Existence and orbital stability of standing waves for nonlinear Schrödinger systems

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Abstract

In this paper we investigate the existence of solutions in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ for nonlinear Schrödinger systems of the form

$$\begin{cases} -\Delta u_1 = \lambda_1 u_1 + \mu_1 |u_1|^{p_1-2} u_1 + r_1 \beta |u_1|^{r_1-2} u_1 |u_2|^{r_2}, \\ -\Delta u_2 = \lambda_2 u_2 + \mu_2 |u_2|^{p_2-2} u_2 + r_2 \beta |u_1|^{r_1} |u_2|^{r_2-2} u_2, \end{cases}$$

under the constraints

$$\int_{\mathbb{R}^N} |u_1|^2 \, dx = a_1 > 0, \quad \int_{\mathbb{R}^N} |u_2|^2 \, dx = a_2 > 0.$$

Here $N \geq 1, \beta > 0, \mu_i > 0, r_i > 1, 2 < p_i < 2+\frac{4}{N}$ for $i = 1, 2$ and $r_1+r_2 < 2+\frac{4}{N}$. This problem is motivated by the search of standing waves for an evolution problem appearing in several physical models. Our solutions are obtained as constrained global minimizers of an associated functional. Note that in the system λ_1 and λ_2 are unknown and will correspond to the Lagrange multipliers. Our main result is the precompactness of the minimizing sequences, up to translation. Assuming the local well posedness of the associated evolution problem we then obtain the orbital stability of the standing waves associated to the set of minimizers.

Keywords: Nonlinear Schrödinger systems, standing waves, orbital stability, minimizing sequences, symmetric-decreasing rearrangements.

1 Introduction

We consider the existence of solutions to a nonlinear Schrödinger system of the form

$$\begin{cases} -\Delta u_1 = \lambda_1 u_1 + \mu_1 |u_1|^{p_1-2} u_1 + r_1 \beta |u_1|^{r_1-2} u_1 |u_2|^{r_2}, \\ -\Delta u_2 = \lambda_2 u_2 + \mu_2 |u_2|^{p_2-2} u_2 + r_2 \beta |u_1|^{r_1} |u_2|^{r_2-2} u_2, \end{cases} \tag{1.1}$$

satisfying the conditions

$$\int_{\mathbb{R}^N} |u_1|^2\,dx = a_1, \quad \int_{\mathbb{R}^N} |u_2|^2\,dx = a_2. \tag{1.2}$$

Here $a_1, a_2 > 0$ are prescribed and we shall assume throughout the paper

(H0) $N \geq 1, \beta > 0, \mu_i > 0, r_i > 1, 2 < p_i < 2 + \frac{4}{N}$ for $i = 1, 2$ and $r_1 + r_2 < 2 + \frac{4}{N}$.

The problem under consideration is associated to the research of standing waves, namely, solutions having the form

$$\Psi_1(t,x) = e^{-i\lambda_1 t} u_1(x), \quad \Psi_2(t,x) = e^{-i\lambda_2 t} u_2(x),$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$, of the nonlinear Schrödinger system

$$\begin{cases} -i\partial_t \Psi_1 = \Delta\Psi_1 + \mu_1 |\Psi_1|^{p_1-2}\Psi_1 + \beta|\Psi_1|^{r_1-2}\Psi_1|\Psi_2|^{r_2}, \\ -i\partial_t \Psi_2 = \Delta\Psi_2 + \mu_2 |\Psi_2|^{p_2-2}\Psi_2 + \beta|\Psi_1|^{r_1}|\Psi_2|^{r_2-2}\Psi_2, \end{cases} \quad \text{in } \mathbb{R} \times \mathbb{R}^N. \tag{1.3}$$

This system comes from mean field models for binary mixtures of Bose-Einstein condensates or for binary gases of fermion atoms in degenerate quantum states (Bose-Fermi mixtures, Fermi-Fermi mixtures), see [2, 12, 22].

One motivation to look for normalized solutions of system (1.1) is that the masses

$$\int_{\mathbb{R}^N} |\Psi_1|^2\,dx \quad \text{and} \quad \int_{\mathbb{R}^N} |\Psi_2|^2\,dx$$

are preserved along the trajectories of (1.3). Our solutions of (1.1)-(1.2) will be obtained as minimizers of the functional

$$J(u_1,u_2) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla u_2|^2\,dx - \int_{\mathbb{R}^N} \frac{\mu_1}{p_1}|u_1|^{p_1} + \frac{\mu_2}{p_2}|u_2|^{p_2} + \beta|u_1|^{r_1}|u_2|^{r_2}\,dx$$

constrained on

$$S(a_1,a_2) := \{(u_1,u_2) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : \|u_1\|_2^2 = a_1, \|u_2\|_2^2 = a_2\}.$$

Namely we are to consider the minimization problem

$$m(a_1,a_2) := \inf_{(u_1,u_2)\in S(a_1,a_2)} J(u_1,u_2). \tag{1.4}$$

It is standard that the minimizers of (1.4) are solutions to (1.1)-(1.2) where λ_1, λ_2 appear as the Lagrange multipliers. Actually the existence of minimizers for (1.4) will be obtained as a consequence of the stronger statement that any minimizing sequence for (1.4) is, up to translation, precompact.

Theorem 1.1. *Assume (H0). Then for any $a_1 > 0$ and $a_2 > 0$ all minimizing sequences for* (1.4) *are precompact in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ after a suitable translation.*

Following some initial works [30,31], the compactness concentration principle of P.L. Lions [19,20] has had, over the last thirty years, a deep influence on solving minimization problems under constraints. Heuristic arguments readily convince that in our problem the compactness of any minimizing sequence holds if the following strict subadditivity conditions are satisfied.

$$m(a_1, a_2) < m(b_1, b_2) + m(a_1 - b_1, a_2 - b_2), \tag{1.5}$$

where $0 \leq b_i < a_i$ for $i = 1, 2$ and $b_1 + b_2 \neq 0$.

To deal with just one constraint, several techniques have been developed to prove strict subadditivity conditions. Most are based on some homogeneity type property. In autonomous case it is also possible to use scaling arguments, see for example [5, 11, 29]. In the case of multiple constraints how to establish strict subadditivity conditions is much less understood. As a matter of fact few papers address the issue of compactness of minimizing sequences for systems as (1.1)-(1.2). Moreover in most of them there is either exactly one constraint [7] or the two constraints cannot be chosen independently [24, 25, 27]. Concerning (1.4) the more complete results seem to be due to [26]. In [26] the precompactness of minimizing sequences is obtained assuming $N = 1$. To exclude the dichotomy the authors crucially applied [1, Lemma 2.10] which depends in turn on original ideas introduced in [6], see also [13]. In [1, Lemma 2.10] it is shown that the $H^1(\mathbb{R})$ norm of some functions are strictly decreasing when the masses of the functions are symmetrically rearranged. See also [4,21] for similar arguments on related problems.

If one is merely interested in the existence of one minimizer, two papers should be mentioned. In [8] the existence of one minimizer had been achieved still for $N = 1$. The restriction on the dimension was subsequently removed in [3] where the existence of a minimizer for (1.4) was obtained in full generality in $H^1(\mathbb{R}^N)$ for $N = 2, 3, 4$ and under some restrictions for $N \geq 5$, see [3, Theorem 2.1] for a precise statement.

In this paper, inspired by [16], we propose an alternatively simple approach to verify the compactness of the minimizing sequences for (1.4) in any dimension. It is standard that any minimizing sequence $\{(u_1^n, u_2^n)\} \subset S(a_1, a_2)$ is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and thus without restriction we can assume that $u_1^n \rightharpoonup u_1$ and $u_2^n \rightharpoonup u_2$ weakly in $H^1(\mathbb{R}^N)$. To demonstrate the strong convergence in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, we first prove the weaker result that, up to translation, $\{(u_1^n, u_2^n)\}$ is strongly convergent in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$, $2 < p < 2^*$. Because we deal with a minimizing sequence it is clear that neither $\{u_1^n\}$ nor $\{u_2^n\}$ can vanish. We also observe that, in contrast to what happens in $L^2(\mathbb{R}^N)$, the non compactness of $\{u_i^n\}$ in $L^p(\mathbb{R}^N)$ implies the existence of at least two bumps going apart one from another. By bumps we mean here exist a $R < \infty$ and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $\liminf_{n\to\infty} \int_{B(y_n,R)} |u_i^n|^p dx > 0$. At this point we make use of a very nice result of M. Shibata [28] as presented in [16, Lemma A.1]. This result, which can somehow be considered as an extension of [1, Lemma 2.10] to any dimension, shows that the existence of two or more bumps for one of the sequence $\{u_i^n\}$ contradicts with its minimizing character. At this point we have proved the compactness of each $\{u_i^n\}$ in $L^p(\mathbb{R}^N)$ and we end the proof of the convergence in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ by showing that the bumps of $\{u_1^n\}$ and $\{u_2^n\}$ cannot move away from each other.

With this convergence, assuming that (u_1, u_2) is the weak limit of one minimizing sequence $\{(u_1^n, u_2^n)\}$, we have that $J(u_1, u_2) \leq m(a_1, a_2)$. Namely our functional is lower semicontinuous on minimizing sequences. If $\|u_1\|_2^2 = a_1$ and $\|u_2\|_2^2 = a_2$ the strong convergence in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ immediately results. Suppose not and assume that $\|u_1\|_2^2 := b_1 < a_1$ or $\|u_2\|_2^2 := b_2 < a_2$. Since $J(u_1, u_2) \leq m(a_1, a_2)$ it follows that $m(b_1, b_2) \leq m(a_1, a_2)$. We then reach a contradiction via observing that the weak version (1.5), where an equality is allowed, always holds and that it implies that the function $(c_1, c_2) \mapsto m(c_1, c_2)$ is strictly decreasing in both arguments. For related observations we refer to [17], see also [16].

Remark 1.2. Note that if one is just interested in the existence of one minimizer for (1.4) a shorter proof can be given. Choosing a minimizing sequence $\{(u_1^n, u_2^n)\} \subset S(a_1, a_2)$ which consists of Schwarz symmetric functions then, thanks to the compact embedding of $H_r^1(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$, $2 < p < 2^*$ (here $H_r^1(\mathbb{R}^N)$ denotes the the subspace of radially symmetric functions of $H^1(\mathbb{R}^N)$), it readily follows that if (u_1, u_2) is the weak limit of $\{(u_1^n, u_2^n)\}$ then $J(u_1, u_2) \leq m(a_1, a_2)$. The rest of the proof is identical to the one of Theorem 1.1. Alternately it is possible to obtain the existence of a minimizer working directly in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$. In that direction we refer to Remark 3.4 later in this paper.

Remark 1.3. The scheme to treat the compactness of minimizing sequences for (1.4) could be carried to deal with n constraints minimization problems on $\mathbb{R}^N$. More precisely,

$$m(a_1, \cdots, a_n) := \inf_{S(a_1, \cdots, a_n)} J(u_1, \cdots, u_n),$$

where $S(a_1, \cdots, a_n) := \{(u_1, \cdots, u_n) \in H^1(\mathbb{R}^N) \times \cdots \times H^1(\mathbb{R}^N) : \|u_i\|_2^2 = a_i > 0 \text{ for } i = 1, \cdots, n\}$,

$$J(u_1, \cdots, u_n) := \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i=1}^n |\nabla u_i|^2 \, dx - \int_{\mathbb{R}^N} \sum_{i=1}^n \frac{\mu_i}{p_i} |u_i|^{p_i} + \frac{1}{2} \sum_{i \neq j}^n \beta_{ij} |u_i|^{r_i} |u_j|^{r_j} \, dx,$$

and $N \geq 1, \mu_i > 0, \beta_{ji} = \beta_{ij} > 0$, $2 < p_i < 2 + \frac{4}{N}, r_i, r_j > 1, r_i + r_j < 2 + \frac{4}{N}$ for $i, j = 1, \cdots, n$.

Remark 1.4. In [28, Theorem 4.1] the author proved the convergence of any minimizing sequences for a minimizing problem of the form

$$E_{a_1,a_2} := \inf_{(u_1,u_2) \in S(a_1,a_2)} \tilde{J}(u_1, u_2). \tag{1.6}$$

where
$$\tilde{J}(u_1,u_2)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_1|^2+|\nabla u_2|^2\,dx-\int_{\mathbb{R}^N}G(|u_1|^2,|u_2|^2)\,dx.$$
The function G satisfies various assumptions, see [28] for a precise statement, and in particular it is required, setting $g_i(s_1,s_2)=\frac{\partial G}{\partial s_i}(s_1,s_2)$ for $i=1,2$ that

(G4) g_i is nondecreasing, that is, $g_i(s_1,s_2)\leq g_i(s_1+h_1,s_2+h_2)$ for $s_i,h_i\geq 0, i=1,2$.

The motivation to study the compactness of the minimizing sequences of (1.6) is to extend the result of [14] where the existence of just one minimizer for (1.6) was obtained. Our minimizing problem (1.4) can be embedded into (1.6) by setting
$$G(s_1,s_2)=\frac{\mu_1}{p_1}s_1^{\frac{p_1}{2}}+\frac{\mu_2}{p_2}s_2^{\frac{p_2}{2}}+\beta s_1^{\frac{r_1}{2}}s_2^{\frac{r_2}{2}},$$
but to satisfy (G4) one then needs $r_1\geq 2$ and $r_2\geq 2$. Since the condition $r_1+r_2<2+\frac{4}{N}$ must hold, this shows that [28, Theorem 4.1] and Theorem 1.1 partially overlap only when $N=1$. In [28, Theorem 4.1] the convergence of an arbitrary minimizing sequence is obtained by proving directly the strong convergence in $L^2(\mathbb{R}^N)\times L^2(\mathbb{R}^N)$. The proof, also by contradiction, starts to establish that for a minimizing sequence $\{(u_1^n,u_2^n)\}\subset S(a_1,a_2)$ with $u_n^1\rightharpoonup u_1$ and $u_n^2\rightharpoonup u_2$ one always has

$$E_{a_1,a_2}=E_{b_1,b_2}+E_{a_1-b_1,a_2-b_2} \tag{1.7}$$

where $b_1:=||u_1||_2^2$ and $b_2:=||u_2||_2^2$. Now if $b_1<a_1$ or $b_2<a_2$ observing that for any $c_1,c_2\in\mathbb{R}^+$ the minimizing sequences of E_{c_1,c_2} are non vanishing one is able, using the results presented in [16, Lemma A.1] (corresponding to [28, Lemma 2.2 and Theorem 2.4]) to show that

$$E_{b_1+d_1,b_2+d_2}<E_{b_1,b_2}+E_{d_1,d_2} \tag{1.8}$$

for some $0\leq d_i\leq a_i-b_i, i=1,2$. At this point a contradiction with (1.7) can be obtained. The need of condition (G4) appears in the proof of (1.8) to control the behaviour of the term $\int_{\mathbb{R}^N}G(|u_1|^2,|u_2|^2)\,dx$ under the rearrangement, see [28, Lemma A.2].

Set
$$G(a_1,a_2):=\{(u_1,u_2)\in S(a_1,a_2):J(u_1,u_2)=m(a_1,a_2)\}.$$

Note that under assumption (H0) it is not known if (1.3) is locally well posed. The point being that when $1<r_i<2$ for $i=1,2$ the interaction part is not Lipchitz continuous and in particular the uniqueness may fail. For a general discussion in that direction we refer to [23]. As a consequence our last result which states the orbital stability of the set of standing waves associated to $G(a_1,a_2)$ is only valid under condition.

Theorem 1.5. *Assume (H0) and the local existence of the Cauchy problem in* (1.3). *Then the set $G(a_1, a_2)$ is orbitally stable, i.e. for any $\varepsilon > 0$, there exists $\delta > 0$ so that if the initial condition $(\psi_1(0), \psi_2(0))$ in system* (1.3) *satisfies*

$$\inf_{(u_1,u_2)\in G(a_1,a_2)} \|(\psi_1(0), \psi_2(0)) - (u_1, u_2)\| \le \delta,$$

then

$$\sup_{t\ge 0} \inf_{(u_1,u_2)\in G(a_1,a_2)} \|(\psi_1(t), \psi_2(t)) - (u_1, u_2)\| \le \varepsilon,$$

where $(\psi_1(t), \psi_2(t))$ is the solution of system (1.3) *corresponding to the initial condition $(\psi_1(0), \psi_2(0))$ and $\|\cdot\|$ denotes the norm in Sobolev space $H^1(\mathbb{R}^N)$.*

Let us pointed out that we do know situations where the local existence holds. For example when $N = 1$ and $r_1 = r_2 := r$ with $2 \le r < 3$, see [23].

This paper is organized as follows: In Section 2, we display some preliminary results. Theorem 1.1 will be completed in Section 3. Section 4 is devoted to Theorem 1.5.

Acknowledgements. The authors thank N. Ikoma for pointing to them that the local well posedness of the Cauchy problem was not known under the assumptions of Theorem 1.1 and S. Bhattarai who indicated to us the reference [23]. They also thank T. Luo for useful observations on a preliminary version. Finally note that this work has been carried out in the framework of the Project NONLOCAL (ANR-14-CE25-0013), funded by the French National Research Agency (ANR).

Notation. In this paper it is understood that all functions, unless otherwise stated, are complex-valued, but for simplicity we write $L^p(\mathbb{R}^N)$, $H^1(\mathbb{R}^N)$..., for any $1 \le p < \infty$, $L^p(\mathbb{R}^N)$ is the usual Lebesgue space with norm

$$\|u\|_p^p := \int_{\mathbb{R}^N} |u|^p \, dx,$$

and $H^1(\mathbb{R}^N)$ the usual Sobolev space endowed with the norm

$$\|u\|^2 := \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 \, dx.$$

We denote by $' \to '$ and $' \rightharpoonup '$ strong convergence and weak convergence, respectively, in corresponding space, and denote by $B(x, R)$ a ball in $\mathbb{R}^N$ of center x and radius $R > 0$.

2 Preliminary results

Firstly, let us observe that the functional J is well defined in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. For $r_1, r_2 > 1, r_1 + r_2 < 2 + \frac{4}{N}$, there is $q > 1$ with $2 < r_1 q, r_2 q' \le 2^*, q' := \frac{q}{q-1}$. Hence

$$\int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2}\, dx \le \|u_1\|_{r_1 q}^{r_1} \|u_2\|_{r_2 q'}^{r_2} < \infty.$$

The Gagliardo-Nirenberg inequality

$$\|u\|_p \le C(N,p) \|\nabla u\|_2^{\alpha} \|u\|_2^{1-\alpha} \quad \text{where } \alpha = \frac{N(p-2)}{2p},$$

which holds for $u \in H^1(\mathbb{R}^N)$ and $2 \le p \le 2^*$, implies for $(u_1, u_2) \in S(a_1, a_2)$:

$$\begin{aligned} \int_{\mathbb{R}^N} |u_1|^{p_1}\, dx &\le C(N, p_1, a_1) \|\nabla u_1\|_2^{\frac{N(p_1-2)}{2}}, \\ \int_{\mathbb{R}^N} |u_2|^{p_2}\, dx &\le C(N, p_2, a_2) \|\nabla u_2\|_2^{\frac{N(p_2-2)}{2}}, \end{aligned} \tag{2.1}$$

and

$$\int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2}\, dx \le \|u_1\|_{r_1 q}^{r_1} \|u_2\|_{r_2 q'}^{r_2} \le C \|\nabla u_1\|_2^{\frac{N(r_1 q - 2)}{2q}} \|\nabla u_2\|_2^{\frac{N(r_2 q' - 2)}{2q'}} \tag{2.2}$$

with $C = C(N, r_1, r_2, a_1, a_2, q)$.

Now recall the rearrangement results of Shibata [28] as presented in [16]. Let u be a Borel measurable function on $\mathbb{R}^N$. It is said to vanish at infinity if $|\{x \in \mathbb{R}^N : |u(x)| > t\}| < \infty$ for every $t > 0$. Here $|A|$ stands for the N-dimensional Lebesgue measure of a Lebesgue mesurable set $A \subset \mathbb{R}^N$. Considering two Borel mesurable functions u, v which vanish at infinity in $\mathbb{R}^N$, we define for $t > 0$, $A^\star(u, v; t) := \{x \in \mathbb{R}^N : |x| < r\}$ where $r > 0$ is chosen so that

$$|B(0,r)| = |\{x \in \mathbb{R}^N : |u(x)| > t\}| + |\{x \in \mathbb{R}^N : |v(x)| > t\}|,$$

and $\{u, v\}^\star$ by

$$\{u, v\}^\star(x) := \int_0^\infty \chi_{A^\star(u,v;t)}(x)\, dt,$$

where $\chi_A(x)$ is a characteristic function of the set $A \subset \mathbb{R}^N$.

Lemma 2.1. *[16, Lemma A.1]*

(i) The function $\{u, v\}^\star$ is radially symmetric, non-increasing and lower semi-continuous. Moreover, for each $t > 0$ there holds $\{x \in \mathbb{R}^N : \{u, v\}^\star > t\} = A^\star(u, v; t)$.

(ii) *Let $\Phi : [0,\infty) \to [0,\infty)$ be non-decreasing, lower semi-continuous, continuous at 0 and $\Phi(0) = 0$. Then $\{\Phi(u), \Phi(v)\}^\star = \Phi(\{u,v\}^\star)$.*

(iii) $\|\{u,v\}^\star\|_p^p = \|u\|_p^p + \|v\|_p^p$ *for $1 \le p < \infty$.*

(iv) *If $u, v \in H^1(\mathbb{R}^N)$, then $\{u,v\}^\star \in H^1(\mathbb{R}^N)$ and $\|\nabla\{u,v\}^\star\|_2^2 \le \|\nabla u\|_2^2 + \|\nabla v\|_2^2$. In addition, if $u, v \in (H^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)) \setminus \{0\}$ are radially symmetric, positive and non-increasing, then*

$$\int_{\mathbb{R}^N} |\nabla\{u,v\}^\star|^2 \, dx < \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} |\nabla v|^2 \, dx.$$

(v) *Let $u_1, u_2, v_1, v_2 \ge 0$ be Borel measurable functions which vanish at infinity, then*

$$\int_{\mathbb{R}^N} (u_1 u_2 + v_1 v_2) \, dx \le \int_{\mathbb{R}^N} \{u_1, v_1\}^\star \{u_2, v_2\}^\star \, dx.$$

3 Proof of Theorem 1.1

Hereafter, we use the same notation $m(a_1, a_2)$ for $a_1, a_2 \ge 0$, namely, one component of (a_1, a_2) may be zero.

In what follows, we collect some basic properties of $m(a_1, a_2)$.

Lemma 3.1. *(i)* *For any $a_1, a_2 \ge 0$ with either $a_1 > 0$ or $a_2 > 0$,*

$$-\infty < m(a_1, a_2) < 0.$$

(ii) *$m(a_1, a_2)$ is continuous with respect to $a_1, a_2 \ge 0$.*

(iii) *For any $a_1 \ge b_1 \ge 0, a_2 \ge b_2 \ge 0, m(a_1, a_2) \le m(b_1, b_2) + m(a_1 - b_1, a_2 - b_2)$.*

Proof. (i) Observe that $\frac{N(p_i-2)}{2} < 2$ by $p_i < 2 + \frac{4}{N}$ for $i = 1, 2$ and that

$$\frac{N(r_1 q - 2)}{2q} + \frac{N(r_2 q' - 2)}{2q'} < 2,$$

owing to $r_1 + r_2 < 2 + \frac{4}{N}$. Thus, it follows from (2.1)-(2.2) that J is coercive and in particular $m(a_1, a_2) > -\infty$. Now taking into account that $\beta > 0$, one has

$$m(a_1, a_2) \le m(a_1, 0) + m(0, a_2).$$

Since $2 < p_i < 2 + \frac{4}{N}$ for $i = 1, 2$, it is standard to show that $m(a_1, 0) < 0$ (if $a_1 > 0$) and $m(0, a_2) < 0$ (if $a_2 > 0$). Thus $m(a_1, a_2) < 0$.

(ii) We assume $(a_1^n, a_2^n) = (a_1, a_2) + o(1)$. From the definition of $m(a_1^n, a_2^n)$, for any $\varepsilon > 0$, there exists $(u_1^n, u_2^n) \in S(a_1^n, a_2^n)$ such that

$$J(u_1^n, u_2^n) \leq m(a_1^n, a_2^n) + \varepsilon. \tag{3.1}$$

Setting

$$v_i^n := \frac{u_i^n}{\|u_i^n\|_2} a_i^{\frac{1}{2}}$$

for $i = 1, 2$, we have that $(v_1^n, v_2^n) \in S(a_1, a_2)$ and

$$m(a_1, a_2) \leq J(v_1^n, v_2^n) = J(u_1^n, u_2^n) + o(1). \tag{3.2}$$

Combining (3.1) and (3.2) we obtain

$$m(a_1, a_2) \leq m(a_1^n, a_2^n) + \varepsilon + o(1).$$

Reversing the argument we obtain similarly that

$$m(a_1^n, a_2^n) \leq m(a_1, a_2) + \varepsilon + o(1).$$

Therefore, since $\varepsilon > 0$ is arbitrary, we deduce that $m(a_1^n, a_2^n) = m(a_1, a_2) + o(1)$.

(iii) By density of $C_0^\infty(\mathbb{R}^N)$ into $H^1(\mathbb{R}^N)$, for any $\varepsilon > 0$, there exist $(\bar{\varphi}_1, \bar{\varphi}_2), (\hat{\varphi}_1, \hat{\varphi}_2) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ with $\|\bar{\varphi}_i\|_2^2 = b_i$, $\|\hat{\varphi}_i\|_2^2 = a_i - b_i$ for $i = 1, 2$ such that

$$J(\bar{\varphi}_1, \bar{\varphi}_2) \leq m(b_1, b_2) + \frac{\varepsilon}{2},$$
$$J(\hat{\varphi}_1, \hat{\varphi}_2) \leq m(a_1 - b_1, a_2 - b_2) + \frac{\varepsilon}{2}.$$

Since J is invariant by translation, without loss of generality, we may assume that $supp\, \bar{\varphi}_i \cap supp\, \hat{\varphi}_i = \emptyset$, and then $\|\bar{\varphi}_i + \hat{\varphi}_i\|_2^2 = \|\bar{\varphi}_i\|_2^2 + \|\hat{\varphi}_i\|_2^2 = a_i$ for $i = 1, 2$, as well as

$$m(a_1, a_2) \leq J(\bar{\varphi}_1 + \hat{\varphi}_1, \bar{\varphi}_2 + \hat{\varphi}_2) \leq m(b_1, b_2) + m(a_1 - b_1, a_2 - b_2) + \varepsilon.$$

Thus

$$m(a_1, a_2) \leq m(b_1, b_2) + m(a_1 - b_1, a_2 - b_2).$$

□

Lemma 3.2. *Assume $r_1, r_2 > 1, r_1 + r_2 < 2 + \frac{4}{N}$. If $(u_1^n, u_2^n) \rightharpoonup (u_1, u_2)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} |u_1^n|^{r_1} |u_2^n|^{r_2} - |u_1^n - u_1|^{r_1} |u_2^n - u_2|^{r_2}\, dx = \int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2}\, dx + o(1).$$

Proof. Since the lemma can be proved following closely the approach of [10, Lemma 2.3], we only provide the outline of the proof. For any $b_1, b_2, c_1, c_2 \in \mathbb{R}$ and $\varepsilon > 0$, set $r := r_1 + r_2$. The mean value theorem and Young's inequality lead to

$$\begin{aligned}&\big||b_1+b_2|^{r_1}|c_1+c_2|^{r_2}-|b_1|^{r_1}|c_1|^{r_2}\big|\\ &\le C\varepsilon\big(|b_1|^r+|c_1|^r+|b_2|^r+|c_2|^r\big)+C_\varepsilon\big(|b_2|^r+|c_2|^r\big).\end{aligned}$$

Denote $b_1 := u_1^n - u_1, c_1 := u_2^n - u_2, b_2 := u_1, c_2 := u_2$. Then

$$\begin{aligned}f_n^\varepsilon :=& \Big[\big||u_1^n|^{r_1}|u_2^n|^{r_2}-|u_1^n-u_1|^{r_1}|u_2^n-u_2|^{r_2}-|u_1|^{r_1}|u_2|^{r_2}\big|\\ &-C\varepsilon(|u_1^n-u_1|^r+|u_2^n-u_2|^r+|u_1|^r+|u_2|^r)\Big]^+\\ &\le |u_1|^{r_1}|u_2|^{r_2}+C_\varepsilon\left(|u_1|^r+|u_2|^r\right),\end{aligned}$$

where $u^+(x) := \max\{u(x), 0\}$, so the dominated convergence theorem implies that

$$\int_{\mathbb{R}^N} f_n^\varepsilon\,dx \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

Since

$$\begin{aligned}&\big||u_1^n|^{r_1}|u_2^n|^{r_2}-|u_1^n-u_1|^{r_1}|u_2^n-u_2|^{r_2}-|u_1|^{r_1}|u_2|^{r_2}\big|\\ &\le f_n^\varepsilon+C\varepsilon\big(|u_1^n-u_1|^r+|u_2^n-u_2|^r+|u_1|^r+|u_2|^r\big),\end{aligned}$$

by the boundedness of $\{(u_1^n, u_2^n)\}$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and (3.3), it follows that

$$\int_{\mathbb{R}^N} |u_1^n|^{r_1}|u_2^n|^{r_2}-|u_1^n-u_1|^{r_1}|u_2^n-u_2|^{r_2}\,dx=\int_{\mathbb{R}^N}|u_1|^{r_1}|u_2|^{r_2}\,dx+o(1).$$

□

Lemma 3.3. *Any minimizing sequence for* (1.4) *is, up to translation, strongly convergent in* $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ *for* $2 < p < 2^*$.

Proof. Assume that $\{(u_1^n, u_2^n)\}$ is a minimizing sequence associated to the functional J on $S(a_1, a_2)$. By the coerciveness of J on $S(a_1, a_2)$, the sequence $\{(u_1^n, u_2^n)\}$ is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. If

$$\sup_{y\in\mathbb{R}^N}\int_{B(y,R)}|u_1^n|^2+|u_2^n|^2\,dx=o(1),$$

for some $R > 0$, then $u_i \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*, i = 1, 2$, see [20, Lemma I. 1]. This is incompatible with the fact that $m(a_1, a_2) < 0$, see Lemma 3.1 (i). Thus, there exist a $\beta_0 > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B(y_n,R)}|u_1^n|^2+|u_2^n|^2\,dx\ge\beta_0>0,$$

and we deduce from the weak convergence in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and the local compactness in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ that $(u_1^n(x-y_n), u_2^n(x-y_n)) \rightharpoonup (u_1, u_2) \neq (0,0)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Our aim is to prove that $w_i^n(x) := u_i^n(x) - u_i(x+y_n) \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*, i = 1,2$ and so we suppose by contradiction that there exists a $2 < q < 2^*$ such that $(w_1^n, w_2^n) \nrightarrow (0,0)$ in $L^q(\mathbb{R}^N) \times L^q(\mathbb{R}^N)$. Note that under this assumption there exists a sequence $\{z_n\} \subset \mathbb{R}^N$ such that

$$(w_1^n(x-z_n), w_2^n(x-z_n)) \rightharpoonup (w_1, w_2) \neq (0,0)$$

in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^N)$. Indeed otherwise

$$\sup_{y\in\mathbb{R}^N} \int_{B(y,R)} |w_1^n|^2 + |w_2^n|^2 \, dx = o(1),$$

which leads to $(w_1^n, w_2^n) \to (0,0)$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ for $2 < p < 2^*$.

Now, combining the Brezis-Lieb Lemma, Lemma 3.2 and the translational invariance we conclude

$$\begin{aligned}
&J(u_1^n, u_2^n) = J(u_1^n(x-y_n), u_2^n(x-y_n))\\
&= J(u_1^n(x-y_n) - u_1 + u_1, u_2^n(x-y_n) - u_2 + u_2)\\
&= J(u_1^n(x-y_n) - u_1, u_2^n(x-y_n) - u_2) + J(u_1,u_2) + o(1)\\
&= J(w_1^n(x-y_n), w_2^n(x-y_n)) + J(u_1,u_2) + o(1)\\
&= J(w_1^n(x-z_n), w_2^n(x-z_n)) + J(u_1,u_2) + o(1)\\
&= J(w_1^n(x-z_n) - w_1 + w_1, w_2^n(x-z_n) - w_2 + w_2) + J(u_1,u_2) + o(1)\\
&= J(w_1^n(x-z_n) - w_1, w_2^n(x-z_n) - w_2) + J(w_1,w_2) + J(u_1,u_2) + o(1),
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
\|u_i^n(x-y_n)\|_2^2 &= \|u_i^n(x-y_n) - u_i + u_i\|_2^2\\
&= \|u_i^n(x-y_n) - u_i\|_2^2 + \|u_i\|_2^2 + o(1)\\
&= \|w_i^n(x-z_n) - w_i + w_i\|_2^2 + \|u_i\|_2^2 + o(1)\\
&= \|w_i^n(x-z_n) - w_i\|_2^2 + \|w_i\|_2^2 + \|u_i\|_2^2 + o(1).
\end{aligned}$$

Thus

$$\begin{aligned}
\|w_i^n(x-z_n) - w_i\|_2^2 &= \|u_i^n(x-y_n)\|_2^2 - \|w_i\|_2^2 - \|u_i\|_2^2 + o(1)\\
&= a_i - \|w_i\|_2^2 - \|u_i\|_2^2 + o(1)\\
&= b_i + o(1),
\end{aligned} \tag{3.5}$$

where $b_i := a_i - \|w_i\|_2^2 - \|u_i\|_2^2$. Noting that

$$\begin{aligned}
\|w_i\|_2^2 &\leq \liminf_{n\to\infty} \|w_i^n(x-z_n)\|_2^2 = \liminf_{n\to\infty} \|u_i(x-y_n) - u_i\|_2^2\\
&= a_i - \|u_i\|_2^2,
\end{aligned}$$

then $b_i \geq 0$ for $i = 1, 2$. Recording that $J(u_1^n, u_2^n) \to m(a_1, a_2)$, in view of (3.5), Lemma 3.1 (ii) and (3.4), we get

$$m(a_1, a_2) \geq m(b_1, b_2) + J(w_1, w_2) + J(u_1, u_2). \tag{3.6}$$

If $J(w_1, w_2) > m(\|w_1\|_2^2, \|w_2\|_2^2)$ or $J(u_1, u_2) > m(\|u_1\|_2^2, \|u_1\|_2^2)$, then, from (3.6) and Lemma 3.1 (iii), it follows

$$m(a_1, a_2) > m(b_1, b_2) + m(\|w_1\|_2^2, \|w_2\|_2^2) + m(\|u_1\|_2^2, \|u_1\|_2^2) \geq m(a_1, a_2),$$

which is impossible. Hence $J(w_1, w_2) = m(\|w_1\|_2^2, \|w_2\|_2^2)$ and $J(u_1, u_2) = m(\|u_1\|_2^2, \|u_1\|_2^2)$. We denote by $\tilde{u}_i, \tilde{w}_i$ the classical Schwarz symmetric-decreasing rearrangement of u_i, w_i for $i = 1, 2,$. Since

$$\|\tilde{u}_i\|_2^2 = \|u_i\|_2^2, \quad \|\tilde{w}_i\|_2^2 = \|w_i\|_2^2,$$

$$J(\tilde{u}_1, \tilde{u}_2) \leq J(u_1, u_2), \quad J(\tilde{w}_1, \tilde{w}_2) \leq J(w_1, w_2)$$

see for example [18], we deduce that

$$J(\tilde{u}_1, \tilde{u}_2) = m(\|u_1\|_2^2, \|u_2\|_2^2), \quad J(\tilde{w}_1, \tilde{w}_2) = m(\|w_1\|_2^2, \|w_2\|_2^2).$$

Therefore, $(\tilde{u}_1, \tilde{u}_2), (\tilde{w}_1, \tilde{w}_2)$ are solutions of the system (1.1) and from standard regularity results we have that $\tilde{u}_i, \tilde{w}_i \in C^2(\mathbb{R}^N)$ for $i = 1, 2$.

At this point Lemma 2.1 comes into play. Without restriction we may assume $u_1 \neq 0$. We divide into two cases.

Case 1: $u_1 \neq 0$ and $w_1 \neq 0$.

By virtue of Lemma 2.1 $(ii), (iv), (v)$,

$$\int_{\mathbb{R}^N} |\nabla \{\tilde{u}_1, \tilde{w}_1\}^\star| \, dx < \int_{\mathbb{R}^N} |\nabla \tilde{u}_1|^2 + |\nabla \tilde{w}_1|^2 \, dx \leq \int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla w_1|^2 \, dx,$$

$$\begin{aligned}\int_{\mathbb{R}^N} |\{\tilde{u}_1, \tilde{w}_1\}^\star|^{r_1} |\{\tilde{u}_2, \tilde{w}_2\}^\star|^{r_2} \, dx &= \int_{\mathbb{R}^N} \{|\tilde{u}_1|^{r_1}, |\tilde{w}_1|^{r_1}\}^\star \{|\tilde{u}_2|^{r_2}, |\tilde{w}_2|^{r_2}\}^\star \, dx, \\ &\geq \int_{\mathbb{R}^N} |\tilde{u}_1|^{r_1} |\tilde{u}_2|^{r_2} + |\tilde{w}_1|^{r_1} |\tilde{w}_2|^{r_2} \, dx \\ &= \int_{\mathbb{R}^N} (|u_1|^{r_1})\tilde{} (|u_2|^{r_2})\tilde{} + (|w_1|^{r_1})\tilde{} (|w_2|^{r_2})\tilde{} \, dx, \\ &\geq \int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2} + |w_1|^{r_1} |w_2|^{r_2} \, dx,\end{aligned}$$

and thus

$$J(u_1, u_2) + J(w_1, w_2) > J(\{\tilde{u}_1, \tilde{w}_1\}^\star, \{\tilde{u}_2, \tilde{w}_2\}^\star). \tag{3.7}$$

Also from Lemma 2.1 (iii), for $i = 1, 2$,

$$\int_{\mathbb{R}^N} |\{\tilde{u}_i, \tilde{w}_i\}^\star|^2\, dx = \int_{\mathbb{R}^N} |\tilde{u}_i|^2 + |\tilde{w}_i|^2\, dx = \int_{\mathbb{R}^N} |u_i|^2 + |w_i|^2\, dx, \tag{3.8}$$

and taking (3.6)-(3.8) and Lemma 3.1 (iii) into consideration, one obtains the contradiction

$$m(a_1, a_2) > m(b_1, b_2) + m(a_1 - b_1, a_2 - b_2) \geq m(a_1, a_2).$$

Case 2: $u_1 \neq 0$, $w_1 = 0$ *and* $w_2 \neq 0$.
If $u_2 \neq 0$, we can reverse the role of u_1, w_1 and u_2, w_2 in *Case 1* to get a contradiction. Thus, we suppose that $u_2 = 0$. Due to Lemma 2.1 (ii)-(v),

$$\begin{aligned} J(\{\tilde{u}_1, 0\}^\star, \{\tilde{w}_2, 0\}^\star) &\leq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla \tilde{u}_1|^2 + |\nabla \tilde{w}_2|^2\, dx - \frac{\mu_1}{p_1}\int_{\mathbb{R}^N} |\tilde{u}_1|^{p_1}\, dx \\ &\quad - \frac{\mu_2}{p_2}\int_{\mathbb{R}^N} |\tilde{w}_2|^{p_2}\, dx - \beta \int_{\mathbb{R}^N} |\tilde{u}_1|^{r_1} |\tilde{w}_2|^{r_2} \\ &< J(\tilde{u}_1, 0) + J(0, \tilde{w}_2) \\ &\leq J(u_1, 0) + J(0, w_2), \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} |\{\tilde{u}_1, 0\}^\star|^2\, dx &= \int_{\mathbb{R}^N} |\tilde{u}_1|^2\, dx = \int_{\mathbb{R}^N} |u_1|^2\, dx, \\ \int_{\mathbb{R}^N} |\{\tilde{w}_2, 0\}^\star|^2\, dx &= \int_{\mathbb{R}^N} |\tilde{w}_2|^2\, dx = \int_{\mathbb{R}^N} |w_2|^2\, dx. \end{aligned} \tag{3.10}$$

Thus using (3.6), (3.9), (3.10) and Lemma 3.1, we also have that

$$m(a_1, a_2) > m(b_1, b_2) + m(a_1 - b_1, a_2 - b_2) \geq m(a_1, a_2).$$

The contradictions obtained in *Cases* 1 and 2 indicate that $w_i^n(x) = u_i^n(x) - u_i(x + y_n) \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*, i = 1, 2$. □

Proof of Theorem 1.1. Let $\{(u_1^n, u_2^n)\}$ be a minimizing sequence for the functional J on $S(a_1, a_2)$. In light of Lemma 3.3, we know that there exists $\{y_n\} \subset \mathbb{R}^N$ such that $u_i^n(x - y_n) \to u_i$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*, i = 1, 2$. Hence by weak convergence

$$J(u_1, u_2) \leq m(a_1, a_2). \tag{3.11}$$

Note that if $||u_1||_2^2 = a_1$ and $||u_2||_2^2 = a_2$ we are done. Indeed the strong convergence of $\{(u_1^n(\cdot - y_n), u_2^n(\cdot - y_n))\}$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ then directly follows. To show that $||u_1||_2^2 = a_1$ and $||u_2||_2^2 = a_2$ we assume by contradiction that $\|u_1\|_2^2 := b_1 < a_1$ or $\|u_2\|_2^2 := b_2 < a_2$. By definition $J(u_1, u_2) \geq m(b_1, b_2)$ and thus it results from (3.11) that $m(b_1, b_2) \leq m(a_1, a_2)$. At this point since from Lemma 3.1 (iii) $m(a_1, a_2) \leq m(b_1, b_2) + m(a_1 - b_1, a_2 - b_2)$ and by Lemma 3.1 (i) $m(a_1 - b_1, a_2 - b_2) < 0$ we have reached a contradiction and Theorem 1.1 is proved. □

Remark 3.4. As indicated in Remark 1.2 a proof for the existence of minimizer for (1.4) can be given working directly in $H^1_r(\mathbb{R}^N)\times H^1_r(\mathbb{R}^N)$. In such space the strong convergence in $L^p(\mathbb{R}^N)\times L^p(\mathbb{R}^N)$ is given for free. Now defining

$$m_r(a_1,a_2) := \inf_{(u_1,u_2)\in S_r(a_1,a_2)} J(u_1,u_2), \tag{3.12}$$

where

$$S_r(a_1,a_2) := \{(u_1,u_2)\in H^1_r(\mathbb{R}^N)\times H^1_r(\mathbb{R}^N) : \|u_1\|_2^2 = a_1, \|u_2\|_2^2 = a_2\}$$

we observe that we still have

$$m_r(a_1,a_2) \le m_r(b_1,b_2) + m_r(a_1-b_1,a_2-b_2), \tag{3.13}$$

where $0\le b_i\le a_i$ for $i=1,2$. Indeed since we can choose a minimizing sequence which consists of Schwarz symmetric functions (which are in particular radially symmetric) it results that $m_r(c_1,c_2)=m(c_1,c_2)$ for any $c_1\ge 0, c_2\ge 0$ and (3.13) follows from Lemma 3.1 (iii). Thus we can end the proof as previously.

Remark 3.5. In [3], (3.13) was not observed and the fact that the weak limit belongs to $S_r(a_1,a_2)$ was proved using Liouville's type arguments, as developed in [16], see also [8, 15]. It is the use of these arguments, which induces the restriction on the dimension N in [3, Theorem 2.1].

4 Proof of Theorem 1.5

Since our proof relies on the classical arguments of [9], we only give a sketch.

Proof of Theorem 1.5. By contradiction, we assume that there is a $\varepsilon_0 > 0$, $\{(\psi_1^n(0),\psi_2^n(0))\}$ and $\{t_n\}\subset\mathbb{R}^+$ such that

$$\inf_{(u_1,u_2)\in G(a_1,a_2)} \|(\psi_1^n(0),\psi_2^n(0))-(u_1,u_2)\|\to 0,$$

and

$$\inf_{(u_1,u_2)\in G(a_1,a_2)} \|(\psi_1^n(t_n),\psi_2^n(t_n))-(u_1,u_2)\|\ge\varepsilon_0. \tag{4.1}$$

Since by the conservation laws,

$$\|\psi_i^n(t_n)\|_2^2 = \|\psi_i^n(0)\|_2^2, \quad J(\psi_1^n(t_n),\psi_2^n(t_n)) = J(\psi_1^n(0),\psi_2^n(0)), \quad \text{for } i=1,2,$$

if we define

$$\hat{\psi}_i^n = \frac{\psi_i^n(t_n)}{\|\psi_i^n(t_n)\|_2^2} a_i^{\frac{1}{2}}, \quad \text{for } i=1,2,$$

we get that

$$\|\hat{\psi}_i^n\|_2^2 = a_i, \quad J(\hat{\psi}_1^n,\hat{\psi}_2^n) = m(a_1,a_2)+o(1).$$

Namely $\{(\hat{\psi}_1^n,\hat{\psi}_2^n)\}$ is a minimizing sequence for (1.4). From Theorem 1.1 it follows that it is precompact in that $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ thus (4.1) fails. □

References

[1] J. Albert, S. Bhattarai: *Existence and stability of a two-parameter family of solitary waves for an NLS-KdV system*, Adv. Differential Equations 18 (2013) 1129-1164.

[2] V.S. Bagnato, D.J. Frantzeskakis, P.G. Kevrekidis, B.A. Malomed, D. Mihalache: *Bose-Einstein condensation: twenty years after*, Roman. Rep. Phys. 67 (2015) 5-50, arXiv:1502.06328.

[3] T. Bartsch, L. Jeanjean: *Normalized solutions for nonlinear Schrödinger systems*, to appear in Proc. Roy. Soc. Edinburgh A, ArXiv identifier 1507.04649.

[4] S. Bhattarai: *Stability of normalized solitary waves for three coupled nonlinear Schrödinger equations*, Discrete Contin. Dyn. Syst- A, 36 (2016) 1789-1811.

[5] J. Bellazzini, G. Siciliano: *Scaling properties of functionals and existence of constrained minimizers*, J. Funct. Anal. 261 (2011) 2486-2507.

[6] J. Byeon: *Effect of symmetry to the structure of positive solutions in nonlinear elliptic problems*, J. Diff. Equa. 163 (2000) 429-474.

[7] R. Cipolatti, W. Zumpichiatti: *Orbital stable standing waves for a system of coupled nonlinear Schrödinger equations*, Nonlinear Anal. 42 (2000) 445-461.

[8] D. Cao, I-L. Chern, J. Wei: *On ground state of spinor Bose-Einstein condensates*, Nonlinear. Differ. Equ. Appl. 18 (2011) 427-445.

[9] T. Cazenave, P.-L. Lions: *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. 85 (1982) 549-561.

[10] Z. Chen, W. Zou: *Existence and symmetry of positive ground states for a doubly critical Schrödinger system*, Trans. Amer. Math. Soc. 367 (2015) 3599-3646.

[11] M. Colin, L. Jeanjean, M. Squassina: *Stability and instability results for standing waves of quasi-linear Schrödinger equations*, Nonlinearity 23 (2010) 1353-1385.

[12] B. D. Esry, C. H. Greene, J. P. Burke Jr, J. L. Bohn: *Hartree-Fock theory for double condensates*, Phys. Rev. Lett. 78 (1997) 3594-3597.

[13] D. Garrisi: *On the orbital stability of standing-wave solutions to a coupled nonlinear Klein-Gordon equation*, Adv. Nonlinear Stud. 12 (2012) 639-658.

[14] H. Hajaiej: *Symmetric ground states solutions of m-coupled nonlinear Schrödinger equation*, Nonlinear Analysis: Methods, Theory and Applications, 71 (2009) 4696-4704.

[15] N. Ikoma: *Existence of Minimizers for Some Coupled Nonlinear Schrödinger Equations*, Geometric properties for parabolic and elliptic PDE's 2 (2013) 143-164.

[16] N. Ikoma: *Compactness of minimizing sequences in nonlinear Schrödinger systems under multiconstraint conditions*, Adv. Nonlinear Studies 14 (2014) 115-136.

[17] L. Jeanjean, M. Squassina: *An approach to minimization under a constraint: The added mass technique*, Calc. Var. Partial Differential Equations 41 (2011) 511-534.

[18] E. H. Lieb, M. Loss: *Analysis*, Second edition, Graduate studies in mathematics, vol. 14, American Mathematical Society, Providence, 2001.

[19] P-L. Lions: *The concentration-compactness principle in the Calculus of Variations. The locally compact case, Part I*, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 109-145.

[20] P-L. Lions: *The concentration-compactness principle in the Calculus of Variations. The locally compact case, Part II*, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 223-283.

[21] C. Liu, N.V. Nguyen, Z-Q. Wang: *Existence and stability of solitary waves of an M-coupled nonlinear Schrödinger system*, Preprint.

[22] B. Malomed: *Multi-component Bose-Einstein condensates: Theory.* In: P.G. Kevrekidis, D.J. Frantzeskakis, R. Carretero-Gonzalez (Eds.), Emergent Nonlinear Phenomena in Bose-Einstein Condensation, Springer-Verlag, Berlin, (2008) 287-305.

[23] N.V. Nguyen, R. Tian, B. Deconinck, N. Sheils: *Global existence of a coupled system of Schrödinger equations with power-type nonlinearities*, J. Math. Phys. 54, (2013), 011503.

[24] N.V. Nguyen, Z.-Q. Wang: *Orbital stability of solitary waves for a nonlinear Schröinger system*, Adv. Differential Equations 16 (2011) 977-1000.

[25] N. V. Nguyen, Z-Q. Wang: *Orbital stability of solitary waves of a 3-coupled nonlinear Schrödinger system*, Nonlinear Analysis 90 (2013) 1-26.

[26] N. V. Nguyen, Z-Q. Wang: *Existence and stability of a two-parameter family of solitary waves for a 2-couple nonlinear schrödinger system*, Discrete Contin. Dyn. Syst. 36 (2016) 1005-1021.

[27] M. Ohta: *Stability of solitary waves for coupled nonlinear Schrödinger equations*, Nonlinear Anal. 26 (1996) 933-939.

[28] M. Shibata: *A new rearrangement inequality and its application for L^2-constraint minimizing problems*, ArXiv:1312.3575 [math.AP].

[29] M. Shibata: *Stable standing waves of nonlinear Schrödinger equations with a general nonlinear term*, Manuscripta math. 143 (2014) 221-237.

[30] C.A. Stuart: *Bifurcation from the continuous spectrum in L^2-theory of elliptic equations on $\mathbb{R}^N$*, Recent Methods in Nonlinear Analysis and Applications, Liguori, Napoli, (1981).

[31] C.A. Stuart: *Bifurcation for Dirichlet problems without eigenvalues*, Proc. London Math. Soc. (3) 45 (1982), 169-192.

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