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DISTRIBUTION OF FUNCTIONS IN ABSTRACT H^1

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I. Introduction

Let A be a *weak*-Dirichlet algebra* i.e. a subalgebra A of $L^\infty(\mu)$ where $(\mathscr{M}, \mu)$ is a probability space such that: μ is multiplicative on A, A contains the constants and $A + \bar{A}$ is weak*-dense in $L^\infty(\mu)$.

The *abstract Hardy spaces* are defined by the following:

$\mathscr{H}^p(\mathscr{M})$ is the closure of A in $L^p(\mu)$, for $1 \leq p < \infty$,
$\mathscr{H}^\infty(\mathscr{M})$ is the weak*-closure of A in $L^\infty(\mu)$.

We also denote by $\mathscr{H}_0^1(\mathscr{M})$ the set of functions in $\mathscr{H}^1(\mathscr{M})$ with $\int_{\mathscr{M}} f d\mu = 0$ and by $\operatorname{Re}\mathscr{H}^1(\mathscr{M})$ the set of real parts of functions in $\mathscr{H}^1(\mathscr{M})$.

These algebras were introduced in [SW], where it was proven that the corresponding abstract Hardy spaces enjoy most of the measure theoretic properties of the original Hardy spaces. Then in [HR] the conjugate function was studied for these weak*-Dirichlet algebras. The conjugation operator is defined for $1 < p < \infty$ by

$$\mathscr{H}: L^p(\mu) \to L^p(\mu)$$

$$f \mapsto \tilde{f} \text{ such that } f + i\tilde{f} \in \mathscr{H}^p(\mathscr{M}) \text{ and } \int_{\mathscr{M}} \tilde{f} d\mu = 0.$$

This operator is bounded on $L^p(\mu), 1 < p < \infty$. For $p = 1$, $\mathscr{H}$ is only bounded from $L^1(\mu)$ into $L^{1,\infty}(\mu)$. So a natural question is to characterize the functions in $L^1(\mu)$ for which $\tilde{f}$ is in $L^1(\mu)$. This is the problem we will investigate here. Note that if $f \geq 0$, Zygmund's theorem (which holds for weak*-Dirichlet algebras, see [HR]) asserts that the condition for $\tilde{f}$ to be in $L^1(\mu)$ is that f is in $L \log_+ L$ (i.e., $\int_{\mathscr{M}} |f| \log_+(|f|)\, d\mu < \infty$).

We will first recall the solution of the problem for the classical Hardy spaces. It was solved on $\mathbf{T}$, $\mathbb{R}$ and $\mathbb{R}^n$ by B. Davis [Da], here is his result for $H^1(\mathbb{R})$. For f a real valued function on $\mathbb{R}$, let f_δ be the signed decreasing function (i.e., non-positive and not increasing on $(-\infty, 0)$, non-negative and not increasing on $(0; \infty)$) which has the same distribution as f and let $M(t) = \int_{-t}^{t} f_\delta(u)\, du$.

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THEOREM [Da]. *A real valued function f in $L^1(\mathbb{R})$ has a rearrangement in* Re $H_0^1(\mathbb{R})$ *if and only if*

$$\int_0^\infty \frac{|M(x)|}{x}\, dx < \infty.$$

Davis's original proof of these results uses probabilistic methods, N. Kalton gives a non probabilistic proof for **T** in [Ka] (Theorem 6.3). His proof is based on a study of the symmetrized Hardy class $H^1_{\text{sym}}(\mathbf{T})$, indeed he shows a characterization of functions in $H_0^1(\mathbf{T})$, which by an equivalence of norms on $H^2_{\text{sym}}(\mathbf{T})$ is equivalent to Davis' condition. We will use the same ideas here. Let's also mention that in [Ka2], N. Kalton gives another proof of Davis' Theorem which is valid for vector-valued H_1-functions.

II. The abstract Hardy space case

Let $\mathscr{M}$ be a Polish space with a non-atomic probability measure μ. Let $\mathscr{H}^1(\mathscr{M})$ be an abstract Hardy space defined from a weak*-Dirichlet algebra A on $\mathscr{M}$. For f a real valued function in $L^1(\mu)$, let f_δ be the signed decreasing function defined on $\mathbb{R}$ which has the same distribution as f and let $M(t) = \int_{-t}^t f_\delta(u)\, du$

THEOREM 1. *If f belongs to* Re $\mathscr{H}_0^1(\mu)$, *then*

$$\int_0^\infty \frac{|M(t)|}{t}\, dt < \infty. \qquad (*)$$

We need the following analog of Kalton's characterization of functions in $H^1(\mathbf{T})$ (Lemma 7.2. in [Ka]).

PROPOSITION 2. *If $f \in \mathscr{H}_0^1(\mathscr{M})$ then*

$$\sup_{\phi \in \mathscr{L}_1^b} \left| \int f\phi(\log|f|)\, d\mu \right| \leq C\|f\|_1, \qquad (**)$$

where $\mathscr{L}_1^b$ is the set of all bounded, 1-Lipschitz functions $\phi\colon \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$.

The proof of this result for $\mathscr{M} = \mathbf{T}$ in [Ka] uses the analyticity of functions in $H^1(\mathbf{T})$, with an argument of plurisubharmonicity. In our abstract setting, we will substitute the following subharmonicity lemma which generalizes Jensen's inequality.

LEMMA 3. *If* $s\colon \mathbb{C} \to \mathbb{R}$ *is subharmonic on* $\mathbb{C}$ *then for any* f *in* $\mathscr{H}^\infty(\mathscr{M})$,

$$\int_{\mathscr{M}} s(f(x))\, d\mu(x) \geq s\left(\int_{\mathscr{M}} f(x)\, d\mu(x)\right).$$

Proof of Lemma 3. To prove this lemma, we will use the Riesz decomposition of a subharmonic function [Ri]. This fact can also be found as an exercise in [Ga, p. 49].

Fact. Let s be a subharmonic function on a domain Ω of $\mathbb{C}$ and let

$$\Omega_\varepsilon = \{z \in \Omega; \operatorname{dist}(z, \partial\Omega) > \varepsilon\}.$$

Then

$$s(z) = \frac{1}{2\pi}\int_{\Omega_\varepsilon} \log|z - w|\, d\Delta s(w) + h_\varepsilon(z)$$

where Δs is the positive Borel measure corresponding to the weak Laplacian of s and h_ε is harmonic on Ω.

Now let f be in $\mathscr{H}^\infty$ and $M = \|f\|_\infty$. We take $\Omega = \{|z| < M + 1\}$ and for some $0 < \varepsilon < 1$ we decompose:

$$s(z) = \frac{1}{2\pi}\int_{\Omega_\varepsilon} \log|z - w|\, d\Delta s(w) + h_\varepsilon(z). \tag{1}$$

Let $\Phi(f) = \int_{\mathscr{M}} s(f(x))\, d\mu(x)$, then

$$\Phi(f) = \frac{1}{2\pi}\int_{\mathscr{M}}\int_{\Omega_\varepsilon} \log|f(x) - w|\, d\Delta s(w)\, d\mu(x) + \int_{\mathscr{M}} h_\varepsilon(f(x))\, d\mu(x). \tag{2}$$

We will now estimate each part of (2).

For the second part, since h_ε is harmonic on Ω and f is in $\mathscr{H}^\infty$ with range included in Ω_ε, it is classical by the analytic functional calculus, because of the multiplicity of the measure μ on $\mathscr{H}^\infty$, that we have

$$\Phi_2(f) = \int_{\mathscr{M}} h_\varepsilon(f(x))\, d\mu(x) = h_\varepsilon\left(\int_{\mathscr{M}} f(x)\, d\mu(x)\right). \tag{3}$$

For the first part, let

$$\Phi_1(f) = \frac{1}{2\pi}\int_{\mathcal{M}}\int_{\Omega_\varepsilon} \log|f(x) - w|d\Delta s(w)\, d\mu(x)$$
$$= \frac{1}{2\pi}\int_{\Omega_\varepsilon}\int_{\mathcal{M}} \log|f(x) - w|\, d\mu(x) d\Delta s(w).$$

We now use Jensen's inequality (this very classical fact in the theory of Hardy spaces on **T** holds in the frame of weak*-Dirichlet algebras, see [SW]):

$$\int_{\mathcal{M}} \log|f|\, d\mu \geq \log\left|\int_{\mathcal{M}} f\, d\mu\right| \quad \text{for } f \text{ in } \mathcal{H}^p(\mathcal{M}).$$

This gives

$$\int_{\mathcal{M}} \log|f(x) - w|\, d\mu(x) \geq \log\left|\int_{\mathcal{M}} (f(x) - w)\, d\mu(x)\right| = \log\left|\int_{\mathcal{M}} f\, d\mu - w\right|.$$

So

$$\Phi_1(f) \geq \frac{1}{2\pi}\int_{\Omega_\varepsilon} \log\left|\int_{\mathcal{M}} f\, d\mu - w\right| d\Delta s(w). \tag{4}$$

Combining (2), (3) and (4) we get

$$\Phi(f) \geq \frac{1}{2\pi}\int_{\Omega_\varepsilon} \log\left|\int_{\mathcal{M}} f\, d\mu - w\right| d\Delta s(x) + h_\varepsilon\left(\int_{\mathcal{M}} f\, d\mu\right).$$

Now, the right hand side is exactly $s(\int_{\mathcal{M}} f\, d\mu)$ decomposed as in (1), which proves the lemma. □

Proof of Proposition 2. The proof is very similar to the proof of Lemma 7.2 from [Ka]. We sketch it here to show the use of Lemma 3, for more details we refer to [Ka].

We first consider $f \in H_0^\infty(\mathcal{M})$. Let $\phi \in \mathcal{L}_b^1$. We want to prove that

$$\left|\int_{\mathcal{M}} f\phi(\log|f|)\, d\mu\right| \leq C\|f\|_1. \tag{5}$$

In fact we will prove it for a function ψ such that:

(6) $|\phi(t) - \psi(t)| < C'$,

(7) $s(z) = \lambda|z| - (\operatorname{Re} z)\psi(\log|z|)$, $s(0) = 0$ is subharmonic on $\mathbb{C}$, for some $\lambda > 0$.

The construction of such a function ψ is the same as in [Ka] and is omited. Now since s is subharmonic on $\mathbb{C}$, by the generalized Jensen's inequality (Lemma 3), we obtain

$$\Phi(f) = \int_{\mathscr{M}} s(f(x))\, d\mu(x) \geq s\left(\int_{\mathscr{M}} f\, d\mu\right) = s(0) = 0;$$

i.e.,

$$\operatorname{Re}\left(\int_{\mathscr{M}} f\psi(\log|f|)\, d\mu\right) \leq \lambda \int_{\mathscr{M}} |f|\, d\mu.$$

Multiplying f by a constant of modulus 1 gives

$$\left|\int_{\mathscr{M}} f\psi(\log|f|)\, d\mu\right| \leq C\|f\|_1;$$

then by (6) the same inequality holds with ϕ instead of ψ, which gives (5).
Now for f in $\mathscr{H}_0^1$, we take (f_n) in $\mathscr{H}_0^\infty$ such that $\|f_n - f\|_1 \to 0$. □

Proof of Theorem 1. Once Proposition 2 is proven, Theorem 1 follows from Kalton's results about the symmetrized Hardy class, which are valid in our setting since in [Ka], $H^1_{\text{sym}}(\mathscr{M})$ was defined for a Polish space $\mathscr{M}$ with a non atomic probability measure μ. In fact Lemma 6.1 and Proposition 7.1 in [Ka] give exactly the equivalence of $(*)$ and $(**)$. □

III. Examples

III.1. Algebras of "analytic" functions on groups with ordered dual. Let G be a compact abelian group, μ its Haar measure, Γ its dual with P a total order on Γ. The algebra of analytic-type functions on G is

$$A = \left\{f \in \mathscr{C}(G), \hat{f}(\xi) = 0, \text{for } \xi \notin P\right\},$$

where $\hat{f}$ is the Fourier transformation of f. Then the measure μ is uniquely representing a multiplicative linear functional on A (see [Ru]). In particular A is a weak*-Dirichlet algebra in $L^\infty(G, \mu)$(see [SW]).

So Theorem 1 holds for $\mathscr{H}^1(G) = \mathscr{H}^1(G, \mu, A)$, the closure of A in $L^1(G, \mu)$. An example of this is the "big disc algebra" on $\mathbf{T}^n$: $G = \mathbf{T}^n$, $\Gamma = \mathbb{Z}^n$ with a total order on it. Another interesting example is $G = \mathbf{T}^{\mathbb{N}}$ the infinite dimensional torus, and $\Gamma = \mathbb{Z}^{(\mathbb{N})}$ with the lexicographic order. This corresponds to the frame of Hardy martingales (see [Gar]).

III.2. Case of the ergodic Hardy spaces. Another type of weak*-Dirichlet algebra is considered in [We]. We suppose that $(\mathcal{M}, \mu)$ is a probability space with $(U_t)_{t \in \mathbb{R}}$ an ergodic flow acting on $\mathcal{M}$. The ergodic Hilbert transform is given by

$$\mathcal{H}_e f(x) = \lim_{\varepsilon \to 0} \int_{\varepsilon < |t| < 1/\varepsilon} \frac{f(U_t x)}{\pi t} \, dt.$$

The ergodic Hardy spaces are defined in the following way: $H_e^\infty(\mathcal{M}, \mu)$ is the subspace of $L^\infty(\mathcal{M}, \mu)$ consisting of functions of the form $f + i\mathcal{H}_e f$, $f \in L^\infty(\mathcal{M}, \mu)$, and $H_e^p(\mathcal{M}, \mu)$ is the closure in $L^p(\mathcal{M}, \mu)$ of $H_e^\infty(\mathcal{M}, \mu) \cap L^p(\mathcal{M}, \mu)$. In [We], it was proven that $H_e^\infty(\mathcal{M}, \mu)$ is a weak*-Dirichlet algebra.

So Theorem 1 applies for the ergodic Hardy space $H_e^1(\mathcal{M}, \mu)$,

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