Multiple complex-valued solutions for nonlinear magnetic Schrödinger equations

Dedicated to Paul Rabinowitz with great admiration and esteem

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Abstract

We study, in the semiclassical limit, the singularly perturbed nonlinear Schrödinger equations

$$L^{\hbar}_{A,V} u = f(|u|^2)u \quad \text{in } \mathbb{R}^N \tag{0.1}$$

where $N \geq 3$, $L^{\hbar}_{A,V}$ is the Schrödinger operator with a magnetic field having source in a C^1 vector potential A and a scalar continuous (electric)

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potential V defined by

$$L^{\hbar}_{A,V} = -\hbar^2\Delta - \frac{2\hbar}{i}A\cdot\nabla + |A|^2 - \frac{\hbar}{i}\operatorname{div} A + V(x). \qquad (0.2)$$

Here f is a nonlinear term which satisfies the so-called Berestycki-Lions conditions. We assume that there exists a bounded domain $\Omega \subset \mathbb{R}^N$ such that

$$m_0 \equiv \inf_{x\in\Omega} V(x) < \inf_{x\in\partial\Omega} V(x)$$

and we set $K = \{x \in \Omega \mid V(x) = m_0\}$. For $\hbar > 0$ small we prove the existence of at least $\operatorname{cupl}(K) + 1$ geometrically distinct, complex-valued solutions to (0.1) whose modula concentrate around K as $\hbar \to 0$.

1 Introduction

In the present work we study, in a semiclassical regime, a nonlinear magnetic Schrödinger equation, which arises in many fields of physics, in particular condensed matter physics and nonlinear optics. More precisely, we are looking for stationary states to the evolution equation

$$i\hbar\frac{\partial\psi}{\partial t} = L^{\hbar}_{A,W}\psi - f(|\psi|^2)\psi \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \qquad (1.1)$$

where $i^2 = -1$ and $L^{\hbar}_{A,W}$ is the Schrödinger operator with a magnetic field B having source in a C^1 vector potential A and a continuous scalar (electric) potential W defined by

$$L^{\hbar}_{A,W} = \left(\frac{\hbar}{i}\nabla - A\right)^2 + W(x) = -\hbar^2\Delta - \frac{2\hbar}{i}A\cdot\nabla + |A|^2 - \frac{\hbar}{i}\operatorname{div} A + W(x). \quad (1.2)$$

Mathematically the transition from Quantum mechanics to Classical mechanics is described letting to zero the Planck constant ($\hbar \to 0$) and the solutions, which exist for small value of $\hbar$, are usually referred *semiclassical bound states*. For the physical background, we refer to [37].

The *ansatz* that the solution $\psi(x,t)$ to (1.1) is a standing wave of the form

$$\psi(t,x) = e^{-iE\hbar^{-1}t}u(x),$$

with $E \in \mathbb{R}$ and $u\colon \mathbb{R}^N \to \mathbb{C}$, leads us to solve the complex-valued semilinear elliptic equation

$$L^{\hbar}_{A,W}u = Eu + f(|u|^2)u \quad \text{in } \mathbb{R}^N. \qquad (1.3)$$

In the work we consider an electric potential $W(x)$ which is bounded from below on $\mathbb{R}^N$, and we choose E such that $V(x) = W(x) - E$ is strictly positive. Hence equation (1.3) becomes

$$L^{\hbar}_{A,V}u = f(|u|^2)u \quad \text{in } \mathbb{R}^N \qquad (1.4)$$

where V is a strictly positive potential.

Concerning the zero external magnetic case, there is an extensive literature starting from the paper by Floer and Weinstein [26] (see for instance [2, 3, 18, 19, 23, 24, 31, 32, 33, 34, 10, 11, 12, 17]).

To our knowledge, the first paper, in which equation (1.4) is considered, is due to Esteban and Lions [25]. The authors proved the existence of standing wave solutions to (1.4) for $\hbar > 0$ fixed and $N = 3$, by a constrained minimization for a constant electric potential and a cubic nonlinearity. Concentration and compactness arguments are applied to solve such minimization problems for special classes of magnetic fields.

Successively, in [29] Kurata proved the existence of least energy solutions to (1.4) for any fixed $\hbar > 0$, under some assumptions linking the magnetic field B and the electric potential V. He also studied the concentration phenomena of a family of least solutions of (1.4) in the semiclassical limit, showing that concentration of the modula of such solutions occurs at global minima of the electric potential V. For periodic scalar and vector potentials, we refer to the paper by Arioli and Szulkin [4].

A first multiplicity result of semiclassical solutions to (1.4), when $f(t) = |t|^{(p-1)/2}$, $1 < p < (N+2)/(N-2)$ for $N > 2$ and $1 < p < +\infty$ if $N = 2$, as $\hbar \to 0$, has been proved by the first author in [15]. Since the nonlinearity satisfies the Nehari monotonicity condition, the problem can be reduced to the search of critical points of a functional constrained to a Nehari manifold and the number of complex-valued solutions to (1.4) can be related to the topology of (global) sublevel of the functional by standard deformation theorems for Hilbert manifold without boundary. Finally by means of an entrance map and a barycenter map, this number is estimated by the topological richness of the set of global minima of the electric potential V.

The existence of complex-valued solutions of the magnetic Schrödinger equations in $\mathbb{R}^3$, whose modula concentrate at local minima of V, has been derived in [21] using a penalization argument (see also [20]) for a cubic nonlinearity. See also [1] for an extention of the results of [20, 21]. Successively, in [16] the existence of multi-peaks solutions for nonlinear Schrödinger equations with an external magnetic field is proved for a more general nonlinearity which satisfies the Berestycki-Lions conditions, using a variational approach developed in [10] for the zero external magnetic case. Recently, the existence of semiclassical cylindrically symmetric solutions, whose moduli concentrate around circles driven by the magnetic and electric fields, has been established in [9], assuming that A and V are cylindrically symmetric potentials.

In this paper we are interested to find a multiplicity result to (1.4) for a large class of magnetic potentials A and under nearly optimal assumptions on the nonlinearity.

Throughout the paper, we assume that $N \geq 3$ and

(A1) $A\colon \mathbb{R}^N \to \mathbb{R}^N$ is a C^1 function such that, for some positive constants C, γ

$$|J_A(x)| \leq C e^{\gamma |x|}, \quad \forall x \in \mathbb{R}^N$$

where J_A denotes the Jacobian matrix of A at x;

(V1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} V(x) = \underline{V} > 0$;

(V2) There is a bounded domain $\Omega \subset \mathbb{R}^N$ such that

$$m_0 = \inf_{x\in\Omega} V(x) < \inf_{x\in\partial\Omega} V(x).$$

On the nonlinearity we require that

(f1) $f\colon [0, +\infty) \to \mathbb{R}$ is continuous;

(f2) $f(0) = \lim_{\xi\to 0^+} f(\xi) = 0$;

(f3) there exists some $1 < p < \frac{N+2}{N-2}$, such that $\lim_{\xi\to\infty} f(\xi^2)/\xi^{p-1} = 0$;

(f4) There exists $\xi_0 > 0$ such that

$$F(|\xi_0|^2) > m_0\xi_0^2 \quad \text{where} \quad F(\xi) = \int_0^\xi f(\tau)\, d\tau.$$

We remark that, under our assumptions of f, the search of solutions to (1.4) cannot be reduced to the study of the critical points of a functional restricted to a Nehari manifold. We obtain multiplicity results of complex-valued solutions to (1.4) by means of a new variational approach developed in the recent paper [17] for the scalar nonlinear Schrödinger equation corresponding to $A = 0$. Precisely, we seek for critical points of the indefinite Euler functional associated to problem (1.4) in a suitable neighborhood of expected solutions, suggested by a complex-valued limiting problem. In order to apply critical point theory, one of the main problem in this approach is the presence of the boundary. However we recognize that this neighborhood is positively invariant for a pseudo gradient flow and we derive a deformation theorem. In this way we relate the number of complex-valued solutions to the relative category of two sublevels of the functional in the neighborhood of expected solutions. Finally this relative category will be estimated by means of the cuplength of the local minima set K of V. To this aim, we need to construct two maps between topological pairs, which involve a barycenter map and a Pohozaev type function. We remark that the presence of the magnetic field produces several additional difficulties, due to the fact that the space $H_{\varepsilon,V,A}(\mathbb{R}^N, \mathbb{C})$, where the Euler functional associated to (1.4) is naturally defined, can not be embedded into $H^1(\mathbb{R}^N, \mathbb{C})$ where the limiting problem (as $\hbar \to 0$) is set up (see [25]). In order to perform our variational arguments, we crucially use of the diamagnetic inequality which, in particular allows to prove that the least energy level of the complex-valued limiting problem coincides with the least energy level of a real-valued problem (see Section 2). This fact enables to construct a barycenter map and a Pohozaev type map, which act only on the modula of the complex-valued functions.

Our main result is the following:

Theorem 1.1. *Suppose $N \geq 3$ and that (A1), (V1)–(V2) and (f1)–(f4) hold. Assume in addition that* $\sup_{x\in\mathbb{R}^N} V(x) < \infty$. *Then letting $K = \{x \in \Omega; V(x) = m_0\}$, for sufficiently small $\varepsilon > 0$, (1.4) has at least* $\operatorname{cupl}(K) + 1$ *geometrically distinct, complex-valued solutions v_ε^i, $i = 1, \ldots, \operatorname{cupl}(K) + 1$, whose modula concentrate as $\varepsilon \to 0$ in K, where* $\operatorname{cupl}(K)$ *denotes the cup-length defined with Alexander-Spanier cohomology with coefficients in the field $\mathbb{F}$.*

We say that two complex-valued solutions to (1.4) are geometrically distinct if their $\mathbb{S}^1$-orbits are different. Our theorem covers the relevant physical case of constant magnetic fields B which leads to vector potentials A, having a polynomial growth on $\mathbb{R}^N$. For instance, if B is the constant magnetic field $(0, 0, b)$, then a suitable vector field is given by $A(x) = \frac{b}{2}(-x_2, x_1, 0)$. In physical literature the potential A corresponds to the so-called *Lorentz gauge* (see [25]).

Remark 1.2. *If $K = S^{N-1}$, the $N - 1$ dimensional sphere in $\mathbb{R}^N$, then* $\operatorname{cupl}(K) + 1 = \operatorname{cat}(K) = 2$. *If $K = T^N$ is the N-dimensional torus, then* $\operatorname{cupl}(K) + 1 = \operatorname{cat}(K) = N + 1$. *However in general* $\operatorname{cupl}(K) + 1 \leq \operatorname{cat}(K)$.

Remark 1.3. *When we say that the solutions v_ε^i, $i = 1, \ldots, \operatorname{cupl}(K) + 1$ of Theorem 1.1 concentrate when $\varepsilon \to 0$ in K, we mean that there exists a maximum point x_ε^i of $|v_\varepsilon^i|$ such that $\lim_{\varepsilon\to 0} dist(x_\varepsilon^i, K) = 0$ and that, for any such x_ε^i, $|w_\varepsilon^i| = |v_\varepsilon^i(\varepsilon(\cdot + x_\varepsilon^i))|$ converges, up to a subsequence, uniformly to a least energy solution of*

$$-\Delta U + m_0 U = f(U), \quad U > 0, \quad U \in H^1(\mathbb{R}^N, \mathbb{R}).$$

We also have

$$|v_\varepsilon^i(x)| \leq C exp(-\frac{c}{\varepsilon}|x - x_\varepsilon^i|) \quad \text{for some } c, C > 0.$$

Remark 1.4. *In addition to condition $(V1)$ the boundedness of V from above is assumed in Theorem 1.1. Arguing as in [10, 12, 16] we could prove Theorem 1.1 without this additional assumption. However, for the sake of simplicity, we assume here the boundedness of V.*

2 The variational framework $H_{\varepsilon,V,A}$

Let $\hbar = \varepsilon$, $v(x) = u(\varepsilon x)$, $A_\varepsilon(x) = A(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$. Then equation (1.4) is equivalent to

$$\left(\frac{1}{i}\nabla - A_\varepsilon(x)\right)^2 v + V_\varepsilon(x)v - f(|v|^2)v = 0, \quad x \in \mathbb{R}^N. \tag{2.1}$$

Let $H_{\varepsilon,V,A}(\mathbb{R}^N, \mathbb{C})$ be the Hilbert space defined by the completion of $C_0^\infty(\mathbb{R}^N, \mathbb{C})$ under the scalar product

$$\langle u, v\rangle_{\varepsilon,V,A} = \operatorname{Re} \int_{\mathbb{R}^N} \left(\frac{1}{i}\nabla u - A_\varepsilon(x)u\right) \left(\overline{\frac{1}{i}\nabla v - A_\varepsilon(x)v}\right) + V_\varepsilon(x)u\overline{v}\, dx. \tag{2.2}$$

We denote by $\|\cdot\|_{\varepsilon,V,A}$ the associated norm and in the special case $V = 1$, we set $\|\cdot\|_\varepsilon := \|\cdot\|_{\varepsilon,1,A}$. Also let H_ε denote the space $H_{\varepsilon,1,A}(\mathbb{R}^N, \mathbb{C})$.

In what follows we use the notations:

$$\begin{aligned} \|u\|_{H^1} &= \left(\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 \, dx \right)^{1/2}, \\ \|u\|_r &= \left(\int_{\mathbb{R}^N} |u|^r \, dx \right)^{1/r} \quad \text{for } r \in [1, \infty). \end{aligned}$$

It is well known that, in general, there is no relationship between the spaces H_ε and $H^1(\mathbb{R}^N, \mathbb{C})$, namely $H_\varepsilon \not\subset H^1(\mathbb{R}^N, \mathbb{C})$ nor $H^1(\mathbb{R}^N, \mathbb{C}) \not\subset H_\varepsilon$ (see [25]).

We now recall the following *diamagnetic inequality*: for every $u \in H_\varepsilon$,

$$\left| \left(\frac{\nabla}{i} - A_\varepsilon \right) u \right| \geq |\nabla |u||, \quad \text{a.e. in } \mathbb{R}^N. \tag{2.3}$$

See [25] for a proof. As a consequence of (2.3), $|u| \in H^1(\mathbb{R}^N, \mathbb{R})$ for any $u \in H_\varepsilon$. We also have for any $r \in [2, \frac{2N}{N-2}]$ there exists $C_r > 0$ independent of ε such that

$$\|u\|_r \leq C_r \|u\|_\varepsilon \quad \text{for all } u \in H_\varepsilon. \tag{2.4}$$

We also note that for any compact set $K \subset \mathbb{R}^N$ there exists a $C_K > 0$ independent of $\varepsilon \in (0, 1]$ such that

$$\|u\|_{H^1(K)} \leq C_K \|u\|_\varepsilon \quad \text{for all } u \in H_\varepsilon. \tag{2.5}$$

See [16, Corollary 2.2]. Thus convergence in H_ε implies convergence in $H^1_{loc}(\mathbb{R}^N, \mathbb{C})$. We can also derive the following lemma.

Lemma 2.1. *For each* $\varepsilon > 0$,

$$H_\varepsilon \to H^1(\mathbb{R}^N, \mathbb{R}); \; u \mapsto |u|$$

is a continuous map.

Proof. It suffices to show for any strongly convergent sequence $u_n \to u$ in H_ε that $|u_n| \to |u|$ strongly in $H^1(\mathbb{R}^N, \mathbb{R})$. Suppose that $(u_n) \subset H_\varepsilon$ converges to $u \in H_\varepsilon$ strongly in H_ε. We infer, by (2.4) that (u_n) strongly converges to u in $L^2(\mathbb{R}^N, \mathbb{C})$ and in particular $|u_n| \to |u|$ in $L^2(\mathbb{R}^N, \mathbb{R})$. Now from (2.3) we deduce that $(|u_n|)$ is bounded in $H^1(\mathbb{R}^N, \mathbb{R})$ and weakly converges in $H^1(\mathbb{R}^N, \mathbb{R})$, up to subsequences. Moreover since strong convergence in H_ε implies strong convergence in $H^1_{loc}(\mathbb{R}^N)$, $(|\nabla |u_n|(x)|)$ converges to $|\nabla |u|(x)|$, a.e in $\mathbb{R}^N$. Using again (2.3), we derive, by Lebesgue's Theorem, that $\||u_n|\|_{H^1(\mathbb{R}^N,\mathbb{R})}$ converges to $\||u|\|_{H^1(\mathbb{R}^N,\mathbb{R})}$ and the conclusion of lemma holds. □

Another continuity property of $u \mapsto |u|$; $H_\varepsilon \to H^1(\mathbb{R}^N)$ will be given in Lemma 2.3 below.

Finally we consider the functional I_ε defined on H_ε by

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left| \left(\frac{1}{i}\nabla - A_\varepsilon(x)\right) u \right|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V_\varepsilon(x)|u|^2\, dx - \frac{1}{2}\int_{\mathbb{R}^N} F(|u|^2)\, dx,$$

where $F(s) = \int_0^s f(t)\, dt$. Without loss of generality, we may assume that $f(\xi) = 0$ for all $\xi \le 0$. It is standard that the functional is C^1 and its critical points are solutions of (2.1).

We notice that the group $\mathbb{S}^1$ of unit complex numbers acts on H_ε by scalar multiplication $(\gamma, u) \mapsto \gamma u$. This action is unitary, that is,

$$\langle \gamma u, \gamma v\rangle_{\varepsilon,V,A} = \langle u, v\rangle_{\varepsilon,V,A} \qquad \forall \gamma \in \mathbb{S}^1,\ u, v \in H_\varepsilon.$$

Since the functional I_ε is $\mathbb{S}^1$-invariant, that is,

$$I_\varepsilon(\gamma u) = I_\varepsilon(u) \qquad \forall u \in H_\varepsilon,\ \gamma \in \mathbb{S}^1$$

then, if $u \in H_\varepsilon$ is a critical point of I_ε, every point γu in the $\mathbb{S}^1$-orbit of u is a critical point of I_ε. We say that two critical points of I_ε are geometrically distinct if their $\mathbb{S}^1$-orbits are different. We shall apply $\mathbb{S}^1$-equivariant Lusternik-Schnirelmann theory to obtain a lower bound for the number of critical $\mathbb{S}^1$-orbits of I_ε.

2.1 Scalar and Complex-valued Limit problems

Let us consider for $a > 0$ the scalar limiting equation of (2.1)

$$-\Delta u + au = f(|u|^2)u, \quad u \in H^1(\mathbb{R}^N, \mathbb{R}). \tag{2.6}$$

(2.6) can be obtained as follows: let $x = y + p/\varepsilon$ in (2.1) and take a (formal) limit as $\varepsilon \to 0$, then we have

$$(\frac{1}{i}\nabla - A(p))^2 v + V(p)v = f(|v|^2)v \quad \text{in } \mathbb{R}^N.$$

Setting $u(x) = e^{-iA(p)x}v(x)$ and considering real-valued solutions, we obtain (2.6) with $a = V(p)$. Solutions to (2.6) correspond to critical points of the functional $L_a\colon H^1(\mathbb{R}^N, \mathbb{R}) \to \mathbb{R}$ defined by

$$L_a(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + a|u|^2\right) dy - \frac{1}{2}\int_{\mathbb{R}^N} F(|u|^2) dy. \tag{2.7}$$

We denote by $E(a)$ the least energy level for (2.6). That is,

$$E(a) = \inf\{L_a(u);\ u \in H^1(\mathbb{R}^N, \mathbb{R}) \setminus \{0\},\ L_a'(u) = 0\}.$$

In [7] it is proved that there exists a least energy solution of (2.6), for any $a > 0$, if (f1)–(f4) are satisfied (here we consider (f4) with $m_0 = a$). Also it is showed that each solution u of (2.6) satisfies the Pohozaev's identity

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + N\int_{\mathbb{R}^N} a\frac{u^2}{2} - \frac{1}{2}F(|u|^2) dx = 0. \tag{2.8}$$

From this we immediately deduce that, for any solution ω of (2.6),

$$\frac{1}{N}\int_{\mathbb{R}^N} |\nabla\omega|^2 dy = L_a(\omega). \tag{2.9}$$

We also consider the complex-valued equation, for $a > 0$,

$$-\Delta u + au = f(|u|^2)u, \quad u \in H^1(\mathbb{R}^N, \mathbb{C}). \tag{2.10}$$

In turn solutions of (2.10) correspond to critical points of the functional $L_a^{\mathbb{C}} : H^1(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$, defined by

$$L_a^{\mathbb{C}}(v) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 + a|v|^2\right) dy - \frac{1}{2}\int_{\mathbb{R}^N} F(|v|^2) dy. \tag{2.11}$$

We denote by $E^{\mathbb{C}}(a)$ the least energy level for (2.10), that is

$$E^{\mathbb{C}}(a) = \inf\{L_a^{\mathbb{C}}(u);\, u \neq 0,\ {L_a^{\mathbb{C}}}'(u) = 0\}.$$

In [36] the Pohozaev's identity (2.8) is established for complex-valued solutions of (2.10) and thus the equivalent of (2.9) holds for such solutions.

In [16, Lemma 2.3], it has been proved that the least energy levels of (2.6) and (2.10) coincide and that any least energy solution U of (2.10) has the form $e^{i\tau}\omega$ where ω is a positive least energy solution of (2.6) and $\tau \in \mathbb{R}$.

Now we introduce the notation

$$\Omega(I) = \{y \in \Omega;\, V(y) - m_0 \in I\}$$

for an interval $I \subset [0, \inf_{x\in\partial\Omega} V(x) - m_0)$. We choose a small $\nu_0 > 0$ such that

(i) $0 < \nu_0 < \inf_{x\in\partial\Omega} V(x) - m_0$;

(ii) $F(|\xi_0|^2) > \frac{1}{2}(m_0 + \nu_0)\xi_0^2$;

(iii) $\Omega([0, \nu_0]) \subset K_d$, where $K = \{x \in \Omega \mid V(x) = m_0\}$ and $d > 0$ is a constant for which Lemma 5.4 (Section 5) holds.

From [7] we note that, under our choice of $\nu_0 > 0$, $E(a)$ is attained for $a \in [m_0, m_0 + \nu_0]$. Clearly $a \mapsto E(a);\, [m_0, m_0 + \nu_0] \to \mathbb{R}$ is continuous and strictly increasing. Choosing $\nu_0 > 0$ smaller if necessary, we may assume

$$E(m_0 + \nu_0) < 2E(m_0).$$

We choose $\ell_0 \in (E(m_0 + \nu_0), 2E(m_0))$ and we set

$$S_{a,\ell_0}^{\mathbb{C}} = \{U \in H^1(\mathbb{R}^N, \mathbb{C})\backslash\{0\};\, (L_a^{\mathbb{C}})'(U) = 0,\ L_a^{\mathbb{C}}(U) \leq \ell_0,\ |U(0)| = \max_{x\in\mathbb{R}^N} |U(x)|\}.$$

We also define

$$\widehat{S}_{\nu_0,\ell_0} = \bigcup_{a\in[m_0, m_0+\nu_0]} S_{a,\ell_0}^{\mathbb{C}}.$$

Following the proof of [10, Proposition 1], we can show that the set $\widehat{S}_{\nu_0,\ell_0}$ is compact in $H^1(\mathbb{R}^N,\mathbb{C})$ and that its elements have a uniform exponential decay. Namely that there exist C, $c>0$ such that

$$|U(x)|+|\nabla U(x)|\le C\exp(-c|x|) \quad \text{for all} \quad U\in \widehat{S}_{l_0,\nu_0}. \tag{2.12}$$

By [16, Lemma 2.3], each element of $S^{\mathbb{C}}_{m_0,E(m_0)}$ is of the form $e^{i\tau}\omega$ where $\tau\in\mathbb{R}$ and ω is a real least energy solution of (2.6). Thus

$$P_0(\omega)=1,$$

where P_0 is defined as

$$P_0(u)=\left(\frac{N\int_{\mathbb{R}^N}\frac{1}{2}F(|u|^2)-\frac{m_0}{2}|u|^2\,dx}{\frac{N-2}{2}\|\nabla|u|\|_2^2}\right)^{\frac{1}{2}}. \tag{2.13}$$

We note that $P_0(\omega(\frac{x}{s}))=s$ and

Lemma 2.2. *Suppose that* $u\in H^1(\mathbb{R}^N,\mathbb{C})$ *satisifes* $P_0(u)\in(0,\sqrt{\frac{N}{N-2}})$. *Then*

$$L_{m_0}(|u|)\ge g(P_0(u))E(m_0),$$

where

$$g(t)=\frac{1}{2}(Nt^{N-2}-(N-2)t^N). \tag{2.14}$$

Proof. By the scaling property

$$L_{m_0}(|u(\frac{x}{s})|)=\frac{s^{N-2}}{2}\|\nabla|u|\|_2^2+s^N\left(\frac{m_0}{2}\|u\|_2^2-\frac{1}{2}\int_{\mathbb{R}^N}F(|u|^2)\,dx\right)$$

and the characterization of $E(m_0)$

$$E(m_0)=\inf\{L_{m_0}(u);\, u\in H^1(\mathbb{R}^N,\mathbb{R})\setminus\{0\},\, P_0(u)=1\},$$

we can deduce the conclusion of Lemma 2.2. See [30] and [17, Lemma 2.1]. □

We claim that, as $\ell\to E(m_0)$ and $\nu\to 0$, $\widehat{S}_{\nu,\ell}$ shrinks to $S^{\mathbb{C}}_{m_0,E(m_0)}$ in $H^1(\mathbb{R}^N,\mathbb{C})$. More precisely we have

$$\lim_{\nu\to 0,\ell\to E(m_0)}\ \sup_{\widetilde{U}\in\widehat{S}_{\nu,\ell}}\ \inf_{U\in S^{\mathbb{C}}_{m_0,E(m_0)}}||U-\widetilde{U}||_{H^1}=0. \tag{2.15}$$

In fact, suppose $\nu_n>0$, $\ell_n>E(m_0)$ and $U_n\in\widehat{S}_{\nu_n,\ell_n}$ satisfy $\nu_n\to 0$ and $\ell_n\to E(m_0)$, then by the compactness of $\widehat{S}_{\nu,\ell}$ for each $\nu\ge 0$, $\ell\ge E(m_0)$, U_n converges to some $U\in S^{\mathbb{C}}_{m_0,E(m_0)}$ in $H^1(\mathbb{R}^N,\mathbb{C})$. Thus (2.15) holds.

As a consequence of (2.15) for ℓ_0 close to $E(m_0)$ and $\nu_0>0$ small, we have

$$P_0(U)\in\left(\frac{1}{2},\sqrt{\frac{N}{N-1}}\right) \quad \text{for all } U\in\widehat{S}_{\nu_0,\ell_0}. \tag{2.16}$$

We fix such ℓ_0 and ν_0 and we write $\widehat{S} = \widehat{S}_{\nu_0,\ell_0}$.

In what follows, we try to find our critical points in the following bounded subsets of H_ε:

$$\mathcal{S}_\varepsilon(r) = \{e^{iA(\varepsilon p)(x-p)}U(x-p) + \varphi(x);\ \varepsilon p \in \overline{\Omega},\ U \in \widehat{S},\ \|\varphi\|_\varepsilon < r\}$$

for a $r > 0$.

2.2 A Pohozaev map in $\mathcal{S}_\varepsilon(r)$

First we give an equi-continuity result of $u \mapsto |u|$; $H_\varepsilon \to H^1(\mathbb{R}^N, \mathbb{R})$.

Lemma 2.3. *For any $r > 0$ there exists $r_{**} > 0$ such that for small $\varepsilon > 0$*

$$\||u(x)| - |U(x-p)|\|_{H^1} < r \tag{2.17}$$

for any $u(x) = e^{iA(\varepsilon p)(x-p)}U(x-p) + \varphi(x) \in \mathcal{S}_\varepsilon(r_{**})$ with $\varepsilon p \in \overline{\Omega}$, $U \in \widehat{S}$, $\|\varphi\|_\varepsilon < r_{**}$.

Proof. It suffices to show

$$\||u_n(x)| - |U_n(x-p_n)|\|_2 \to 0, \tag{2.18}$$

$$\|\nabla|u_n(x)| - \nabla|U_n(x-p_n)|\|_2 \to 0 \tag{2.19}$$

for $u_n = e^{iA(\varepsilon_n p_n)(x-p_n)}U_n(x-p_n) + \varphi_n(x) \in \mathcal{S}_{\varepsilon_n}(r_n)$ with $\varepsilon_n \to 0$, $r_n \to 0$, $\varepsilon_n p_n \in \overline{\Omega}$, $U_n \in \widehat{S}$ and

$$\|\varphi_n\|_{\varepsilon_n} < r_n \to 0. \tag{2.20}$$

Since $\widehat{S}$ is compact in $H^1(\mathbb{R}^N, \mathbb{C})$, we may assume that $U_n \to U_0 \in \widehat{S}$ and $\varepsilon_n p_n \to p_0 \in \overline{\Omega}$ as $n \to \infty$. We proceed in several steps.

Step 1: (2.18) *holds.*

By (2.4) and (2.20),

$$\|u_n(x) - e^{iA(\varepsilon_n p_n)(x-p_n)}U_n(x-p_n)\|_2 = \|\varphi_n\|_2 \le \|\varphi_n\|_{\varepsilon_n} \to 0.$$

Since $u \mapsto |u|$; $L^2(\mathbb{R}^N, \mathbb{C}) \to L^2(\mathbb{R}^N, \mathbb{R})$ is continuous, we have

$$\||u_n(x)| - |U_n(x-p_n)|\|_2 = \||u_n(x)| - |e^{iA(\varepsilon_n p_n)(x-p_n)}U_n(x-p_n)|\|_2 \to 0.$$

Step 2: $\|e^{iA(\varepsilon_n p_n)(x-p_n)}(U_n(x-p_n) - U_0(x-p_n)) + \varphi_n(x)\|_{\varepsilon_n} \to 0$.

Observe that

$$\begin{aligned}
&\left\|\left(\frac{1}{i}\nabla - A(\varepsilon_n x)\right)(e^{iA(\varepsilon_n p_n)(x-p_n)}(U_n(x-p_n) - U_0(x-p_n)))\right\|_2 \\
= &\left\|\left(\frac{1}{i}\nabla - A(\varepsilon_n x + \varepsilon_n p_n)\right)(e^{iA(\varepsilon_n p_n)x}(U_n(x) - U_0(x)))\right\|_2 \\
= &\|A(\varepsilon_n p_n)e^{iA(\varepsilon_n p_n)x}(U_n - U_0) + \frac{1}{i}e^{iA(\varepsilon_n p_n)x}(\nabla U_n - \nabla U_0) \\
&-A(\varepsilon_n x + \varepsilon_n p_n)e^{iA(\varepsilon_n p_n)x}(U_n - U_0)\|_2 \\
= &\|(A(\varepsilon_n p_n) - A(\varepsilon_n x + \varepsilon_n p_n))(U_n - U_0) + \frac{1}{i}(\nabla U_n - \nabla U_0)\|_2 \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Here we have use the fact that $U_n \to U_0$ in $H^1(\mathbb{R}^N, \mathbb{C})$ and that the elements in $\widehat{S}$ have a uniform exponential decay. Clearly also

$$\|e^{iA(\varepsilon_n p_n)(x-p_n)}(U_n(x-p_n) - U_0(x-p_n))\|_2 = \|U_n - U_0\|_2 \to 0.$$

By (2.20), we have the conclusion of Step 2.

Step 3: $|e^{iA(\varepsilon_n p_n)(x)}U_n(x) + \varphi_n(x+p_n)| \to |U_0(x)|$ *in* $H^1_{loc}(\mathbb{R}^N, \mathbb{R})$.
In particular, after taking a subsequence

$$\nabla |e^{iA(\varepsilon_n p_n)(x)}U_n(x) + \varphi_n(x+p_n)| \to \nabla |U_0(x)| \quad \text{a.e. } in\ \mathbb{R}^N.$$

Using notation

$$\|v\|^2_{\varepsilon_n,1,A(\cdot+\varepsilon_n p_n)} = \|(\frac{1}{i}\nabla - A(\varepsilon_n x + \varepsilon_n p_n))v\|_2^2 + \|v\|_2^2,$$

we have by Step 2

$$\begin{aligned} & \|e^{iA(\varepsilon_n p_n)x}(U_n(x) - U_0(x)) + \varphi_n(x+p_n)\|_{\varepsilon_n,1,A(\cdot+\varepsilon_n p_n)} \\ = \ & \|e^{iA(\varepsilon_n p_n)(x-p_n)}(U_n(x-p_n) - U_0(x-p_n)) + \varphi_n(x)\|_{\varepsilon_n} \\ \to \ & 0 \qquad \text{as } n \to \infty. \end{aligned} \tag{2.21}$$

As in (2.4), we get

$$e^{iA(\varepsilon_n p_n)x}(U_n(x) - U_0(x)) + \varphi_n(x+p_n) \to 0 \quad \text{in } H^1_{loc}(\mathbb{R}^N, \mathbb{C}),$$

from which the conclusion of Step 3 follows.

Step 4: (2.19) *holds.*

By (2.21),

$$\|\left(\frac{1}{i}\nabla - A(\varepsilon_n x + \varepsilon_n p_n)\right)(e^{iA(\varepsilon_n p_n)x}(U_n(x) - U_0(x)) + \varphi_n(x+p_n))\|_2 \to 0.$$

Thus $\left(\frac{1}{i}\nabla - A(\varepsilon_n x + \varepsilon_n p_n)\right)(e^{iA(\varepsilon_n p_n)x}U_n(x) + \varphi_n(x+p_n))$ converges to $\left(\frac{1}{i}\nabla - A(p_0)\right)(e^{iA(p_0)x}U_0(x))$ in $L^2(\mathbb{R}, \mathbb{C})$. Therefore, there exists a $h(x) \in L^2(\mathbb{R}^N)$ such that

$$\left|\left(\frac{1}{i}\nabla - A(\varepsilon_n x + \varepsilon_n p_n)\right)(e^{iA(\varepsilon_n p_n)x}U_n(x) + \varphi_n(x+p_n))\right| \le h(x).$$

By the diamagnetic inequality, we have

$$\begin{aligned} & \left|\nabla |e^{iA(\varepsilon_n p_n)x}U_n(x) + \varphi_n(x+p_n))| - \nabla |U_0(x)|\right| \\ \le \ & \left|\left(\frac{1}{i}\nabla - A(\varepsilon_n x + \varepsilon_n p_n)\right)(e^{iA(\varepsilon_n p_n)x}U_n(x) + \varphi_n(x+p_n))\right| + |\nabla U_0(x)| \\ \le \ & h(x) + |\nabla U_0(x)| \in L^2(\mathbb{R}^N). \end{aligned}$$

Therefore, by Lebesgue theorem, it follows from Step 3 that

$$\begin{aligned}\|\nabla|u_n(x+p_n)| - \nabla|U_0(x)|\|_2 &= \|\nabla|e^{iA(\varepsilon_n p_n)x}U_n(x) + \varphi_n(x+p_n))| - \nabla|U_0(x)|\|_2 \\ &\to 0.\end{aligned}$$

which is nothing but (2.19). □

For a later use we have the following

Corollary 2.4. *There exists $r_0 > 0$ such that for $\varepsilon > 0$ small, $P_0 : \mathcal{S}_\varepsilon(r_0) \to \mathbb{R}$ is continuous and*

$$P_0(u) \in \Big(0, \sqrt{\frac{N}{N-1}}\Big) \quad \text{for all } u \in \mathcal{S}_\varepsilon(r_0). \tag{2.22}$$

Proof. Since $\||u|-|U|\|_{H^1} \to 0$ implies $\int_{\mathbb{R}^N} F(|u|^2)\,dx \to \int_{\mathbb{R}^N} F(|U|^2)\,dx$, $\|u\|_2^2 \to \|U\|_2^2$ and $\|\nabla|u|\|_2^2 \to \|\nabla|U|\|_2^2$, by Lemma 2.3 there exists $r_0 > 0$ such that (2.16) holds for $\varepsilon > 0$ small. The continuity of $P_0 : \mathcal{S}_\varepsilon(r_0) \to \mathbb{R}$ follows from Lemma 2.1. □

Remark 2.5. *We remark that there does not exist a constant $C > 0$ with the following property:*

$$\|\nabla|u+\varphi| - \nabla|u|\|_2 \le C(\|\nabla|\varphi|\|_2 + \|\varphi\|_2) \quad \text{for all } u, \varphi \in H_\varepsilon.$$

Thus Lemma 2.3 is not a direct consequence from the diamagenetic inequality. To see such an inequality does not hold, for $n \in \mathbb{N}$ we set u, $\varphi_n \in H^1(\mathbb{R}, \mathbb{C})$ by

$$u(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 2-|x| & \text{if } |x| \in (1,2], \\ 0 & \text{otherwise,} \end{cases} \qquad \varphi_n(x) = \begin{cases} e^{2\pi i n x} & \text{if } |x| \le 1, \\ 2-|x| & \text{if } |x| \in (1,2], \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have

$$\|\nabla|u+\varphi_n| - \nabla|u|\|_{L^2(\mathbb{R})} \to \infty, \quad \sup_{n\in\mathbb{N}}(\|\nabla|\varphi_n|\|_{L^2(\mathbb{R})} + \|\varphi_n\|_{L^2(\mathbb{R})}) < \infty.$$

We can easily extend this example to general dimension N.

2.3 A barycenter map in $\mathcal{S}_\varepsilon(r)$

Following [12, 13] we introduce a center of mass in $\mathcal{S}_\varepsilon(r)$.

Lemma 2.6. *There exist r_0, R_0, $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$ there exists a function $\Upsilon_\varepsilon : \mathcal{S}_\varepsilon(r_0) \to \mathbb{R}^N$ such that*

$$|\Upsilon_\varepsilon(u) - p| \le 2R_0$$

for all $u(x) = e^{iA(\varepsilon p)(x-p)}U(x-p) + \varphi(x) \in \mathcal{S}_\varepsilon(r_0)$ with $p \in \mathbb{R}^N$, $\varepsilon p \in \overline{\Omega}$, $U \in \widehat{S}$, $\|\varphi\|_\varepsilon \le r_0$. Moreover, Υ_ε has the following properties *(i)* Υ_ε *is shift equivariant, that is,*

$$\Upsilon_\varepsilon(u(x-y)) = \Upsilon_\varepsilon(u(x)) + y \quad \textit{for all } u \in \mathcal{S}_\varepsilon(r_0) \textit{ and } y \in \mathbb{R}^N.$$

(ii) Υ_ε *is* $\mathbb{S}^1$*-invariant, that is,*

$$\Upsilon_\varepsilon(e^{i\tau}u) = \Upsilon_\varepsilon(u) \quad \textit{for all } u \in \mathcal{S}_\varepsilon(r_0) \textit{ and } e^{i\tau} \in \mathbb{S}^1.$$

(iii) $\Upsilon_\varepsilon : \mathcal{S}_\varepsilon(r_0) \subset H_\varepsilon \to \mathbb{R}^N$ *is a continuous function. Moreover,* Υ_ε *is a locally Lipschitz continuous function of* $|u|$ *in the following sense:* Υ_ε *satisfies* $\Upsilon_\varepsilon(u) = \Upsilon_\varepsilon(|u|)$ *for all* $u \in \mathcal{S}_\varepsilon(r_0)$ *and there exist constants* C_1*,* $C_2 > 0$ *such that*

$$|\Upsilon_\varepsilon(u) - \Upsilon_\varepsilon(v)| \le C_1 \| |u| - |v| \|_{H^1} \quad \textit{for all } u, v \in \mathcal{S}_\varepsilon(r_0) \textit{ with } \| |u| - |v| \|_{H^1} \le C_2. \tag{2.23}$$

Proof. We set $r_* = \min_{U \in \widehat{S}} \| |U| \|_{H^1} > 0$ and choose $R_0 > 1$ such that for $U \in \widehat{S}$

$$\| \, |U| \, \|_{H^1(|x| \le R_0)} > \frac{3}{4} r_* \quad \text{and} \quad \| \, |U| \, \|_{H^1(|x| \ge R_0)} < \frac{1}{8} r_*.$$

This is possible by the uniform exponential decay (2.12). For $u \in H^1(\mathbb{R}^N, \mathbb{C})$ and $q \in \mathbb{R}^N$, we define

$$d(q,u) = \psi\left(\inf_{\tilde{U} \in \widehat{S}} \| |u(x)| - |\tilde{U}(x-q)| \|_{H^1(|x-q| \le R_0)} \right),$$

where $\psi \in C_0^\infty(\mathbb{R}, \mathbb{R})$ is such that

$$\psi(r) = \begin{cases} 1 & r \in [0, \frac{1}{4} r_*], \\ 0 & r \in [\frac{1}{2} r_*, \infty), \end{cases}$$
$$\psi(r) \in [0,1] \quad \text{for all } r \in [0, \infty).$$

Now by Lemma 2.3, there exists $r_{**} \in (0, \frac{1}{8} r_*]$ such that for $\varepsilon > 0$ small

$$\| \, |u(x)| - |U(x-p)| \, \|_{H^1} < \frac{1}{8} r_* \tag{2.24}$$

for $u(x) = e^{iA(\varepsilon p)(x-p)} U(x-p) + \varphi(x) \in \mathcal{S}_\varepsilon(r_{**})$. We set

$$\Upsilon_\varepsilon(u) = \frac{\displaystyle\int_{\mathbb{R}^N} q \, d(q,u) \, dq}{\displaystyle\int_{\mathbb{R}^N} d(q,u) \, dq} \quad \text{for} \quad u \in \mathcal{S}_\varepsilon(r_{**}).$$

We shall show that Υ_ε has the desired property.

Let $u \in \mathcal{S}_\varepsilon(r_{**})$ and write $u(x) = e^{iA(\varepsilon p)(x-p)}U(x-p) + \varphi(x)$ ($p \in \mathbb{R}^N$, $\varepsilon p \in \overline{\Omega}$, $U \in \widehat{S}$, $\|\varphi\|_\varepsilon \le r_{**}$).

Taking into account that $\||\varphi|\|_{H^1} \le \|\varphi\|_\varepsilon$, we have that for $|q-p| \ge 2R_0$ and $\tilde{U} \in \widehat{S}$, we have

$$\begin{aligned} & \||u(x)| - |\tilde{U}(x-q)|\|_{H^1(|x-q|\le R_0)} \\ \ge\ & \||\tilde{U}(x-q)|\|_{H^1(|x-q|\le R_0)} - \||u(x)|\|_{H^1(|x-q|\le R_0)} \\ \ge\ & \||\tilde{U}(x-q)|\|_{H^1(|x-q|\le R_0)} - \||U(x-p)|\|_{H^1(|x-q|\le R_0)} - \||u(x)| - |U(x-p)|\|_{H^1} \\ \ge\ & \||\tilde{U}(x-q)|\|_{H^1(|x-q|\le R_0)} - \||U(x-p)|\|_{H^1(|x-p|\ge R_0)} - \frac{1}{8}r_* \\ >\ & \frac{3}{4}r_* - \frac{1}{8}r_* - \frac{1}{8}r_* = \frac{1}{2}r_*. \end{aligned}$$

Thus $d(q,u) = 0$ for $|q-p| \ge 2R_0$. We can also see that, for small $r > 0$

$$d(q,u) = 1 \quad \text{for } |q-p| < r.$$

Thus $B(p,r) \subset \operatorname{supp} d(\cdot,u) \subset B(p,2R_0)$. Therefore $\Upsilon_\varepsilon(u)$ is well-defined and we have

$$\Upsilon_\varepsilon(u) \in B(p,2R_0) \quad \text{for} \quad u \in \mathcal{S}_\varepsilon(r_{**}).$$

It is clear from the definition that $\Upsilon_\varepsilon(u) = \Upsilon_\varepsilon(|u|)$ for all $u \in \mathcal{S}_\varepsilon(r_{**})$. Its shift equivariance, $\mathbb{S}^1$-invariance and locally Lipschitz continuity (2.23) can be checked easily. Thus continuity of $\Upsilon_\varepsilon : \mathcal{S}_\varepsilon(r_{**}) \to \mathbb{R}^N$, where the topology of $\mathcal{S}_\varepsilon(r_{**})$ is induced from H_ε, follows from Lemma 2.1. □

Using this lemma we have

Lemma 2.7. *There exist $\delta_1 > 0$, $r_1 \in (0,r_0)$ and $\nu_1 \in (0,\nu_0)$ such that for $\varepsilon > 0$ small*

$$I_\varepsilon(u) \ge E(m_0) + \delta_1$$

for all $u \in \mathcal{S}_\varepsilon(r_1)$ with $\varepsilon\Upsilon_\varepsilon(u) \in \Omega([\nu_1,\nu_0])$.

Proof. We set $\underline{M} = \inf_{U\in\widehat{S}} \|U\|_2^2$, $\overline{M} = \sup_{U\in\widehat{S}} \|U\|_2^2$. It follows from the compactness of $\widehat{S}$ that $0 < \underline{M} \le \overline{M} < \infty$. For later use in (2.28) below, we choose $\nu_1 \in (0,\nu_0)$ such that

$$E(m_0+\nu_1) - \frac{1}{2}(\nu_0-\nu_1)\overline{M} > E(m_0). \tag{2.25}$$

First we claim that for some $\delta_1 > 0$

$$\inf_{U\in\widehat{S}} L^{\mathbb{C}}_{m_0+\nu_1}(U) \ge E(m_0) + 3\delta_1. \tag{2.26}$$

Indeed, on one hand, if $U \in S^{\mathbb{C}}_{a,\ell_0}$ with $a \in [m_0, m_0+\nu_1]$, we have

$$\begin{aligned} L^{\mathbb{C}}_{m_0+\nu_1}(U) &= L^{\mathbb{C}}_a(U) + \frac{1}{2}(m_0+\nu_1-a)\|U\|_2^2 \\ &\ge E(a) + \frac{1}{2}(m_0+\nu_1-a)\underline{M} \end{aligned}$$

and thus

$$\inf_{U \in \bigcup_{a \in [m_0, m_0+\nu_1]} S^{\mathbb{C}}_{a,\ell_0}} L^{\mathbb{C}}_{m_0+\nu_1}(U) > E(m_0). \tag{2.27}$$

On the other hand, if $U \in S^{\mathbb{C}}_{a,\ell_0}$ with $a \in [m_0+\nu_1, m_0+\nu_0]$,

$$\begin{aligned} L^{\mathbb{C}}_{m_0+\nu_1}(U) &= L^{\mathbb{C}}_a(U) + \frac{1}{2}(m_0+\nu_1-a)\|U\|_2^2 \\ &\geq E(a) - \frac{1}{2}(\nu_0-\nu_1)\overline{M} \\ &\geq E(m_0+\nu_1) - \frac{1}{2}(\nu_0-\nu_1)\overline{M} \end{aligned}$$

and using (2.25), it follows that

$$\inf_{U \in \bigcup_{a \in [m_0+\nu_1, m_0+\nu_0]} S^{\mathbb{C}}_{a,\ell_0}} L^{\mathbb{C}}_{m_0+\nu_1}(U) > E(m_0). \tag{2.28}$$

Choosing $\delta_1 > 0$ small enough, (2.26) follows from (2.27) and (2.28).

Now observe that, since elements in $\widehat{S}$ have uniform exponential decays,

$$|I_\varepsilon(e^{iA(\varepsilon p)(x-p)}U(x-p)) - L^{\mathbb{C}}_{V(\varepsilon p)}(U)| \to 0 \quad \text{as } \varepsilon \to 0$$

uniformly in $U \in \widehat{S}$, $\varepsilon p \in \overline{\Omega}$. Thus, by (2.26), for $U \in \widehat{S}$, $\varepsilon p \in \Omega([\nu_1, \nu_0])$

$$\begin{aligned} I_\varepsilon(e^{iA(\varepsilon p)(x-p)}U(x-p)) &= L^{\mathbb{C}}_{V(\varepsilon p)}(U) + o(1) \geq L^{\mathbb{C}}_{m_0+\nu_1}(U) + o(1) \\ &\geq E(m_0) + 2\delta_1 \qquad \text{for } \varepsilon > 0 \text{ small}. \end{aligned} \tag{2.29}$$

If we suppose that $u(x) = e^{iA(\varepsilon p)(x-p)}U(x-p) + \varphi(x) \in \mathcal{S}_\varepsilon(r_0)$ satisfies $\varepsilon\Upsilon_\varepsilon(u) \in \Omega([\nu_1, \nu_0])$, then by Lemma 2.6, εp belongs to a $2\varepsilon R_0$-neighborhood of $\Omega([\nu_1, \nu_0])$. Thus by (2.29) it follows that

$$I_\varepsilon(e^{iA(\varepsilon p)(x-p)}U(x-p)) \geq E(m_0) + \frac{3}{2}\delta_1 \quad \text{for } \varepsilon > 0 \text{ small}.$$

Finally we observe that I'_ε is bounded on bounded sets uniformly in $\varepsilon \in (0,1]$ and that by the compactness of $\widehat{S}$, $\{e^{iA(\varepsilon p)(x-p)}U(x-p);\ U \in \widehat{S},\ \varepsilon p \in \overline{\Omega}\}$ is bounded in H_ε. Thus choosing $r_1 \in (0, r_0)$ small, if $u(x) = e^{iA(\varepsilon p)(x-p)}U(x-p) + \varphi(x) \in \mathcal{S}_\varepsilon(r_1)$, we have

$$I_\varepsilon(e^{iA(\varepsilon p)(x-p)}U(x-p)+\varphi(x)) \geq I_\varepsilon(e^{iA(\varepsilon p)(x-p)}U(x-p)) - \frac{1}{2}\delta_1 \geq E(m_0)+\delta_1.$$

Thus, the conclusion of lemma holds. □

3 A penalization on the modulus

For technical reasons, we introduce a penalized functional J_ε following [10]. Without restriction we can assume that $\partial\Omega$ is smooth and for $h > 0$ we set

$$\Omega_h = \{x \in \mathbb{R}^N \setminus \Omega;\ \mathrm{dist}\,(x, \partial\Omega) < h\} \cup \Omega.$$

We choose a small $h_0 > 0$ such that

$$V(x) > m_0 \quad \text{for all } x \in \overline{\Omega_{2h_0} \setminus \Omega}.$$

Let

$$Q_\varepsilon(u) = \left(\varepsilon^{-2}\|u\|^2_{L^2(\mathbb{R}^N \setminus (\Omega_{2h_0}/\varepsilon))} - 1\right)_+^{\frac{p+1}{2}}$$

and

$$J_\varepsilon(u) = I_\varepsilon(u) + Q_\varepsilon(u).$$

Observe that J_ε is $\mathbb{S}^1$-invariant and we say that two critical points of J_ε are geometrically distinct if their $\mathbb{S}^1$-orbits are different. In Proposition 3.2 we prove that a $\mathbb{S}^1$ critical orbit of J_ε is also a $\mathbb{S}^1$ critical orbit of I_ε for ε small enough. Note that the penalization term Q_ε forces the concentration of the modula to occur on Ω. A motivation to introduce J_ε is that it satisfies a useful estimate from below given in Lemma 3.4.

Now we define

$$\widehat{\rho}_\varepsilon(u) = \inf\{\|u - e^{iA(\varepsilon p)(x-p)}U(x-p)\|_\varepsilon;\ p \in \mathbb{R}^N, \varepsilon p \in \overline{\Omega},\ U \in \widehat{S}\} : \mathcal{S}_\varepsilon(r_0) \to \mathbb{R}.$$

In the following proposition we derive a crucial uniform estimate of $\|J'_\varepsilon\|_{(H_\varepsilon)^*}$ in an annular neighborhood of a set of expected solutions.

Proposition 3.1. *There exists $r_2 \in (0, r_1)$ with the following property: for any $0 < \rho_1 < \rho_0 \le r_2$, there exists $\delta_2 = \delta_2(\rho_0, \rho_1) > 0$ such that for $\varepsilon > 0$ small*

$$\|J'_\varepsilon(u)\|_{(H_\varepsilon)^*} \ge \delta_2$$

for all $u \in \mathcal{S}_\varepsilon(r_2)$ with $J_\varepsilon(u) \le E(m_0 + \nu_1)$ and $(\widehat{\rho}_\varepsilon(u), \varepsilon\Upsilon_\varepsilon(u)) \in ([0, \rho_0] \times \Omega([0, \nu_0])) \setminus ([0, \rho_1] \times \Omega([0, \nu_1]))$.

Proof. By (f1)–(f3), for any $a > 0$ there exists $C_a > 0$ such that

$$|f(\xi^2)| \le a + C_a|\xi|^{p-1} \quad \text{for all } \xi \in \mathbb{R}. \tag{3.1}$$

We fix a $a_0 \in (0, \frac{1}{2}\underline{V})$ and compute

$$\begin{aligned}
I'_\varepsilon(u)u &= \int_{\mathbb{R}^N} |(\frac{1}{i}\nabla - A(\varepsilon x))u|^2\,dx + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^2\,dx - \int_{\mathbb{R}^N} f(|u|^2)|u|^2\,dx \\
&\ge \int_{\mathbb{R}^N} |(\frac{1}{i}\nabla - A(\varepsilon x))u|^2\,dx + \underline{V}\|u\|_2^2 - a_0\|u\|_2^2 - C_{a_0}\|u\|_{p+1}^{p+1} \\
&\ge \int_{\mathbb{R}^N} |(\frac{1}{i}\nabla - A(\varepsilon x))u|^2\,dx + \frac{1}{2}\underline{V}\|u\|_2^2 - C_{a_0}\|u\|_{p+1}^{p+1}.
\end{aligned}$$

Now choosing $r_2 > 0$ small enough there exists $c > 0$ such that

$$\int_{\mathbb{R}^N} |(\frac{1}{i}\nabla - A(\varepsilon x))u|^2\, dx + \frac{1}{2}\underline{V}\|u\|_2^2 - 2^p C_{a_0}\|u\|_{p+1}^{p+1} \geq c\|u\|_\varepsilon^2 \quad \text{for all} \quad \|u\|_\varepsilon \leq 2r_2. \tag{3.2}$$

(For a technical reason, especially to get (3.23) later, we add "2^p" in front of C_{a_0}.) In particular, we have

$$I'_\varepsilon(u)u \geq c\|u\|_\varepsilon^2 \quad \text{for all} \quad \|u\|_\varepsilon \leq 2r_2. \tag{3.3}$$

Now we set

$$n_\varepsilon = \left[\frac{h_0}{\varepsilon}\right] - 1$$

and for each $i = 1, 2, \cdots, n_\varepsilon$ we fix a function $\varphi_{\varepsilon,i} \in C_0^\infty(\Omega)$ such that

$$\varphi_{\varepsilon,i}(x) = \begin{cases} 1 & \text{if } x \in \Omega_{\varepsilon,i}, \\ 0 & \text{if } x \notin \Omega_{\varepsilon,i+1}, \end{cases}$$
$$\varphi_{\varepsilon,i}(x) \in [0,1], \ |\varphi'_{\varepsilon,i}(x)| \leq 2 \quad \text{for all } x \in \Omega.$$

Here we denote for $\varepsilon > 0$ and $h \in (0, 2h_0/\varepsilon]$

$$\begin{aligned} \Omega_{\varepsilon,h} &= (\Omega_{\varepsilon h})/\varepsilon \\ &= \{x \in \mathbb{R}^N \setminus (\Omega/\varepsilon); \operatorname{dist}(x, (\partial\Omega)/\varepsilon) < h\} \cup (\Omega/\varepsilon). \end{aligned}$$

Now suppose that a sequence $(u_\varepsilon) \subset \mathcal{S}_\varepsilon(r_2)$ satisfies for $0 < \rho_1 < \rho_0 < r_2$

$$J_\varepsilon(u_\varepsilon) \leq E(m_0 + \nu_1), \tag{3.4}$$
$$\widehat{\rho}_\varepsilon(u_\varepsilon) \in [0, \rho_0], \tag{3.5}$$
$$\varepsilon\Upsilon_\varepsilon(u_\varepsilon) \in \Omega([0, \nu_0]), \tag{3.6}$$
$$\|J'_\varepsilon(u_\varepsilon)\|_{(H_\varepsilon)^*} \to 0. \tag{3.7}$$

We shall prove, in several steps, that for $\varepsilon > 0$ small

$$\widehat{\rho}_\varepsilon(u_\varepsilon) \in [0, \rho_1] \quad \text{and} \quad \varepsilon\Upsilon_\varepsilon(u_\varepsilon) \in \Omega([0, \nu_1]), \tag{3.8}$$

from which the conclusion of Proposition 3.1 follows.

Step 1: There exists a $i_\varepsilon \in \{1, 2, \cdots, n_\varepsilon\}$ such that

$$\|u_\varepsilon\|^2_{H_\varepsilon(\Omega_{\varepsilon,i_\varepsilon+1} \setminus \Omega_{\varepsilon,i_\varepsilon})} \leq \frac{4r_2^2}{n_\varepsilon}. \tag{3.9}$$

Here we use notation:

$$\|u\|^2_{H_\varepsilon(K)} = \int_K \left|\left(\frac{1}{i}\nabla - A_\varepsilon\right)u\right|^2 + |u|^2\, dx$$

for $u \in H_\varepsilon$ and $K \subset \mathbb{R}^N$.

Indeed we can write $u_\varepsilon = e^{iA(\varepsilon p_\varepsilon)(x-p_\varepsilon)}U_\varepsilon(x-p_\varepsilon)+\varphi_\varepsilon(x)$ with $\|\varphi_\varepsilon\|_\varepsilon \le r_2$ and by (3.6) and the uniform exponential decay of $\widehat{S}$, we have

$$\|u_\varepsilon\|_{H_\varepsilon(\mathbb{R}^N\setminus(\Omega/\varepsilon))} \le \|e^{iA(\varepsilon p_\varepsilon)(x-p_\varepsilon)}U_\varepsilon(x-p_\varepsilon)\|_{H_\varepsilon(\mathbb{R}^N\setminus(\Omega/\varepsilon))}+\|\varphi_\varepsilon\|_{H_\varepsilon(\mathbb{R}^N\setminus(\Omega/\varepsilon))} \le 2r_2$$

for $\varepsilon>0$ small. Thus

$$\sum_{i=1}^{n_\varepsilon}\|u_\varepsilon\|^2_{H_\varepsilon(\Omega_{\varepsilon,i+1}\setminus\Omega_{\varepsilon,i})} \le \|u_\varepsilon\|^2_{H_\varepsilon(\Omega_{\varepsilon,h_0/\varepsilon}\setminus(\Omega/\varepsilon))} \le 4r_2^2$$

and there exists $i_\varepsilon \in \{1,2,\cdots,n_\varepsilon\}$ satisfying (3.9).

Step 2: For the i_ε obtained in Step 1, we set

$$u_\varepsilon^{(1)}(x)=\varphi_{\varepsilon,i_\varepsilon}(x)u_\varepsilon(x) \quad \textit{and} \quad u_\varepsilon^{(2)}(x)=u_\varepsilon(x)-u_\varepsilon^{(1)}(x).$$

Then we have, as $\varepsilon\to 0$,

$$I_\varepsilon(u_\varepsilon^{(1)}) = J_\varepsilon(u_\varepsilon)+o(1), \tag{3.10}$$
$$\|I'_\varepsilon(u_\varepsilon^{(1)})\|_{(H_\varepsilon)^*}\to 0, \tag{3.11}$$
$$\|u_\varepsilon^{(2)}\|_\varepsilon \to 0, \tag{3.12}$$
$$Q_\varepsilon(u_\varepsilon^{(2)})\to 0. \tag{3.13}$$

Observe that

$$I_\varepsilon(u_\varepsilon) = I_\varepsilon(u_\varepsilon^{(1)})+I_\varepsilon(u_\varepsilon^{(2)})+o(1). \tag{3.14}$$

Indeed, by (3.9)

$$\begin{aligned} & I_\varepsilon(u_\varepsilon)-(I_\varepsilon(u_\varepsilon^{(1)})+I_\varepsilon(u_\varepsilon^{(2)})) \\ = \;& Re\int_{\Omega_{\varepsilon,i_\varepsilon+1}\setminus\Omega_{\varepsilon,i_\varepsilon}} (\frac{1}{i}\nabla - A(\varepsilon x))(\varphi_{\varepsilon,i_\varepsilon}u_\varepsilon)\overline{\left(\frac{1}{i}\nabla - A(\varepsilon x)\right)((1-\varphi_{\varepsilon,i_\varepsilon})u_\varepsilon)} \\ & +V(\varepsilon x)\varphi_{\varepsilon,i_\varepsilon}(1-\varphi_{\varepsilon,i_\varepsilon})|u_\varepsilon|^2\,dx \\ & -\frac{1}{2}\int_{\Omega_{\varepsilon,i_\varepsilon+1}\setminus\Omega_{\varepsilon,i_\varepsilon}} F(|u_\varepsilon|^2)-F(|u_\varepsilon^{(1)}|^2)-F(|u_\varepsilon^{(2)}|^2)\,dx \\ \to \;& 0 \qquad \text{as } \varepsilon\to 0. \end{aligned}$$

Thus

$$J_\varepsilon(u_\varepsilon) = I_\varepsilon(u_\varepsilon^{(1)})+I_\varepsilon(u_\varepsilon^{(2)})+Q_\varepsilon(u_\varepsilon^{(2)})+o(1). \tag{3.15}$$

We can also see that

$$\|I'_\varepsilon(u_\varepsilon)-I'_\varepsilon(u_\varepsilon^{(1)})-I'_\varepsilon(u_\varepsilon^{(2)})\|_{(H_\varepsilon)^*}\to 0 \quad \text{as } \varepsilon\to 0. \tag{3.16}$$

In a similar way, it follows from (3.7) that, since $\|u_\varepsilon^{(2)}\|_\varepsilon$ is bounded, that

$$I'_\varepsilon(u_\varepsilon^{(2)})u_\varepsilon^{(2)}+Q'_\varepsilon(u_\varepsilon^{(2)})u_\varepsilon^{(2)} = J'_\varepsilon(u_\varepsilon)u_\varepsilon^{(2)}+o(1)=o(1). \tag{3.17}$$

We note that $\|u_\varepsilon^{(2)}\|_\varepsilon \le 2r_2$ and $(p+1)Q_\varepsilon(u) \le Q'_\varepsilon(u)u$ for all $u \in H_\varepsilon$. Thus by (3.3)

$$c\|u_\varepsilon^{(2)}\|_\varepsilon^2 + (p+1)Q_\varepsilon(u_\varepsilon^{(2)}) \to 0 \quad \text{as } \varepsilon \to 0,$$

which implies (3.12) and (3.13). Now (3.12) implies that $I_\varepsilon(u_\varepsilon^{(2)}) \to 0$ and thus (3.10) follows from (3.15).

Finally we show (3.11). We choose a function $\widetilde{\varphi} \in C_0^\infty(\mathbb{R}^N)$ such that

$$\widetilde{\varphi}(x) = \begin{cases} 1 & \text{for } x \in \Omega_{h_0}, \\ 0 & \text{for } x \in \mathbb{R}^N \setminus \Omega_{2h_0}. \end{cases}$$

Then we have, for all $w \in H_\varepsilon$,

$$\begin{aligned} I'_\varepsilon(u_\varepsilon^{(1)})w &= I'_\varepsilon(u_\varepsilon^{(1)})(\widetilde{\varphi}(\varepsilon x)w) \\ &= I'_\varepsilon(u_\varepsilon)(\widetilde{\varphi}(\varepsilon x)w) - (I'_\varepsilon(u_\varepsilon) - I'_\varepsilon(u_\varepsilon^{(1)}))(\widetilde{\varphi}(\varepsilon x)w) \\ &= J'_\varepsilon(u_\varepsilon)(\widetilde{\varphi}(\varepsilon x)w) - (I'_\varepsilon(u_\varepsilon) - I'_\varepsilon(u_\varepsilon^{(1)}))(\widetilde{\varphi}(\varepsilon x)w) \end{aligned}$$

and it follows that

$$|I'_\varepsilon(u_\varepsilon^{(1)})w| \le \|J'_\varepsilon(u_\varepsilon)\|_{(H_\varepsilon)^*}\|\widetilde{\varphi}(\varepsilon x)w\|_\varepsilon + \|I'_\varepsilon(u_\varepsilon) - I'_\varepsilon(u_\varepsilon^{(1)})\|_{(H_\varepsilon)^*}\|\widetilde{\varphi}(\varepsilon x)w\|_\varepsilon.$$

We note that by (3.12) and (3.16), $\|I'_\varepsilon(u_\varepsilon) - I'_\varepsilon(u_\varepsilon^{(1)})\|_{(H_\varepsilon)^*} \to 0$. Therefore, by (3.7), $\|I'_\varepsilon(u_\varepsilon^{(1)})\|_{(H_\varepsilon)^*} \to 0$, that is (3.11) holds true.

Step 3: After extracting a subsequence — still we denoted by ε —, there exist a sequence $(\tilde{p}_\varepsilon) \subset \mathbb{R}^N$ and $\tilde{U} \in \widehat{S}$ such that

$$\begin{aligned} &\varepsilon\tilde{p}_\varepsilon \to \tilde{p}_0 \quad \text{for some } \tilde{p}_0 \in \Omega([0,\nu_1]), && (3.18)\\ &\|u_\varepsilon^{(1)} - e^{iA(\varepsilon\tilde{p}_\varepsilon)(x-\tilde{p}_\varepsilon)}\tilde{U}(x-\tilde{p}_\varepsilon)\|_\varepsilon \to 0, && (3.19)\\ &I_\varepsilon(u_\varepsilon^{(1)}) \to L_{V(\tilde{p}_0)}(\tilde{U}) \quad \text{as } \varepsilon \to 0. && (3.20) \end{aligned}$$

Let $q_\varepsilon = \Upsilon_\varepsilon(u_\varepsilon)$. We may assume that

$$e^{-iA(\varepsilon q_\varepsilon)(x+q_\varepsilon)}u_\varepsilon^{(1)}(x+q_\varepsilon) \rightharpoonup U(x) \quad \text{weakly in } H^1(\mathbb{R}^N) \tag{3.21}$$

for some $U \in H^1(\mathbb{R}^N) \setminus \{0\}$ and also that $\varepsilon q_\varepsilon \to q_0 \in \overline{\Omega}$. In fact, noting $A_\varepsilon(x)$ is uniformly bounded on $\operatorname{supp} u_\varepsilon^{(1)} \subset \Omega_{\varepsilon,n_\varepsilon+1} \subset \Omega_{h_0}/\varepsilon$, boundedness of $\|u_\varepsilon^{(1)}\|_\varepsilon$ implies boundedness of $\|u_\varepsilon^{(1)}\|_{H^1}$. Thus, $e^{-iA(\varepsilon q_\varepsilon)(x+q_\varepsilon)}u_\varepsilon^{(1)}(x+q_\varepsilon)$ is also bounded in $H^1(\mathbb{R}^N)$. Taking a subsequence if necessary, we have (3.21).

From the definition of Υ_ε and (3.11), it follows that $(L^{\mathbb{C}}_{V(q_0)})'(U) = 0$ and $U \ne 0$. In particular, $U(x)$ decays exponentially as $|x| \to \infty$. Setting

$$w_\varepsilon(x) = u_\varepsilon^{(1)}(x+q_\varepsilon) - e^{iA(\varepsilon q_\varepsilon)\,x}U(x),$$

we have w_ε is bounded in $H^1(\mathbb{R}^N)$ and $e^{-iA(\varepsilon q_\varepsilon)(x+q_\varepsilon)}w_\varepsilon \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$ and thus $w_\varepsilon \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$. We shall prove that $\|w_\varepsilon(x-q_\varepsilon)\|_\varepsilon \to 0$.

We have from the exponential decay of $U(x)$ that

$$
\begin{aligned}
& I_\varepsilon'(u_\varepsilon^{(1)})w_\varepsilon(x-q_\varepsilon) \\
= \ & I_\varepsilon'(e^{iA(\varepsilon q_\varepsilon)(x-q_\varepsilon)}U(x-q_\varepsilon)+w_\varepsilon(x-q_\varepsilon))w_\varepsilon(x-q_\varepsilon) \\
= \ & \mathrm{Re}\int_{\mathbb{R}^N}\left(\frac{1}{i}\nabla - A(\varepsilon x+\varepsilon q_\varepsilon)\right)(e^{iA(\varepsilon q_\varepsilon)\,x}U+w_\varepsilon)\overline{\left(\frac{1}{i}\nabla - A(\varepsilon x+\varepsilon q_\varepsilon)\right)w_\varepsilon}\,dx \\
& +\mathrm{Re}\int_{\mathbb{R}^N}V(\varepsilon x+\varepsilon q_\varepsilon)(e^{iA(\varepsilon q_\varepsilon)\,x}U+w_\varepsilon)\overline{w_\varepsilon}\,dx \\
& -\mathrm{Re}\int_{\mathbb{R}^N}f(|e^{iA(\varepsilon q_\varepsilon)x}U+w_\varepsilon|^2)(e^{iA(\varepsilon q_\varepsilon)x}U+w_\varepsilon)\overline{w_\varepsilon}\,dx \\
= \ & \int_{\mathbb{R}^N}|\left(\frac{1}{i}\nabla - A(\varepsilon x+\varepsilon q_\varepsilon)\right)w_\varepsilon|^2+V(\varepsilon x+\varepsilon q_\varepsilon)|w_\varepsilon|^2\,dx \\
& +\mathrm{Re}\int_{\mathbb{R}^N}\left(\frac{1}{i}\nabla - A(\varepsilon x+\varepsilon q_\varepsilon)\right)(e^{iA(\varepsilon q_\varepsilon)x}U)\overline{\left(\frac{1}{i}\nabla - A(\varepsilon x+\varepsilon q_\varepsilon)\right)w_\varepsilon}\,dx \\
& +\mathrm{Re}\int_{\mathbb{R}^N}V(\varepsilon x+\varepsilon q_\varepsilon)e^{iA(\varepsilon q_\varepsilon)x}U\overline{w_\varepsilon}\,dx \\
& -\mathrm{Re}\int_{\mathbb{R}^N}f(|e^{iA(\varepsilon q_\varepsilon)x}U+w_\varepsilon|^2)(e^{iA(\varepsilon q_\varepsilon)x}U+w_\varepsilon)\overline{w_\varepsilon}\,dx \\
= \ & \int_{\mathbb{R}^N}|\left(\frac{1}{i}\nabla - A(\varepsilon x+\varepsilon q_\varepsilon)\right)w_\varepsilon|^2+V(\varepsilon x+\varepsilon q_\varepsilon)|w_\varepsilon|^2\,dx \\
& +\mathrm{Re}\int_{\mathbb{R}^N}\left(\frac{1}{i}\nabla - A(\varepsilon q_\varepsilon)\right)(e^{iA(\varepsilon q_\varepsilon)x}U)\overline{\left(\frac{1}{i}\nabla - A(\varepsilon q_\varepsilon)\right)w_\varepsilon}\,dx \\
& +\mathrm{Re}\int_{\mathbb{R}^N}V(\varepsilon q_\varepsilon)e^{iA(\varepsilon q_\varepsilon)x}U\overline{w_\varepsilon}\,dx+o(1) \\
& -\mathrm{Re}\int_{\mathbb{R}^N}f(|e^{iA(\varepsilon q_\varepsilon)x}U+w_\varepsilon|^2)(e^{iA(\varepsilon q_\varepsilon)x}U+w_\varepsilon)\overline{w_\varepsilon}\,dx \\
= \ & \int_{\mathbb{R}^N}|\left(\frac{1}{i}\nabla - A(\varepsilon x+\varepsilon q_\varepsilon)\right)w_\varepsilon|^2+V(\varepsilon x+\varepsilon q_\varepsilon)|w_\varepsilon|^2\,dx \\
& +(L^{\mathbb{C}}_{V(\varepsilon q_\varepsilon)})'(U)(e^{-iA(\varepsilon q_\varepsilon)x}w_\varepsilon)+\mathrm{Re}\int_{\mathbb{R}^N}f(|U|^2)U\overline{e^{-iA(\varepsilon q_\varepsilon)x}w_\varepsilon}\,dx \\
& -\mathrm{Re}\int_{\mathbb{R}^N}f(|e^{iA(\varepsilon q_\varepsilon)x}U+w_\varepsilon|^2)(e^{iA(\varepsilon q_\varepsilon)x}U+w_\varepsilon)\overline{w_\varepsilon}\,dx+o(1) \\
= \ & \int_{\mathbb{R}^N}|\left(\frac{1}{i}\nabla - A(\varepsilon x+\varepsilon q_\varepsilon)\right)w_\varepsilon|^2+V(\varepsilon x+\varepsilon q_\varepsilon)w_\varepsilon^2\,dx \\
& +(L^{\mathbb{C}}_{V(\varepsilon q_\varepsilon)})'(U)(e^{-iA(\varepsilon q_\varepsilon)x}w_\varepsilon)+(I)-(II)+o(1). \qquad (3.22)
\end{aligned}
$$

Since $(L^{\mathbb{C}}_{V(\varepsilon q_\varepsilon)})'(U)\to(L^{\mathbb{C}}_{V(p_0)})'(U)=0$, we have

$$(L^{\mathbb{C}}_{V(\varepsilon q_\varepsilon)})'(U)(e^{-iA(\varepsilon q_\varepsilon)x}w_\varepsilon)\to 0.$$

Now, by (3.1),

$$\begin{aligned}
|(I)|+|(II)| &\leq \int_{\mathbb{R}^N} (a_0(|U|+|e^{iA(\varepsilon q_\varepsilon)x}U+w_\varepsilon|)+C_{a_0}(|U|^p+|e^{iA(\varepsilon q_\varepsilon)x}U+w_\varepsilon|^p))|w_\varepsilon|\,dx \\
&\leq \int_{\mathbb{R}^N} a_0|w_\varepsilon|^2+2^pC_{a_0}|w_\varepsilon|^{p+1}+(2a_0|U|+(1+2^p)C_{a_0}|U|^p)|w_\varepsilon|\,dx \\
&\leq \int_{\mathbb{R}^N} a_0|w_\varepsilon|^2+2^pC_{a_0}|w_\varepsilon|^{p+1}\,dx+o(1).
\end{aligned}$$

Here we used the fact that $w_\varepsilon \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$. Thus, by (3.22) and (3.11), we have

$$\int_{\mathbb{R}^N} |\left(\frac{1}{i}\nabla - A(\varepsilon x+\varepsilon q_\varepsilon)\right) w_\varepsilon|^2+\underline{V}\|w_\varepsilon\|_2^2 \leq a_0\|w_\varepsilon(x-q_\varepsilon)\|_\varepsilon^2+2^pC_{a_0}\|w_\varepsilon\|_{p+1}^{p+1}+o(1)$$

from which we deduce, using (3.2), that

$$\|w_\varepsilon(x-q_\varepsilon)\|_\varepsilon \to 0. \tag{3.23}$$

At this point we have obtained (3.19), (3.20) where $\tilde{p}_\varepsilon$, $\tilde{p}_0$ and $\tilde{U}$ are replaced with q_ε, q_0 and U. Since

$$I_\varepsilon(u_\varepsilon) = I_\varepsilon(u_\varepsilon^{(1)})+o(1) = J_\varepsilon(u_\varepsilon)+o(1) \leq E(m_0+\nu_1)+o(1) \tag{3.24}$$

implies

$$E(V(q_0)) \leq L_{V(q_0)}(U) \leq E(m_0+\nu_1),$$

we have $\tilde{p}_0 = q_0 \in \Omega([0,\nu_1])$ and U belongs to $S_{V(\tilde{p}_0)} \subset \widehat{S}$ after a suitable shift, that is, $U(x) := \widetilde{U}(x+y_0) \in \widehat{S}$ for some $y_0 \in \mathbb{R}^N$. Setting $\tilde{p}_\varepsilon = q_\varepsilon + y_0$, we get (3.18)–(3.20).

Step 4: Conclusion

In Steps 1–3, we have shown that a sequence $(u_n) \subset \mathcal{S}_\varepsilon(r_2)$ with (3.4)–(3.7) satisfies, up to a subsequence, and for some $U \in \widehat{S}$ (3.19)–(3.20) with $\tilde{p}_\varepsilon = \Upsilon_\varepsilon(u_\varepsilon)+y_0$. This implies that

$$\begin{gathered}
\varepsilon\Upsilon_\varepsilon(u_\varepsilon) \to \tilde{p}_0 \in \Omega([0,\nu_1]), \\
\|u_\varepsilon(x) - e^{iA(\varepsilon\tilde{p}_\varepsilon)\,(x-\tilde{p}_\varepsilon)}U(x-\tilde{p}_\varepsilon)\|_\varepsilon \to 0.
\end{gathered}$$

In particular since $\widehat{\rho}_\varepsilon(u_\varepsilon) \to 0$ as $\varepsilon \to 0$, we have $\widehat{\rho}_\varepsilon(u_\varepsilon) \in [0,\rho_1]$ and (3.8) holds. This ends the proof of the Proposition. □

Proposition 3.2. *There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0,\varepsilon_0]$ if $u_\varepsilon \in \mathcal{S}_\varepsilon(r_2)$ satisfies*

$$J'_\varepsilon(u_\varepsilon) = 0, \tag{3.25}$$

$$J_\varepsilon(u_\varepsilon) \leq E(m_0+\nu_1), \tag{3.26}$$

$$\varepsilon\Upsilon_\varepsilon(u_\varepsilon) \in \Omega([0,\nu_0]), \tag{3.27}$$

then

$$Q_\varepsilon(u_\varepsilon) = 0 \quad \textit{and} \quad I'_\varepsilon(u_\varepsilon) = 0. \tag{3.28}$$

That is, u_ε is a solution of (2.1).

Proof. Suppose that u_ε satisfies (3.25)–(3.27). Since u_ε satisfies (3.25) we have

$$\left(\frac{1}{i}\nabla - A_\varepsilon\right)^2 u_\varepsilon + \Big(V_\varepsilon + (p+1)(\varepsilon^{-2}\|u_\varepsilon\|^2_{L^2(\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon))} - 1)_+^{\frac{p-1}{2}} \\ \times \varepsilon^{-2}\chi_{\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon)}(x)\Big)u_\varepsilon = f(|u_\varepsilon|^2)u_\varepsilon, \tag{3.29}$$

where $\chi_{\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon))}(x)$ is the characteristic function of the set $\mathbb{R}^N \setminus (\Omega_{2h_0}/\varepsilon)$.

Clearly u_ε satisfies (3.4)–(3.7) and thus, by the proof of Proposition 3.1, we have

$$\|u_\varepsilon\|_{H_\varepsilon(\mathbb{R}^N\setminus(\Omega_{h_0}/\varepsilon))} \le \|u_\varepsilon^{(2)}\|_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0.$$

From Moser's iteration scheme, it follows that

$$\|u_\varepsilon\|_{L^\infty(\mathbb{R}^N\setminus(\Omega_{\frac{3}{2}h_0}/\varepsilon))} \to 0 \quad \text{as } \varepsilon \to 0$$

and using a comparison principle, we deduce that for some c, $c' > 0$

$$|u_\varepsilon(x)| \le c' \exp(-c\,\mathrm{dist}\,(x, \Omega_{\frac{3}{2}h_0}/\varepsilon)).$$

In particular then

$$\|u_\varepsilon\|_{L^2(\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon))} < \varepsilon \quad \text{for} \quad \varepsilon > 0 \text{ small}$$

and we have (3.28). □

To find critical points of J_ε, we need the following.

Proposition 3.3. *For any fixed $\varepsilon > 0$, the Palais-Smale condition holds for J_ε in $\{u \in \mathcal{S}_\varepsilon(r_2);\ \varepsilon\Upsilon_\varepsilon(u) \in \Omega([0,\nu_0])\}$. That is, if a sequence $(u_j) \subset H_\varepsilon$ satisfies for some $c > 0$*

$$\begin{aligned} &u_j \in \mathcal{S}_\varepsilon(r_2), \\ &\varepsilon\Upsilon_\varepsilon(u_j) \in \Omega([0,\nu_0]), \\ &\|J'_\varepsilon(u_j)\|_{(H_\varepsilon)^*} \to 0, \\ &J_\varepsilon(u_j) \to c \quad \textit{as } j \to \infty, \end{aligned}$$

then (u_j) has a strongly convergent subsequence in H_ε.

Proof. Since $\mathcal{S}_\varepsilon(r_2)$ is bounded in H_ε, after extracting a subsequence if necessary, we may assume $u_j \rightharpoonup u_0$ weakly in H_ε for some $u_0 \in H_\varepsilon$. We will show that $u_j \to u_0$ strongly in H_ε. Denoting $B_R = \{x \in \mathbb{R}^N;\, |x| < R\}$, it suffices to show that

$$\lim_{R\to\infty} \lim_{j\to\infty} \|u_j\|^2_{H_\varepsilon(\mathbb{R}^N\setminus B_R)} = 0. \tag{3.30}$$

To show (3.30) we first we note that, since $\varepsilon > 0$ is fixed, $\|u_j\|_{H_\varepsilon(\mathbb{R}^N\setminus B_L)} < 2r_2$ for a large $L > 1$. In particular, for any $n \in \mathbb{N}$

$$\sum_{i=1}^n \|u_j\|^2_{H_\varepsilon(D_i)} < 4r_2^2,$$

where $D_i = B_{L+i} \setminus B_{L+i-1}$.

Thus, for any $j \in \mathbb{N}$, there exists $i_j \in \{1, 2, \cdots, n\}$ such that

$$\|u_j\|^2_{H_\varepsilon(D_{i_j})} < \frac{4r_2^2}{n}.$$

Now we choose $\zeta_i \in C^1(\mathbb{R}, \mathbb{R})$ such that $\zeta_i(r) = 1$ for $r \le L + i - 1$, $\zeta_i(r) = 0$ for $r \ge L + i$ and $\zeta_i'(r) \in [-2, 0]$ for all $r > 0$. We set

$$\widetilde{u}_j(x) = (1 - \zeta_{i_j}(|x|))u_j(x).$$

We have, for a constant $C > 0$ independent of n, j

$$\begin{aligned}
&J_\varepsilon'(u_j)\widetilde{u}_j = I_\varepsilon'(u_j)\widetilde{u}_j + Q_\varepsilon'(u_j)\widetilde{u}_j,\\
&I_\varepsilon'(u_j)\widetilde{u}_j = I_\varepsilon'(\widetilde{u}_j)\widetilde{u}_j + \mathrm{Re}\int_{D_{i_j}} \left(\frac{1}{i}\nabla - A(\varepsilon x)\right)(\zeta_{i_j}u_j)\overline{\left(\frac{1}{i}\nabla - A(\varepsilon x)\right)((1-\zeta_{i_j})u_j)}\,dx\\
&+\int_{D_{i_j}} V(\varepsilon x)\zeta_{i_j}(1-\zeta_{i_j})|u_j|^2 + [f(|(1-\zeta_{i_j})u_j|^2)(1-\zeta_{i_j}) - f(|u_j|^2)](1-\zeta_{i_j})|u_j|^2\,dx\\
&\ge I_\varepsilon'(\widetilde{u}_j)\widetilde{u}_j - \frac{C}{n}, &(3.31)\\
&Q_\varepsilon'(u_j)\widetilde{u}_j = (p+1)\left(\varepsilon^{-2}\|u_j\|^2_{L^2(\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon))} - 1\right)_+^{\frac{p-1}{2}}\\
&\times\varepsilon^{-2}\int_{\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon))} (1-\zeta_{i_j})|u_j|^2\,dx \ge 0. &(3.32)
\end{aligned}$$

Since $J_\varepsilon'(u_j)\widetilde{u}_j \to 0$, it follows from (3.31)–(3.32) that

$$I_\varepsilon'(\widetilde{u}_j)\widetilde{u}_j \le \frac{C}{n} + o(1) \quad \text{as } j \to \infty.$$

Now recording that $\|\widetilde{u}_j\|_\varepsilon < 2r_2$ we have by (3.3) for some $C > 0$

$$\|\widetilde{u}_j\|_\varepsilon^2 \le \frac{C}{n} + o(1).$$

Thus, from the definition of $\widetilde{u}_j$, we deduce that

$$\|u_j\|^2_{H_\varepsilon(\mathbb{R}^N\setminus B_{L+n})} \le \frac{C}{n} + o(1).$$

That is, (3.30) holds and (u_j) strongly converges. $\square$

The following lemma will be useful to compute the relative category.

Lemma 3.4. *There exists $C_0 > 0$ independent of $\varepsilon > 0$ such that*

$$J_\varepsilon(u) \ge L_{m_0}(|u|) - C_0\varepsilon^2 \quad \text{for all } u \in \mathcal{S}_\varepsilon(r_1). \tag{3.33}$$

Proof.

$$\begin{aligned} J_\varepsilon(u) &\ge L_{m_0}(|u|) + \frac{1}{2}\int_{\mathbb{R}^N} (V(\varepsilon x) - m_0)|u|^2\, dx + Q_\varepsilon(u) \\ &\ge L_{m_0}(|u|) - \frac{1}{2}(m_0 - \underline{V})\|u\|^2_{L^2(\mathbb{R}^N\setminus(\Omega/\varepsilon))} + Q_\varepsilon(u). \end{aligned}$$

We distinguish the two cases: (a) $\|u\|^2_{L^2(\mathbb{R}^N\setminus(\Omega/\varepsilon))} \le 2\varepsilon^2$, (b) $\|u\|^2_{L^2(\mathbb{R}^N\setminus(\Omega/\varepsilon))} \ge 2\varepsilon^2$.

If case (a) occurs, we have

$$J_\varepsilon(u) \ge L_{m_0}(|u|) - (m_0 - \underline{V})\varepsilon^2$$

and (3.33) holds. If case (b) takes place, we have

$$Q_\varepsilon(u) \ge \left(\frac{1}{2}\varepsilon^{-2}\|u\|^2_{L^2(\mathbb{R}^N\setminus(\Omega/\varepsilon))}\right)^{\frac{p+1}{2}} \ge \frac{1}{2}\varepsilon^{-2}\|u\|^2_{L^2(\mathbb{R}^N\setminus(\Omega/\varepsilon))}$$

and thus

$$\begin{aligned} J_\varepsilon(u) &\ge L_{m_0}(|u|) + \frac{1}{2}(\varepsilon^{-2} - (m_0 - \underline{V}))\|u\|^2_{L^2(\mathbb{R}^N\setminus(\Omega/\varepsilon))} \\ &\ge L_{m_0}(|u|) \quad \text{for } \varepsilon > 0 \text{ small.} \end{aligned}$$

Therefore (3.33) also holds. $\square$

4 A $\mathbb{S}^1$-invariant neighborhood of expected solutions

In order to find critical points of the penalized functional J_ε, we need to find a $\mathbb{S}^1$-invariant neighborhood $\mathcal{N}_{\varepsilon,\delta}$ of expected solutions, which is positively invariant under a $\mathbb{S}^1$-equivariant pseudo-gradient flow.

We fix $0 < \rho_1 < \rho_0 < r_2$ and we then choose δ_1, $\delta_2 > 0$ according to Lemma 2.7 and Proposition 3.1. We set for $\delta \in (0, \min\{\frac{\delta_2}{4}(\rho_0 - \rho_1), \delta_1\})$,

$$\mathcal{N}_{\varepsilon,\delta} = \{u \in \mathcal{S}_\varepsilon(\rho_0);\ \varepsilon\Upsilon_\varepsilon(u) \in \Omega([0,\nu_0]),\ J_\varepsilon(u) \le E(m_0) + \delta - \frac{\delta_2}{2}(\widehat{\rho}_\varepsilon(u) - \rho_1)_+\}.$$

We notice that $\mathcal{N}_{\varepsilon,\delta}$ is $\mathbb{S}^1$-invariant, namely if $u \in \mathcal{N}_{\varepsilon,\delta}$ then $\gamma u \in \mathcal{N}_{\varepsilon,\delta}$ for any $\gamma \in \mathbb{S}^1$. We shall try to find $\mathbb{S}^1$-orbits of critical points of J_ε in $\mathcal{N}_{\varepsilon,\delta}$. In this aim first note that

(a) $u \in \mathcal{S}_\varepsilon(\rho_0)$ and $\varepsilon\Upsilon_\varepsilon(u) \in \Omega([\nu_1, \nu_0])$ imply, by Lemma 2.7, that

$$J_\varepsilon(u) \ge I_\varepsilon(u) \ge E(m_0) + \delta_1 > E(m_0) + \delta. \tag{4.1}$$

In particular,

$$\varepsilon\Upsilon_\varepsilon(u) \in \Omega([0,\nu_1)) \quad \text{for } u \in \mathcal{N}_{\varepsilon,\delta}.$$

(b) For $u \in \mathcal{N}_{\varepsilon,\delta}$, if $\widehat{\rho}_\varepsilon(u) = \rho_0$, i.e., $u \in \partial\mathcal{S}_\varepsilon(\rho_0)$, then by the choice of δ

$$J_\varepsilon(u) \le E(m_0) + \delta - \frac{\delta_2}{2}(\rho_0 - \rho_1) < E(m_0) - \delta. \tag{4.2}$$

4.1 A $\mathbb{S}^1$-equivariant deformation theorem

Now we consider a deformation flow defined by

$$\begin{cases} \frac{d\eta}{d\tau} = -\phi(\eta)\frac{\mathcal{V}(\eta)}{\|\mathcal{V}(\eta)\|_{H^1}}, \\ \eta(0,u) = u, \end{cases} \tag{4.3}$$

where $\mathcal{V}(u) : \{u \in H_\varepsilon;\ J'_\varepsilon(u) \neq 0\} \to H_\varepsilon$ is a locally Lipschitz continuous $\mathbb{S}^1$-equivariant, pseudo-gradient vector field satisfying

$$\|\mathcal{V}(u)\|_\varepsilon \le \|J'_\varepsilon(u)\|_{(H_\varepsilon)^*}, \quad J'_\varepsilon(u)\mathcal{V}(u) \ge \frac{1}{2}\|J'_\varepsilon(u)\|^2_{(H_\varepsilon)^*}$$

and $\phi(u) : H_\varepsilon \to [0,1]$ is a locally Lipschitz continuous function. We require that $\phi(u)$ satisfies $\phi(u) = 0$ if $J_\varepsilon(u) \notin [E(m_0) - \delta, E(m_0) + \delta]$.

Arguing as in [17, Proposition 4.1] (see also [8, Theorem 1.8]), we can derive the following deformation theorem in a neighborhood of expected solutions $\mathcal{N}_{\varepsilon,\delta}$.

Proposition 4.1. *For any $c \in (E(m_0) - \delta, E(m_0) + \delta)$ and for any $\mathbb{S}^1$-invariant neighborhood O of $\mathcal{K}_c \equiv \{u \in \mathcal{N}_{\varepsilon,\delta};\ J'_\varepsilon(u) = 0,\ J_\varepsilon(u) = c\}$ ($O = \emptyset$ if $\mathcal{K}_c = \emptyset$), there exist $d > 0$ and a $\mathbb{S}^1$-equivariant deformation $\eta(\tau, u) : [0,1] \times (\mathcal{N}_{\varepsilon,\delta} \setminus O) \to \mathcal{N}_{\varepsilon,\delta}$ such that*

(i) $\eta(0,u) = u$ *for all* u.

(ii) $\eta(\tau,u) = u$ *for all* $\tau \in [0,1]$ *if* $J_\varepsilon(u) \notin [E(m_0) - \delta, E(m_0) + \delta]$.

(iii) $J_\varepsilon(\eta(\tau,u))$ *is a non-increasing function of* τ *for all* u.

(iv) $J_\varepsilon(\eta(1,u)) \le c - d$ *for all* $u \in \mathcal{N}_{\varepsilon,\delta} \setminus O$ *satisfying* $J_\varepsilon(u) \le c + d$.

4.2 Two maps between topological pairs

Now for $c \in \mathbb{R}$, we set

$$\mathcal{N}^c_{\varepsilon,\delta} = \{u \in \mathcal{N}_{\varepsilon,\delta};\ J_\varepsilon(u) \le c\}.$$

For $\hat{\delta} > 0$ small, using relative $\mathbb{S}^1$-equivariant category, we shall estimate the change of topology between $\mathcal{N}^{E(m_0)+\hat{\delta}}_{\varepsilon,\delta}$ and $\mathcal{N}^{E(m_0)-\hat{\delta}}_{\varepsilon,\delta}$.

We recall that $K = \{x \in \Omega; V(x) = m_0\}$. For $s_0 \in (0,1)$ small we introduce two maps:

$$\begin{aligned}
&\tilde{\Phi}_\varepsilon : ([1-s_0, 1+s_0] \times K, \{1 \pm s_0\} \times K) \to (\mathcal{N}^{E(m_0)+\hat{\delta}}_{\varepsilon,\delta}, \mathcal{N}^{E(m_0)-\hat{\delta}}_{\varepsilon,\delta});\\
&\tilde{\Psi}_\varepsilon : (\mathcal{N}^{E(m_0)+\hat{\delta}}_{\varepsilon,\delta}, \mathcal{N}^{E(m_0)-\hat{\delta}}_{\varepsilon,\delta}) \to \\
&\qquad ([1-s_0, 1+s_0] \times \Omega([0,\nu_1]), ([1-s_0, 1+s_0] \setminus \{1\}) \times \Omega([0,\nu_1])).
\end{aligned}$$

Here we use notation from algebraic topology: $f : (A, B) \to (A', B')$ means $B \subset A$, $B' \subset A'$, $f : A \to A'$ is continuous and $f(B) \subset B'$.

Definition of $\tilde{\Phi}_\varepsilon$:
Fix a least energy solution $U_0 \in \widehat{S}$ of $-\Delta u + m_0 u = f(u)$ and set

$$\tilde{\Phi}_\varepsilon(s, p) = e^{iA(p)(\frac{x-p/\varepsilon}{s})} U_0(\frac{x - p/\varepsilon}{s}).$$

Let us show that $\tilde{\Phi}_\varepsilon$ is well-defined for a suitable choice of s_0 and $\hat{\delta}$ and assuming $\varepsilon > 0$ small enough.

By the exponential decay of U_0, we can find $s_0 \in (0,1)$ small such that

$$\|e^{iA(p)(\frac{x-p/\varepsilon}{s})} U_0(\frac{x - p/\varepsilon}{s}) - e^{iA(p)(x-p/\varepsilon)} U_0(x - p/\varepsilon)\|_\varepsilon < \rho_1$$

for all $p \in K$, $s \in [1-s_0, 1+s_0]$ and small $\varepsilon > 0$. Therefore, using the first property of Lemma 2.6, that is

$$|\Upsilon_\varepsilon(u) - p| \le 2R_0$$

for $u(x) = e^{iA(\varepsilon p)(x-p)} U(x-p) + \varphi(x) \in \mathcal{S}_\varepsilon(\rho_0)$, we get

$$\big|\Upsilon_\varepsilon(e^{iA(p)(\frac{x-p/\varepsilon}{s})} U_0(\frac{x - p/\varepsilon}{s})) - p/\varepsilon\big| \le 2R_0.$$

It follows that, for $p \in K$, $s \in [1-s_0, 1+s_0]$

$$\varepsilon \Upsilon_\varepsilon(e^{iA(p)(\frac{x-p/\varepsilon}{s})} U_0(\frac{x - p/\varepsilon}{s})) = p + o(1) \tag{4.4}$$

and so $\varepsilon \Upsilon_\varepsilon \tilde{\Phi}_\varepsilon(s,p) \in \Omega([0,\nu_0])$ for $\varepsilon > 0$ small enough.

Since U_0 is a least energy solution, we note that $|U_0|$ satisfies the Pohozaev identity (2.7) and thus $P_0(|U_0|) = 1$ and $P_0(|U_0(\frac{x}{s})|) = s$. Also we have, by Lemma 2.2, for $p \in K$ and $s \in [1-s_0, 1+s_0]$

$$\begin{aligned} J_\varepsilon(e^{iA(p)(\frac{x-p/\varepsilon}{s})}U_0(\frac{x-p/\varepsilon}{s})) &= L_{m_0}(|U_0(\frac{x-p/\varepsilon}{s})|) + o(1) \\ &= g(P_0(U_0(\frac{x-p/\varepsilon}{s})))E(m_0) + o(1) \\ &= g(s)E(m_0) + o(1), \end{aligned}$$

where $g : [0, \infty) \to \mathbb{R}$ is defined in (2.14). Note that g satisfies $g(t) \le 1$ for all $t > 0$ and that $g(t) = 1$ holds if and only if $t = 1$. Thus choosing $\hat{\delta} > 0$ small so that $g(1 \pm s_0)E(m_0) < E(m_0) - \hat{\delta}$, we have $\tilde{\Phi}_\varepsilon(\{1 \pm s_0\} \times K) \subset \mathcal{N}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}$ and thus $\tilde{\Phi}_\varepsilon$ is well-defined.

Definition of $\tilde{\Psi}_\varepsilon$:
We introduce the continuous function $\tilde{P}_0 : \mathcal{S}_\varepsilon(r_0) \to \mathbb{R}$ by

$$\tilde{P}_0(u) = \begin{cases} 1+s_0 & \text{if } P_0(u) \ge 1+s_0, \\ 1-s_0 & \text{if } P_0(u) \le 1-s_0, \\ P_0(u) & \text{otherwise,} \end{cases}$$

where P_0 is given in (2.13) and we define our operator $\tilde{\Psi}_\varepsilon$ by

$$\tilde{\Psi}_\varepsilon(u) = (\tilde{P}_0(u), \varepsilon\Upsilon_\varepsilon(u)) \quad \text{for } u \in \mathcal{N}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}.$$

Let us show that $\tilde{\Psi}_\varepsilon$ is well-defined for $\varepsilon > 0$ small enough. By definition $\tilde{\Psi}_\varepsilon(\mathcal{N}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}) \subset [1-s_0, 1+s_0] \times \Omega([0,\nu_1])$. Now if $u \in \mathcal{N}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}$ we have by Lemma 3.4,

$$L_{m_0}(|u|) \le J_\varepsilon(u) + C_0\varepsilon^2 \le E(m_0) - \hat{\delta} + C_0\varepsilon^2.$$

Thus, for $\varepsilon > 0$ small enough,

$$L_{m_0}(|u|) < E(m_0). \tag{4.5}$$

At this point we recall, see [30] for a proof, that $E(m_0)$ can be characterized as

$$E(m_0) = \inf\{L_{m_0}(u);\, u \neq 0,\, P_0(u) = 1\}. \tag{4.6}$$

Thus (4.5) implies that $P_0(u) \neq 1$ and $\tilde{\Psi}_\varepsilon$ is well-defined.

Now setting $\Phi_\varepsilon(s,p) := \mathbb{S}^1\tilde{\Phi}_\varepsilon(s,p)$ for each $(s,p) \in [1-s_0, 1+s_0] \times K$, it results that Φ_ε is well-defined as a map

$$\Phi_\varepsilon : ([1-s_0, 1+s_0] \times K, \{1 \pm s_0\} \times K) \to (\mathcal{N}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}/\mathbb{S}^1, \mathcal{N}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}/\mathbb{S}^1).$$

Similarly setting$\Psi_\varepsilon(\mathbb{S}^1 u) := \tilde{\Psi}_\varepsilon(u)$ for any $u \in \mathcal{N}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}$, we see that Ψ_ε is well-defined as a map

$$\begin{aligned}\Psi_\varepsilon :\ &(\mathcal{N}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}/\mathbb{S}^1, \mathcal{N}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}/\mathbb{S}^1) \to \\ &([1-s_0, 1+s_0] \times \Omega([0,\nu_1]), ([1-s_0, 1+s_0] \setminus \{1\}) \times \Omega([0,\nu_1])).\end{aligned}$$

Finally we derive the following topological lemma.

Proposition 4.2.

$$\begin{aligned}\Psi_\varepsilon \circ \Phi_\varepsilon :\ &([1-s_0, 1+s_0] \times K, \{1 \pm s_0\} \times K) \\ &\to ([1-s_0, 1+s_0] \times \Omega([0,\nu_1]), ([1-s_0, 1+s_0] \setminus \{1\}) \times \Omega([0,\nu_1]))\end{aligned}$$

is homotopic to the embedding $j(s,p) = (s,p)$. That is, there exists a continuous map

$$\eta :\ [0,1] \times [1-s_0, 1+s_0] \times K \to [1-s_0, 1+s_0] \times \Omega([0,\nu_1])$$

such that

$$\begin{aligned}&\eta(0,s,p) = (\Psi_\varepsilon \circ \Phi_\varepsilon)(s,p), \\ &\eta(1,s,p) = (s,p) \quad \textit{for all } (s,p) \in [1-s_0, 1+s_0] \times K, \\ &\eta(t,s,p) \in ([1-s_0, 1+s_0] \setminus \{1\}) \times \Omega([0,\nu_1]) \\ &\qquad \textit{for all } t \in [0,1] \textit{ and } (s,p) \in \{1 \pm s_0\} \times K.\end{aligned}$$

Proof. By the definitions of Φ_ε and Ψ_ε, we have

$$\begin{aligned}(\Psi_\varepsilon \circ \Phi_\varepsilon)(s,p) &= \Big(\tilde{P}_0(\varepsilon^{iA(p)(\frac{x-p/\varepsilon}{s})} U_0(\frac{x-p/\varepsilon}{s})), \varepsilon\Upsilon_\varepsilon(e^{iA(p)(\frac{x-p/\varepsilon}{s})} U_0(\frac{x-p/\varepsilon}{s}))\Big) \\ &= \Big(s, \varepsilon\Upsilon_\varepsilon(e^{iA(p)(\frac{x-p/\varepsilon}{s})} U_0(\frac{x-p/\varepsilon}{s}))\Big).\end{aligned}$$

We set

$$\eta(t,s,p) = \Big(s, (1-t)\varepsilon\Upsilon_\varepsilon(e^{iA(p)(\frac{x-p/\varepsilon}{s})} U_0(\frac{x-p/\varepsilon}{s})) + tp\Big).$$

Recalling (4.4), we see that for $\varepsilon > 0$ small $\eta(t,s,p)$ has the desired properties and $\Psi_\varepsilon \circ \Phi_\varepsilon$ is homotopic to the embedding j. □

5 Proof of Theorem 1.1

In order to prove our theorem, we shall need some topological tools that we now present for the reader convenience. Following [6], see also [27, 28], we define

Definition 5.1. *Let $B \subset A$ and $B' \subset A'$ be topological spaces and $f : (A, B) \to (A', B')$ be a continuous map, that is $f : A \to A'$ is continuous and $f(B) \subset B'$. The category* $\mathrm{cat}(f)$ *of f is the least integer $k \geq 0$ such that there exist open sets A_0, A_1, $\cdots$, A_k with the following properties:*

(a) $A = A_0 \cup A_1 \cup \cdots \cup A_k$.

(b) $B \subset A_0$ *and there exists a map* $h_0 : [0, 1] \times A_0 \to A'$ *such that*

$$\begin{aligned} h_0(0, x) = f(x) \quad & \text{for all } x \in A_0, \\ h_0(1, x) \in B' \quad & \text{for all } x \in A_0, \\ h_0(t, x) = f(x) \quad & \text{for all } x \in B \text{ and } t \in [0, 1]. \end{aligned}$$

(c) For $i = 1, 2, \cdots, k$, $f|_{A_i} : A_i \to A'$ *is homotopic to a constant map.*

We also introduce the cup-length of $f : (A, B) \to (A', B')$. Let H^* denote Alexander-Spanier cohomology with coefficients in the field $\mathbb{F}$. We recall that the cup product $\smile$ turns $H^*(A)$ into a ring with unit 1_A, and it turns $H^*(A, B)$ into a module over $H^*(A)$. A continuous map $f : (A, B) \to (A', B')$ induces a homomorphism $f^* : H^*(A') \to H^*(A)$ of rings as well as a homomorphism $f^* : H^*(A', B') \to H^*(A, B)$ of abelian groups. We also use notation:

$$\tilde{H}^n(A') = \begin{cases} 0 & \text{for } n = 0, \\ H^n(A') & \text{for } n > 0. \end{cases}$$

For more details on algebraic topology we refer to [35].

Definition 5.2. *For $f : (A, B) \to (A', B')$ the cup-length,* $\mathrm{cupl}(f)$ *is defined as follows; when $f^* : H^*(A', B') \to H^*(A, B)$ is not a trivial map,* $\mathrm{cupl}(f)$ *is defined as the maximal integer $k \geq 0$ such that there exist elements α_1, $\cdots$, $\alpha_k \in \tilde{H}^*(A')$ and $\beta \in H^*(A', B')$ with*

$$\begin{aligned} f^*(\alpha_1 \smile \cdots \smile \alpha_k \smile \beta) &= f^*(\alpha_1) \smile \cdots \smile f^*(\alpha_k) \smile f^*(\beta) \\ &\neq 0 \text{ in } H^*(A, B). \end{aligned}$$

When $f^* = 0 : H^*(A', B') \to H^*(A, B)$, we define $\mathrm{cupl}(f) = -1$.

We note that $\mathrm{cupl}(f) = 0$ if $f^* \neq 0 : H^*(A', B') \to H^*(A, B)$ and $\tilde{H}^*(A') = 0$.

Finally we recall

Definition 5.3. *For a set (A, B), we define the relative category* $\mathrm{cat}(A, B)$ *and the relative cup-length* $\mathrm{cupl}(A, B)$ *by*

$$\begin{aligned} \mathrm{cat}(A, B) &= \mathrm{cat}(id_{(A,B)} : (A, B) \to (A, B)), \\ \mathrm{cupl}(A, B) &= \mathrm{cupl}(id_{(A,B)} : (A, B) \to (A, B)). \end{aligned}$$

We also set

$$\mathrm{cat}(A) = \mathrm{cat}(A, \emptyset), \quad \mathrm{cupl}(A) = \mathrm{cupl}(A, \emptyset).$$

The following lemma which is due to Bartsch [5] (see [17] for a proof) is one of the keys of our proof and we make use of the continuity property of Alexander-Spanier cohomology.

Lemma 5.4. *Let $K \subset \mathbb{R}^N$ be a compact set. For a d-neighborhood $K_d = \{x \in \mathbb{R}^N; \operatorname{dist}(x, K) \leq d\}$ and $I = [0,1]$, $\partial I = \{0,1\}$, we consider the inclusion*

$$j : (I \times K, \partial I \times K) \to (I \times K_d, \partial I \times K_d)$$

defined by $j(s,x) = (s,x)$. Then for $d > 0$ small,

$$\operatorname{cupl}(j) \geq \operatorname{cupl}(K).$$

Now we have all the ingredients to give the

Proof of Theorem 1.1. Using Proposition 4.1, we can apply $\mathbb{S}^1$-invariant L.S. theory and derive that for $\varepsilon > 0$ small the number of critical $\mathbb{S}^1$-orbits of J_ε in $\mathcal{N}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}} \backslash \mathcal{N}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}$ is at least $\mathbb{S}^1$-cat$(\mathcal{N}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}, \mathcal{N}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}})$ (see [27, Theorem 4.2] and [14, Theorem 1.1]).

Since $\mathbb{S}^1$ acts freely on $H_\varepsilon \setminus \{0\}$, that is $\gamma u \neq u$ for all $u \in H_\varepsilon \setminus \{0\}$, $\gamma \in \mathbb{S}^1$, $\gamma \neq 1$, we have

$$\mathbb{S}^1 - \operatorname{cat}(\mathcal{N}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}, \mathcal{N}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}) = \operatorname{cat}(\mathcal{N}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}/\mathbb{S}^1, \mathcal{N}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}/\mathbb{S}^1).$$

Finally, using Lemma 5.4, we can argue as in the proof of Theorem 1.1 in [17] and we deduce that

$$\operatorname{cat}(\mathcal{N}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}/\mathbb{S}^1, \mathcal{N}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}/\mathbb{S}^1) \geq \operatorname{cupl}(K) + 1.$$

Thus we conclude that J_ε has at least cupl(K)+1 critical $\mathbb{S}^1$ orbits in $\mathcal{N}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}} \setminus \mathcal{N}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}$. Recalling Proposition 3.2, this completes the proof. □

Proof of Remark 1.3. From the proof of Proposition 3.1 we know that for any $\nu_0 > 0$ small enough the critical points u_ε^j, $j = 1, \ldots, \operatorname{cupl}(K) + 1$ satisfy

$$||u_\varepsilon^j(x) - e^{iA(x_j)(x-x_j)} U^j(x - x_\varepsilon^j)||_\varepsilon \to 0$$

where $\varepsilon x_\varepsilon^j = \varepsilon \Upsilon_\varepsilon(u_\varepsilon^j) + o(1) \to x_0^j \in \Omega([0, \nu_0])$ and $U^i \in \widehat{S}$. Thus $w_\varepsilon^j(x) = u_\varepsilon^j(x + x_\varepsilon^j)$ converges to $e^{iA(x_j)(x_j)} U^j \in \widehat{S}$. Now observing that these results holds for any $\nu_0 > 0$ and any $\ell_0 > E(m_0 + \nu_0)$ we deduce, considering sequences $\nu_0^n \to 0$, $\ell_0^n \to E(m_0)$ and making a diagonal process, that it is possible to assume that each w_ε^j converges to a least energy solution of

$$-\Delta U + m_0 U = f(U), \quad U > 0, \quad U \in H^1(\mathbb{R}^N, \mathbb{C}).$$

Clearly also

$$|u_\varepsilon^j(x)| \leq C exp(-c|x - x_\varepsilon^j|), \quad \text{for some } c, C > 0$$

and this ends the proof. □

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