_i\rangle$$

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Consider $\langle q_{j+1}| \, e^{-\frac{i\delta}{\hbar}(p^2/2m)} \, |q_j\rangle = \int \frac{dp}{2\pi} \langle q_{j+1}| \, e^{-\frac{i\delta}{\hbar}(p^2/2m)} \, |p\rangle \langle p|q_j\rangle$

$$= \int \frac{dp}{2\pi} e^{-i(p^2/2m)} e^{ip(q_{j+1} - q_j)}$$

Where $H = p^2/2m$ and $\int \frac{dp}{2\pi} |p\rangle \langle p| = 1$

Path integral

Take the limit $N \to \infty$ and use Gaussian integral we have:

$$\langle q_f | e^{-\frac{iT}{\hbar}H} | q_i \rangle = \boxed{\int D[q] e^{(\frac{i}{\hbar}S)}}$$

Where $S = \int dtL$ is the action of the theory.

$$\boxed{\int D[q(t)] = \lim_{N\to\infty} \left(\frac{-i2\pi m}{\delta}\right)^{\frac{N}{2}} \prod_{j=0}^{N-1} \int dq_j}$$

Probability Amplitude

- What is $\int D[q]e^{(\frac{i}{\hbar}S(q))}q(t_1)$? where $t_i < t_1 < t_f$

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- Correspondence Principal

$$\begin{aligned}\int D[q]e^{(\frac{i}{\hbar}S(q))}q(t_1) &= \int dq_1 \overbrace{\int D[q]e^{(\frac{i}{\hbar}S(q))}}^{p_1\to p_f} \times \overbrace{\int D[q]e^{(\frac{i}{\hbar}S(q))}}^{p_i\to p_1} \times q(t_1) \\ &= \int dq_1 \langle q_f|U(t_f - t_1)|q_1\rangle\, q(t_1)\, \langle q_1|U(t_1 - t_i)|q_i\rangle \\ &= \langle q_f; t_f|\, Q(t_1)\, |q_i; t_i\rangle\end{aligned}$$

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- It automatically oder the operators in time!

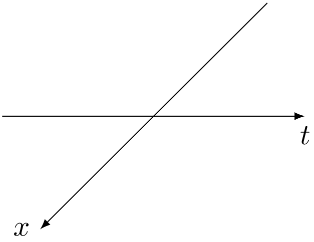
$$\int D[q]e^{(\frac{i}{\hbar}S(q))}q(t_1)q(t_2)$$

$$= \int dq_1 dq_2 \overbrace{\int D[q]e^{(\frac{i}{\hbar}S(q))}}^{p_2 \to p_f} \times \overbrace{\int D[q]e^{(\frac{i}{\hbar}S(q))}}^{p_1 \to p_2} \times \overbrace{\int D[q]e^{(\frac{i}{\hbar}S(q))}}^{p_i \to p_1} \times q(t_1)q(t_2)$$

$$= \int dq_1 dq_2 \langle q_f|U(t_f - t_2)|q_2\rangle\, q(t_2)\, \langle q_2|U(t_2 - t_1)|q_1\rangle\, q(t_1)\, \langle q_1|U(t_1 - t_i)|q_i\rangle$$

$$= \langle q_f; t_f|\, Q(t_2)Q(t_1)\, |q_i; t_i\rangle$$

Wick Rotation



- Minkovski Spacetime
  $ds^2 = -dt^2 + d\vec{x}^2$

$$U(t) = \sum e^{\frac{t}{i\hbar}E_n} |n\rangle \langle n|$$
$$= \sum \underbrace{e^{-\frac{\tau}{\hbar}E_n}}_{\text{convergent only if } \tau > 0} |n\rangle \langle n|$$

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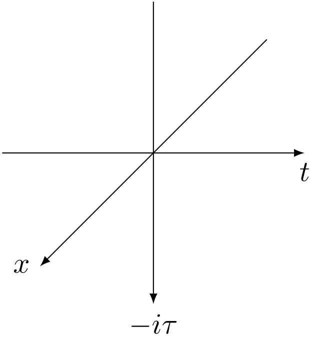

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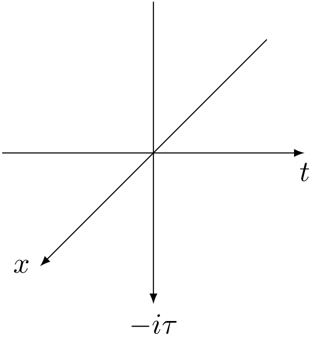

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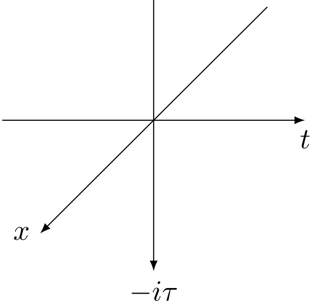

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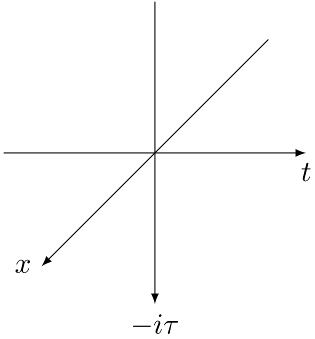

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- We will see $\tau = \beta\hbar = \frac{\hbar}{k_B T}$

Why Euclidian time?

- $$S_C(q) = \int \frac{m}{2}\dot{q}^2 - V(q) \longrightarrow \ddot{q}(t) = -\frac{\partial}{\partial q}V(q(t))$$
  $$S_E(q) = \int \frac{m}{2}\dot{q}^2 + V(q) \longrightarrow \ddot{q}(\tau) = \frac{\partial}{\partial q}V(q(\tau))$$

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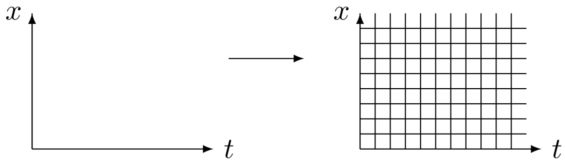
- Path integral for Euclidian time:
  $$\mathrm{K}(\mathrm{q}_f, \tau_f; q_i, \tau_i) = \langle q_f | U_E(\tau_f - \tau_i) | q_i \rangle$$
  $$= \int_{\substack{q(\tau_i)=q_i \\ q(\tau_f)=q_f}} D[q] \exp\left(-\frac{1}{\hbar} S_E(q)\right)$$

Field Theory

Difficulty: $\phi(x,t)$ has infinitly many degrees of freedom
Solution: Discretization



The idea is to treat x as a label of ϕ, just as the subscript i of q_i.

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The functional integral of field theory in Euclidian time is:

$$Z = \int \mathcal{D}[\phi(x,\tau)] \exp\left(-\frac{1}{\hbar} S_E[\phi]\right)$$

Thermodynamics

- Partition Function: $Z = Tr(\exp(-\beta H))$
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- Replace time by an imaginary number (Wick rotation) in the evolving operator $U(t)$: $t \to -i\tau$

$$U(t) = e^{-\frac{it}{\hbar}H} \to \boxed{U(-i\tau) = e^{-\frac{\tau}{\hbar}H} = e^{-\beta H}}$$

where $\tau = \beta\hbar$

Partition Function

- Partition function of a quantum system:

$$Z = Tr(U_E(\beta\hbar)) = \sum \langle n| U_E(\beta\hbar) |n\rangle = \sum e^{\beta E_n}$$

where we denote $U(-i\tau) := U_E(\tau)$

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- $$\begin{aligned} \text{Z} &= \sum \langle n| U_E(\beta\hbar) |n\rangle \\ &= \int dq \, \langle q| U_E(\beta\hbar) |q\rangle \\ &= \int_{q(\beta\hbar)=q(0)} D[q] \exp\left(-\frac{1}{\hbar} S_E(q)\right) \end{aligned}$$

Thermodynamics

- $$\overbrace{\int_{\substack{q(\tau_i)=q_i \\ q(\tau_f)=q_f}} D[q] \exp\left(-\frac{1}{\hbar} S_E(q)\right)}^{Euclidian PI} \leftrightarrow \overbrace{\int_{q(\beta\hbar)=q(0)} D[q] \exp\left(-\frac{1}{\hbar} S_E(q)\right)}^{\text{Partition Function}}$$

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- $\boxed{q(\beta\hbar) = q(0)} \longleftrightarrow$ boundary condition
- $\tau = \beta\hbar = \frac{\hbar}{k_B T}$
- Finite temperature $\longleftrightarrow$ Euclidian period

Ground State

- Go back to Euclidian time:

$$U_E(\tau) = \exp\left(-\frac{\tau}{\hbar}H\right)$$

where $H|n\rangle = E_n|n\rangle$ and $E_0 < E_1 < ...$

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  where $H\left|n\right\rangle = E_n \left|n\right\rangle$ and $E_0 < E_1 < \ldots$

- $U_E(\tau) = \sum \exp\left(-\tfrac{\tau}{\hbar} E_n\right) \left|n\right\rangle \left\langle n\right| \simeq \exp\left(-\tfrac{\tau}{\hbar} E_0\right) \left|0\right\rangle \left\langle 0\right|$

  when $\tau \to \infty$

Ground state

- A thermal state:

$$\langle F(q)\rangle_\beta = \frac{Tr(F(q)U_E(\beta\hbar))}{Tr(U_E(\hbar\beta))} = \frac{\int_{q(\beta\hbar)=q(0)} D[q]\exp\left(-\frac{1}{\hbar}S_E(q)\right)F(q)}{\int_{q(\beta\hbar)=q(0)} D[q]\exp\left(-\frac{1}{\hbar}S_E(q)\right)}$$

with periodic boundary condition.

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  with periodic boundary condition.

- In oder to get the vacuum expectation value $\langle 0|F(q)|0\rangle$, we can use the previous result and take the limit $\tau \to \infty$

"Coordinate Bethe Ansatz" for $XXX_{1/2}$ Heisenberg spin chain

Eigenstate of the Hamiltonian:

$$\text{H} = -\sum_{i=1}^{L} \vec{\sigma_i} \cdot \vec{\sigma_{i+1}} = L - 2\sum_{i=1}^{L} \mathcal{P}_{i,i+1}$$

- vacuum: $\underbrace{|\downarrow\downarrow\downarrow \cdots \downarrow\downarrow\rangle}_{L\text{sites}}$

$\mathcal{H} = (\mathbb{C}^2)^{\otimes L}$; with boundary condition $\vec{\sigma_{L+1}} = \vec{\sigma_1}$ where $\mathcal{P}_{i,i+1}$ is permutation operator.

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- vacuum: $\underbrace{|\downarrow\downarrow\downarrow \cdots \downarrow\downarrow\rangle}_{L \text{sites}}$
- single particle state: $|\psi\rangle \propto \sum_k e^{ikp} |\{k\}\rangle$ with $e^{2ipL} = 1$

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- where $S = -\frac{1+e^{i(p_1+p_2)}-2e^{ip_2}}{1+e^{i(p_1+p_2)}-2e^{ip_1}}$

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  inorder to make $|\psi\rangle$ an eigenstate, need
- $\mathcal{A}_\sigma \propto (-1)^\sigma \prod_{j<k} (1 + e^{i(p_{\sigma(j)} + p_{\sigma(k)})} - 2e^{ip_{\sigma(k)}})$

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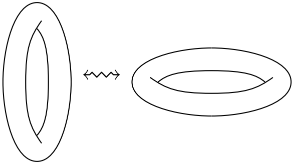
$$\mathsf{H} = -\sum_{i=1}^{L} \vec{\sigma_i} \cdot \vec{\sigma_{i+1}} = L - 2\sum_{i=1}^{L} \mathcal{P}_{i,i+1}$$

$\mathcal{H} = (\mathbb{C}^2)^{\otimes L}$; with boundary condition $\vec{\sigma_{L+1}} = \vec{\sigma_1}$

- n particle state: $|\psi\rangle = \sum_{1\leq j_1<j_2\cdots j_n\leq L} \sum_{\sigma\in\mathfrak{S}_n} \mathcal{A}_\sigma e^{i\sum_k p_{\sigma(k)} j_k} |\{j_1, j_2, \ldots, j_n\}\rangle$ with $e^{iLp_2} = S = e^{-iLp_1}$
  inorder to make $|\psi\rangle$ an eigenstate, need
- $\mathcal{A}_\sigma \propto (-1)^\sigma \prod_{j<k}(1 + e^{i(p_{\sigma(j)}+p_{\sigma(k)})} - 2e^{ip_{\sigma(k)}})$
- $\forall j, e^{iLp_j} = \prod_{k\neq j} S(p_j, p_k)$ where $S(p,p') = -\frac{1+e^{i(p+p')}-2e^{ip}}{1+e^{i(p+p')}-2e^{ip'}}$

$1 + 1$D Field Theory

After exchange the space and time:



- Set of relativistic particles strongly separated to avoid interaction (off-mass-shell effect) .

$1 + 1$D Field Theory

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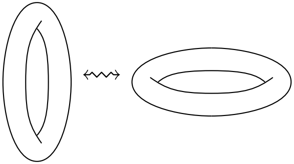
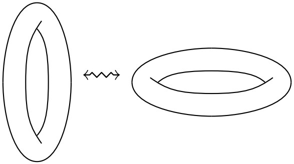


- Set of relativistic particles strongly separated to avoid interaction (off-mass-shell effect) .
- Can use coordinates x and momenta p, further introduce wave function $\psi(x_1, \ldots, x_N)$

$1 + 1$D Field Theory

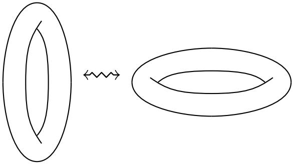
After exchange the space and time:



- These wave functions are Bethe wave functions.

$1+1$D Field Theory

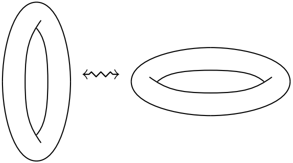
After exchange the space and time:



- These wave functions are Bethe wave functions.
- Denote the free region as $(i_1, i_2, \ldots, i_N)$ if $x_{i_1} < x_{i_2} < \cdots < x_{i_N}$.

$1+1$D Field Theory

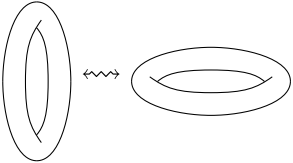
After exchange the space and time:



- In purely elastic scattering case, every transition, $(i_1, \ldots, i_r, i_{r+1}, \ldots, i_N) \rightarrow (i_1, \ldots, i_{r+1}, i_r, \ldots, i_N)$ will contribute a scattering amplitude to the wave function.

1 + 1D Field Theory

After exchange the space and time:

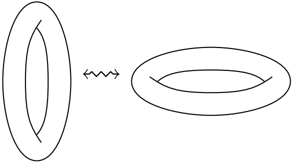


The Bethe wave function is eigenstate of the theory if:

- The space is one dimensional and there is periodic boundary condition

$1+1$D Field Theory

After exchange the space and time:

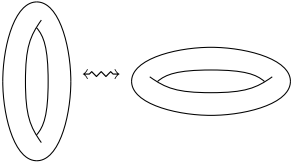


The Bethe wave function is eigenstate of the theory if:

- The space is one dimensional and there is periodic boundary condition
- No off mass shell effect

$1+1$D Field Theory

After exchange the space and time:



The Bethe wave function is eigenstate of the theory if:

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- Factorization formula must holds

$1+1$D Field Theory

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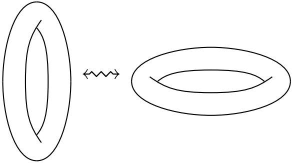

The Bethe wave function is eigenstate of the theory if:

- The space is one dimensional and there is periodic boundary condition
- No off mass shell effect
- Factorization formula must holds
- Need infinitely many conserved charges.

Thank you!