Conservation laws with point constraints: analysis, approximation, modeling of road and pedestrian traffic

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based upon joint works with
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Classical LWR and ARZ models

LWR model

Macroscopic models of trafic

General principle of local conservation of vehicles on a road (modeled by the real line):

$$\rho_t + (v\rho)_x = 0$$

where at every point (t, x) of time-space ($t \geq 0$, $x \in \mathbb{R}$) the unknowns are (ρ, v):

ρ is density of trafic;
v is the traffic speed.

Different models are derived by specifying closure relations $\rho \mapsto v[\rho]$.

Lighthill-Whitham and Richards model (LWR)

The dependence $\rho \mapsto v$ is local , typically, $v = v(\rho) = V_{max}\left(1 - \frac{\rho}{\rho_{max}}\right)$.

Outcome: LWR writes as

$$\rho_t + f(\rho)_x = 0,$$

with $f : [0, \rho_{max}] \to \mathbb{R}^+$, $f(\rho) = v(\rho)\rho$.

The nonlinearity f is called fundamental diagram .
Typically, it is unimodal ("bell-shaped"), with $f(0) = 0 = f(\rho_{max})$.

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Framework for analysis of LWR. Main aspects of the theory.

Classical solutions: do not exist for general data; do not reflect typical nature of road traffic. Weak solutions (with shocks): non-unique.

Kruzhkov entropy solutions are usually accepted to interpret LWR :

$$\forall k \in \mathbb{R} \;\; |\rho - k|_t + Q(\rho, k)_x \leq 0 \;\; \text{in } \mathcal{D}', \quad Q(\rho, k) := \operatorname{sign}(\rho - k)(f(\rho) - f(k)).$$

Kinetic formulation: alternative characterization + additional information.

Entropy solutions $\rightsquigarrow$ order-preserving contractive semigroup on $L^1(\mathbb{R})$:

$$\rho_0 \leq \widehat{\rho}_0 \quad \Longrightarrow \quad \forall t > 0 \;\; \rho(t, \cdot) \leq \widehat{\rho}(t, \cdot),$$
$$\|\rho(t, \cdot) - \widehat{\rho}(t, \cdot)\|_{L^1} \leq \|\rho_0 - \widehat{\rho}_0\|_{L^1}.$$

Admissibility encoded:
shocks in entropy solutions of LWR increase car density ($\rho_- \leq \rho_+$)
and decrease car velocity ($v(\rho_-) \geq v(\rho_+)$) : shock = sudden braking .

Existence, Numerics :
solutions of LWR can be obtained as limits of Finite Volume approximations

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\big(F(\rho_i, \rho_{i+1}) - F(\rho_{i-1}, \rho_i)\big),$$

with consistent and monotone numerical flux functions :

$$F(k, k) = f(k) \;\; \text{for all } k, \quad F(\cdot, b) \text{ is } \nearrow, \quad F(a, \cdot) \text{ is } \searrow \;\; \text{for all } a, b.$$

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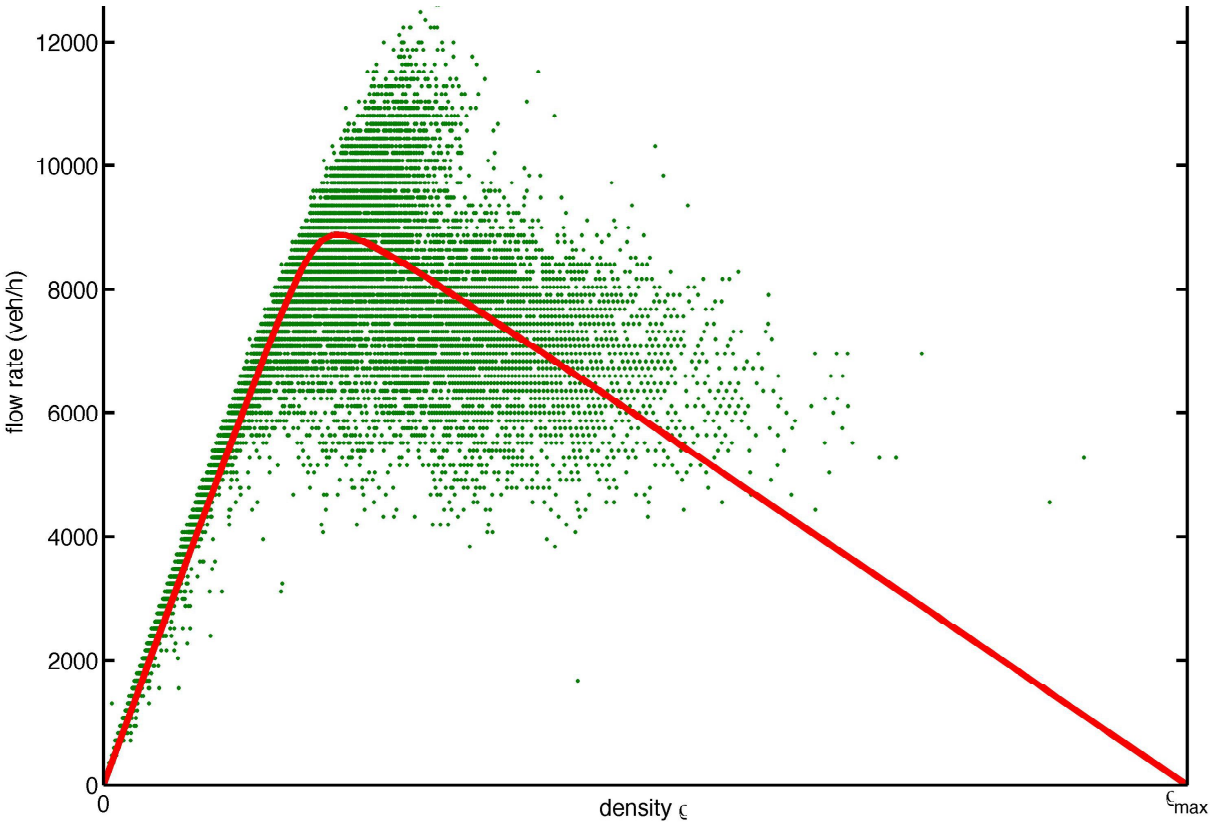
Shortcomings of LWR

LWR does not exhibit all observed kinds of traffic behavior. Moreover, it fits quite poorly experimental data.



Popular improvement of LWR: the “second-order” ARZ model obtained by enriching the closure relation $v = v[\rho]$.

ARZ model

Aw-Rascle and Zhang model (ARZ)

The dependence $\rho \mapsto v$ is governed by a PDE ,
via the relation $w = v + p(\rho)$. Here w is a "lagrangian marker" .
This means that w is merely transported along the flow: $w_t + vw_x = 0$.

Since $\rho_t + (v\rho)_x = 0$, this can be rewritten in conservative form:

$$(\rho w)_t + (v\rho w)_x = \rho(w_t + vw_x) = 0.$$

Nonlinearity p ("pressure") is of the kind $p(\rho) = \rho^\gamma$, $\gamma > 0$.

ARZ model writes as $\begin{cases} \rho_t + (v\rho)_x = 0 \\ (\rho w)_t + (v\rho w)_x = 0, \end{cases} \quad w = v + p(\rho).$

Conservative variables are $(\rho, \rho w)$;
the most convenient variables are $(v, w) \in \mathbb{R}^+ \times \mathbb{R}^+$.

Nature of ARZ:
Hyperbolic system of conservation laws (except at vacuum $\rho = 0$);
Admissibility of solutions: via Riemann solver (unstable at vacuum);
Except at vacuum, admissibility characterized via entropy inequalities.

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Colombo-Goatin model :

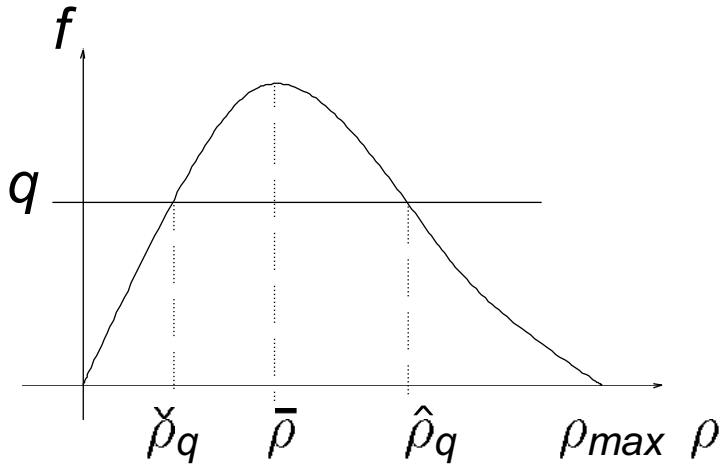
LWR with point constraint

Colombo-Goatin model : LWR with point constraint on the flux

Colombo-Goatin model: in the context of road lights, pay tolls, small-scale construction sites, one may consider the formal model

$$\begin{cases} \rho_t + f(\rho)_x = 0 & \text{(LWR)} \\ f(\,\rho(t,0^{\pm})\,) \leq q(t) & \text{(point constraint)} \end{cases}$$

where the map $t \mapsto q(t)$
(point constraint at $x = 0$, given beforehand)
prescribes the maximal possible value
of the car flow $f(\rho(t,0^+)) \equiv f(\rho(t,0^-))$.



Riemann solver at $x = 0$:
if the flow at $x = 0$ "wants to be" above q (for unconstrained LWR),
then constrained $\rho(t,\cdot)$ jumps from $\rho_- = \hat{\rho}_q$ to $\rho_+ = \check{\rho}_q$, and $f(\rho_\pm) = q$.

Colombo-Goatin model: rigorous notion of solution

Analytical framework:

- entropy inequalities adapted to the constraint via a remainder term:

$$\forall k \in [0, \rho_{max}] \quad |\rho - k|_t + q(\rho, k)_x \leq 2\big(f(k) - \min\{f(k), q(t)\}\big)\ \delta_{x=0}.$$

These inequalities are Riemann-solver compatible, in particular they

$\rightsquigarrow$ allow for clasical jumps in ρ at interface with flow level $\leq q(t)$ ...

$\rightsquigarrow$ allow for the non-classical jump from $\hat{\rho}_q$ to $\check{\rho}_q$, at flow level $= q(t)$.

- weak form of constraint "$f(\rho(t, 0^{\pm})) \leq q(t)$":

$\forall \psi \in \mathcal{D}(\mathbb{R}_*^+ \times \mathbb{R}^-)$

$$\int_0^{+\infty} f(\rho(t, 0^-))\psi(t, 0)\, dt = \int_0^{+\infty}\int_{-\infty}^{0} (\rho\psi_t + f(\rho)\psi_x)\, dtdx \leq \int_0^{+\infty} q(t)\psi(t, 0)\, dt$$

(weak formulation of LWR + Green-Gauss integration-by-parts used).

$\rightsquigarrow$ forbids classical jumps in ρ at interface with flow level $> q(t)$.

NB: adapted entropy inequalities + weak constraint "pass to the limit"

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Alternative : [A, Karlsen, Risebro '11], [A, Goatin, Seguin '10]
One can exploit the "unified" theory of discontinuous-flux SCL .
· Interface coupling conditions related to the modeling assumption
· $\rightsquigarrow$ "germ" G_q and another kind of adapted entropy inequalities
· Yields stability results (convergence of approx.; variable $q(\cdot)$).

Summary of theoretical results

- given $q(\cdot)$, existence of a unique admissible solution
- given $q(\cdot)$, L^1 contraction and order preservation
- FV scheme with constrained numerical flux at interface $x = 0$

$$F_q(a,b) := \min\{\, F(a,b)\,,\, q\,\}, \quad \text{where } F \text{ is any classical (monotone, consistent with } f\text{) numerical flux}$$

is convergent. The scheme is structure-preserving :
discrete L^1 contraction + order preservation hold.

- Lipschitz dependence on $q(\cdot)$ wrt the L^1 topology:

$$\|\rho(t,\cdot) - \widehat{\rho}(t,\cdot)\|_{L^1(\mathbb{R})} \leq \|\rho_0 - \widehat{\rho}_0\|_{L^1(\mathbb{R})} + 2\int_0^t |q(s) - \widehat{q}(s)|\, ds$$

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Rosini et al. model :

LWR with non-local point constraint

Modeling capacity drop at the exit

Capacity drop and its avatars (Braess Paradox, Faster Is Slower)
Order-preservation: a key feature of LWR. Real road traffic / pedestrian flows: non-monotone behavior observed. Capacity drop (localized at an "exit"): high density upstream the exit $\rightsquigarrow$ clogging $\rightsquigarrow$ small densities downstream.

Non-locally defined constraint [A., Donadello, Rosini '14]
One computes a subjective density $\xi(\cdot)$ upstream the exit $x = 0$:

$$\xi(t) = \int_{-\infty}^{0} w(x)\rho(t,x)\,dx \quad \text{where } w \geq 0, \int_{-\infty}^{0} w(x)\,dx = 1.$$

The weight w (assumed Lipschitz & compactly supported on $\mathbb{R}^-$) and a nonlinearity (constraint function) $p(\cdot)$ define

$$\text{non-local point constraint} \quad q(t) := p(\xi(t)).$$

Capacity drop is modeled by a positive, non-increasing $p(\cdot)$.

Rosini et al. model = Colombo-Goatin model + non-local constraint:

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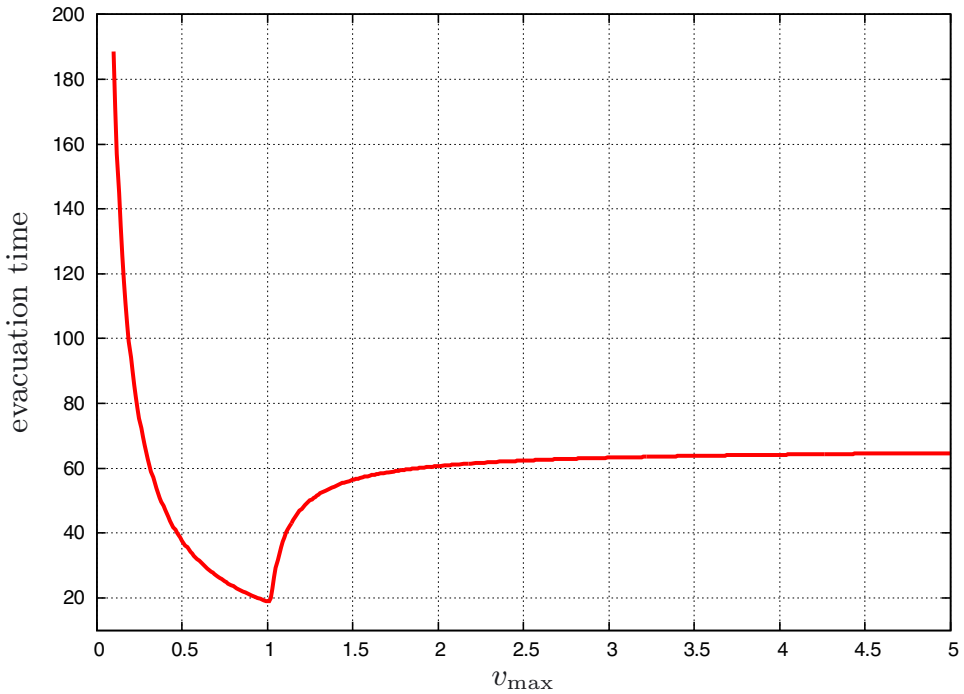
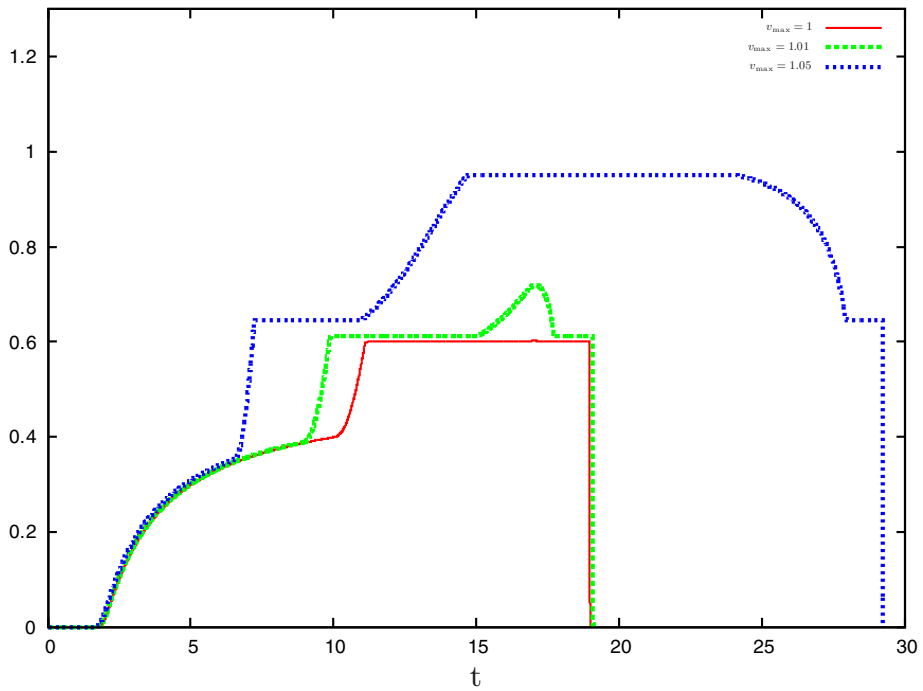
Numerical evidence: Faster Is Slower

Faster is Slower: [A., Donadello, Razafison, Rosini '16]

For same initial densities and constraint functions, we make vary the parameter V_{max} in the fundamental diagram $f : \rho \mapsto V_{max}\rho(1-\rho)$.



Evacuation time as function of V_{max}



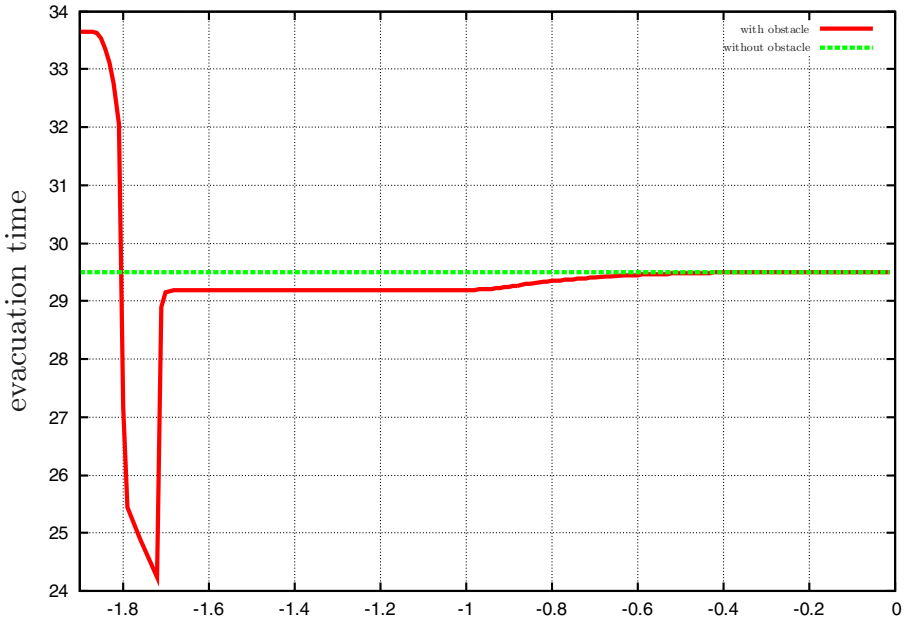
Evolution of the traffic density at the exit

Increasing traffic velocity may not accelerate car evacuation !

Numerical evidence: Braess' paradox

Braess' Paradox: [A., Donadello, Razafison, Rosini '16]

An obstacle (a slow-down zone or a "pre-exit" with 15% higher passing capacity) is introduced at some distance upstream of the exit. The position of the obstacle is optimized numerically.



Evacuation time as function of the position of the obstacle

An obstacle may decrease exit densities and evacuation time !

Wave-Front Tracking (BV-based) : analysis of interactions

Wave-Front Tracking: Construction of approximate solutions ρ_h by

- solving Riemann problems
- discretizing rarefactions by a sequence of small shocks
- repeating the procedure recursively when shock fronts meet
- estimating number of interactions + increase of TotVar in space of ρ_h
  (*ad hoc* Glimm/Temple kind functional $\Psi(\rho_h(t,\cdot))$ decreases with t)
- compactness of approx. solutions ρ_h + passage to the limit $h \to 0$
  $\rightsquigarrow$ convergence (up to a subsequence) [Holden, Risebro '02,'11]

Specificity of the Rosini et al. model: [A., Donadello, Rosini '14]

- presence of "non-local interactions" at $x = 0$
  (constraint handled via splitting , updated at fractional times $n\Delta t$)
- interactions at $x = 0$: find a suitable Glimm/Temple functional Ψ

$\rightsquigarrow$ use of Colombo-Goatin ideas + controlled increase of $\Psi(\rho_h(t,\cdot))$

Discretization of constraint: [A., Donadello, Razafison, Rosini '15]

NB: "small-amplitude non-uniqueness" for Riemann problem ,
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Uniqueness for Rosini et al. model

Uniqueness claim: There exists a unique solution of Rosini et al. model provided $p : [0, \rho_{max}]$ is Lipschitz continuous .

NB: $\rightsquigarrow$ the whole sequence of WFT approx. converges (BV data)

Technique: Lipschitz dependencies + Gronwall

· Lipschitz dependence of ρ wrt constraint level q:

$$\|\rho - \widehat{\rho}\|_{L^\infty((0,T);L^1(\mathbb{R}))} \leq 2\|q - \widehat{q}\|_{L^1(0,T)}$$

· Lipschitz dependence of the subjective density marker ξ wrt ρ:

$$|\xi(t) - \widehat{\xi}(t)| \leq \|w\|_\infty \|\rho(t,\cdot) - \widehat{\rho}(t,\cdot)\|_{L^1(\mathbb{R})}$$

· Lipschitz dependence of the constraint level $q = p(\xi)$ wrt ξ:

$$|q(t) - \widehat{q}(t)| \leq \|p\|_{W^{1,\infty}} |\xi(t) - \widehat{\xi}(t)|$$

· straightforward application of Gronwall inequality

Possibility to use contractive fixed-point:
one can get well-posedness by local in time use of Banach-Picard

Uniqueness for Rosini et al. model

Uniqueness claim: There exists a unique solution of Rosini et al. model provided $p : [0, \rho_{max}]$ is Lipschitz continuous .

NB: $\rightsquigarrow$ the whole sequence of WFT approx. converges (BV data)

Technique: Lipschitz dependencies + Gronwall

· Lipschitz dependence of ρ wrt constraint level q:

$$\|\rho - \widehat{\rho}\|_{L^\infty((0,T);L^1(\mathbb{R}))} \leq 2\|q - \widehat{q}\|_{L^1(0,T)}$$

· Lipschitz dependence of the subjective density marker ξ wrt ρ:

$$|\xi(t) - \widehat{\xi}(t)| \leq \|w\|_\infty \|\rho(t,\cdot) - \widehat{\rho}(t,\cdot)\|_{L^1(\mathbb{R})}$$

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Convergence of "pure splitting" approximations (L^∞-based)

Possibility to use splitting:

- constraint freezed at $t = 0$, Colombo-Goatin model solved for $t \in [0, \Delta t]$
- constraint updated at $t = \Delta t$: $q^n_{\Delta t} = \int_{\mathbb{R}_-} w(x)\rho(\Delta t, x)\, dx$
- procedure repeated recursively $\rightsquigarrow$ sequence of approx. solutions $\rho_{\Delta t}$ corresponding to constraints $q_{\Delta t}$ (piecewise constant)
- L^1_{loc} compactness of $(\rho_{\Delta t})_{\Delta t}$ (from kinetic theory)
  $\rightsquigarrow$ convergence (up to subseq.) of $\rho_{\Delta t}$
  $\rightsquigarrow$ convergence (up to subseq.) of $q_{\Delta t}$
  $\rightsquigarrow$ passage to the limit $\rightsquigarrow$ existence for Rosini et al. model (L^∞ data)

Numerical counterpart: [A., Donadello, Razafison, Rosini '16]

- use constraint level at time t^{n-1} to approximate ρ on $[t^{n-1}, t^n]$
- update the constraint level and use it on next time interval
- Repeat recursively
  $\rightsquigarrow$ the explicit FV splitting scheme used to validate Braess/FIS phenomena
- Convergence ? The first step (compactification for $(\rho_{\Delta t})_{\Delta t}$) is delicate
  $\rightsquigarrow$ alternative: what if we study compactness of $(q_{\Delta t})_{\Delta t}$ first ?

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Convergence of splitting approximations: Finite Volume scheme

Rewriting the constraint: assume w Lipschitz on $\mathbb{R}_-$.

Observation: being ρ solution of $\rho_t + f(\rho)_x = 0$, the subjective density ξ defined by $\xi(t) = \int_{\mathbb{R}_-} w(x)\rho(t,x)\,dx$ actually solves the ODE Cauchy pb.

$$\dot{\xi}(t) = \int_{\mathbb{R}_-} w'(x)\big[f(\rho(t,x)) - f(\rho(t,0^-))\big]\,dx, \quad \xi(0) = \int_{\mathbb{R}^-} w(x)\rho_0(x)\,dx.$$

Consequence: since $\rho \in L^\infty(0,T;L^1(\mathbb{R}))$ and w' is compactly supported, we get $\dot{\xi} \in L^\infty([0,T]) \rightsquigarrow q = p(\xi) \in BV([0,T])$ with an *a priori* bound .

Convergence of splitting FV scheme: [A., Donadello, Razafison, Rosini '16]

· discrete analogue of the rewriting argument
  $\rightsquigarrow$ uniform BV bound on $q_{\Delta t} \rightsquigarrow$ compactness of $(q_{\Delta t})_{\Delta t}$

· compare $\rho_\Delta t$ to the FV approx. of Colombo-Goatin model for $q_* = \lim_{\Delta t \to 0} q_{\Delta t}$
  using the discrete counterpart of the continuous $q(\cdot)$-dependence estimate:

$$\|\rho^n_{\Delta t} - \widehat{\rho}^n_{\Delta t}\|_{L^1(\mathbb{R})} \le 2|q_{\Delta t} - \widehat{q}_{\Delta t}|\,\Delta t.$$

· get convergence of ρ_Δ towards the solution ρ_* of Colombo-Goatin, given $q_*(\cdot)$

· inherit $q_* = p\big(\int_{\mathbb{R}_-} w(x)\rho_*(t,x)\,dx\big)$ from $q^n_{\Delta t} = p(\sum_i w_i \rho_i^{n-1}\Delta x)$.

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Rosini et al. model :

general class of constraints,

data acquisition, self-organization

Vehicles on highway: data acquisition. Self-organization.

Nonlocal constraints: data acquisition

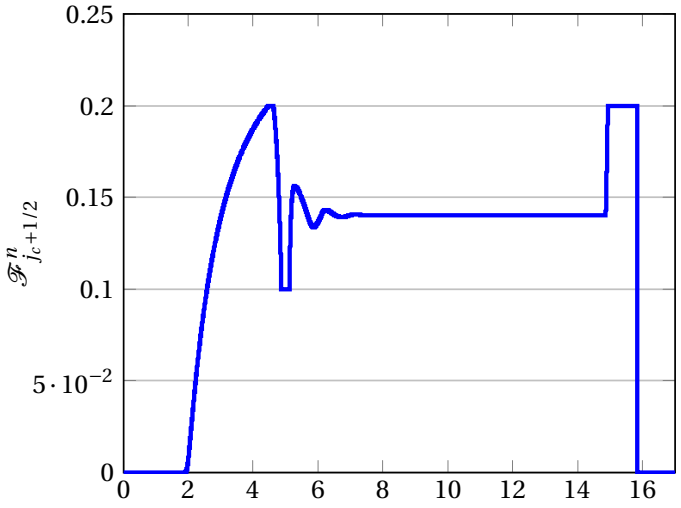
· Possibility to adapt passing capacity of a paytoll, etc. to observations of traffic: snapshots / video /... Space and time non-locality involved.

· Inertia and memory effects (time non-locality), relaxation of constraint,...

$\rightsquigarrow$ different expressions for subjective density / constraint level

Non-local constraints: self-organization

In real (non-panic) flows, evidence of self-organization at partially clogged exit. Basic Rosini et al. model: unable to reproduce self-organization.



Evolution of the flux at the exit (self-organization)
(simulation with a toy model combining memory and subjective density)

Constraint operators and examples

Functional setting

- two unknowns: ρ (density) and q (constraint level)
- q possibly computed from an auxiliary unknown ξ (subjective density)
- $\rho \in C([0,T], L^1(\mathbb{R}))$ equipped with $\|\cdot\|_{L^1([0,T]\times\mathbb{R})}$ or $\|\cdot\|_{L^\infty([0,T];L^1(\mathbb{R}))}$
- $q \in L^1([0,T])$ equipped with $\|\cdot\|_{L^1}$ or with $\|\cdot\|_{L^\infty}$

Constraint operator

$$\mathcal{Q} : \rho \in C([0,T]; L^1(\mathbb{R})) \mapsto q = \mathcal{Q}[\rho] \in L^1([0,T])$$

Assumed "history-dependent" , i.e. $\mathcal{Q}[\rho](t)$ depends only on $\rho|_{[0,t]\times\mathbb{R}}$.
$\mathcal{Q} : \rho \mapsto q$ is to be coupled with Colombo-Goatin LWR+point constraint model

Examples [A., Donadello, Razafison, Rosini '17+ε?]

- $\xi(t)$ computed from the ODE (cf. reformulation of Rosini et al. model) enriched with inertia or relaxation effects
- $\xi(t)$ computed from weighted average of ρ in $s \in [0,t]$, $x \in \mathbb{R}_-$
- $\xi(t)$ computed from weighted average of $(\rho(t_i,\cdot))_i$, $t_i < t$
- $q(t)$ computed from weighted average of $(f(\rho)(\cdot, y_j))_j$, $y_j \leq 0$
- ...

Well-posedness by fixed-point methods

Properties of constraint operator [A., Donadello, Razafison, Rosini '17+ε?]

- Continuity .....
- Compactness (not always necessary)

Often, can be recovered due to a reformulation of constraint
(notion of equivalence of constraints)

- Lipschitz continuity (for uniqueness only) .....

Well-posedness by contractive fixed-point

Under Lipschitz continuity assumption,
applying the Banach-Picard theorem on short time intervals

Well-posedness by Schauder fixed-point

Combining continuity and compactness
· either compactification of ρ due to kinetic theory
· or compactness assumption on $\mathcal{Q}$

Splitting Finite Volume approximation

· a specific notion of consistency for discretizations of $\mathcal{Q}$
· convergence strategy analogous to the basic Rosini et al. model

ARZ model
with point constraints

Important facts about ARZ

ARZ model

$$\begin{cases} \rho_t + (v\rho)_x = 0 \\ (\rho w)_t + (v\rho w)_x = 0, \end{cases} \qquad w = v + p(\rho)$$

Setting for WFT for ARZ model [Godvik, Hanche-Olsen '08]

Interactions are best understood in variables (v, w), $v \geq 0$, $w \geq v$:

· L^∞ and BV stability of both v and w at interactions

· unphysical "multiple vacuum states": $\rho = 0 \Leftrightarrow w = v \in \mathbb{R}_+$.

Entropies for ARZ

• [Panov'08] Renormalization for w : for all Borel function g,

$$\text{weak sol. of ARZ fulfill} \quad (\rho g(w))_t + (v\rho\, g(w))_x = 0$$

NB: this corresponds to a family of entropies that are exactly conserved

• [A.,Donadello,Rosini'16] new(?) family of "Kruzhkov-like entropy-flux pairs"

$$\mathcal{E}_k(v, w) = \mathrm{sign}^+(v - k)\left(1 - \frac{p^{-1}(w-v)}{p^{-1}(w-k)}\right), \quad \mathcal{Q}_k(v, w) = \dots$$

· Except at vacuum, the entropy inequalities

$$\forall k > 0 \quad \mathcal{E}_k(v, w)_t + \mathcal{Q}_k(v, w)_x \leq 0 \text{ in } \mathcal{D}'$$

select the desired (from modeling viewpoint) solutions of Riemann problems

· normalization for Riemann solver at vacuum; but heredity 100% unclear.

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Fixed constraint : model and Wave-Front Tracking approximation

ARZ with point constraint: [Garavello, Goatin '11]

Given $q : [0, T] \to \mathbb{R}_+$,

$$\begin{cases} \rho_t + (v\rho)_x = 0, \\ Q := v\rho \ \text{ fullfills } \ Q(t, 0^\pm) \le q(t)\,; \\ (\rho w)_t + (v\rho w)_x = 0, \ \text{ where } w = v + p(\rho) \end{cases}$$

Riemann solver at $x = 0$:

· w does not jump across the interface except if $Q(v_\pm, w_\pm) = 0$

· For given $w \ge 0$ and $q \ge 0$ one defines $\check{v}_q$ and $\hat{v}_q$ like for constrained LWR

· if the flow at $x = 0$ "wants to be" above $q(t)$ (for unconstrained ARZ), in constrained ARZ $v(t, \cdot)$ jumps from $v_- = \hat{v}_q$ to $v_+ = \check{v}_q$, and $Q(v_\pm, w) = q$.

WFT (for q constant in time only) [A., Donadello, Rosini '16]

· standard WFT constructions with specific Riemann solver at $x = 0$

· interactions at $x \ne 0$ treated following [Godvik,Hanche-Olsen '08]

· for interactions at $x = 0$, compensations detected $\rightsquigarrow$ Glimm-kind functional $\rightsquigarrow$ uniform BV estimates $\rightsquigarrow$ compactness (in pointwise a.e. sense)

Extension to general time-dependent q ?

· Immediate, for piecewise constant in time constraint

· Loss of BV control, if q varies continuously (due to singular dependence of the Glimm functional wrt q).

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Fixed constraint : adapted entropy formulation

Passage to the limit in WFT approximations

- weak formulation is easily inherited ($\Rightarrow$ renormalization prop. for w)
- away from $x = 0$, entropy inequalities $\mathcal{E}_k(v, w)_t + \mathcal{Q}_k(v, w)_x \leq 0$ are encoded in Riemann solver; they pass to the a.e. limit
- the constraint $Q(v, w)(t, 0^{\pm}) \leq q$ makes sense and passes to the limit , due to the conservation $\rho(v, w)_t + Q(v, w)_x = 0$ + Green-Gauss theorem
- one can write adapted entropy inequalities (involving $x = 0$) and pass to the limit combining renormalization for w + Green-Gauss

Adapted entropy formulation of constrained ARZ

- for $k > 0$, set "renormalization functions" $g_k(q, w) := \left[\frac{k}{q} - \frac{1}{p^{-1}([w-k]^+)}\right]^+$;
- adapted (to the constraint) entropy inequalities write

$$\mathcal{E}_k(v, w)_t + \mathcal{Q}_k(v, w)_x \leq Q(v, w) g_k(q, w)\, \delta_{x=0} \text{ in } \mathcal{D}'$$

- these inequalities + constraint $Q(v, w)|_{x=0} \leq q$ select as admissible at $x = 0$ precisely the desired solutions to constrained Riemann problems
- renormalization prop. $\rightsquigarrow$ conservation $(\rho g_k(q, w))_t + (Q(v, w) g_k(q, w))_x = 0$
  $\rightsquigarrow$ Green-Gauss yields a "distributed reformulation" for $Q(v, w) g_k(q, w)\, \delta_{x=0}$
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Towards variable constraint (modeling capacity drop and self-organization)

Work in progress and open research directions

- constraints in so-called "two-phase models"
  combining LWR near vacuum and ARZ elsewhere
  [Benyahia, Rosini et al.'16,'17,...]

- Study of existence for ARZ with non-local constraints

· splitting, fixed point and constraint reformulation arguments (cf. LWR case)

· compactness via BV bound on w + L^∞ bound on v (both are available)
  + compensated compactness, using the "Kruzhkov-like" entropies

- Numerics of constrained ARZ (with tricks from [Chalons, Goatin '07]):
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