A framework for interface coupling of conservation laws

Boris Andreianov
Institut Denis Poisson, Université de Tours

based on joint works
with Clément Cancès (INRIA Lille) and many others

former ANR project CoToCoLa

Institut Camille Jordan Lyon, June 18, 2019

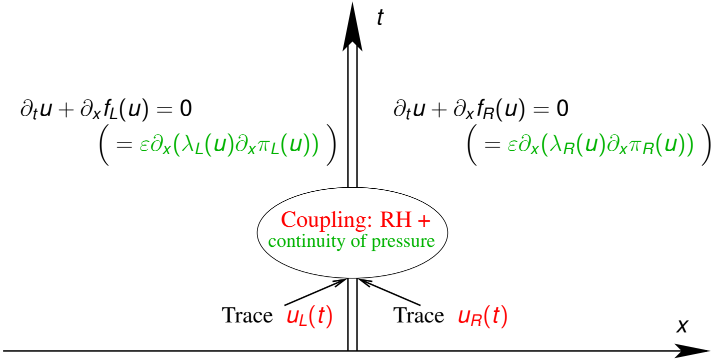
1. **Example from porous media: two-rocks interface**
2. **Example from road/pedestrian traffic modelling**
3. **Boundary Conditions revisited**
4. **Interface Coupling Conditions**
5. **Examples of Interface Coupling Conditions in Applications. Approximation by Finite Volume schemes**
   - Coupling by transmission maps
   - Flux limitation/velocity limitation in traffic models
6. **Conclusions**

Example from porous media

Example: Buckley-Leverett equation as vanishing capillarity limit

Consider Buckley-Leverett equation in 1D medium
constituted of two rocks with distinct physical properties

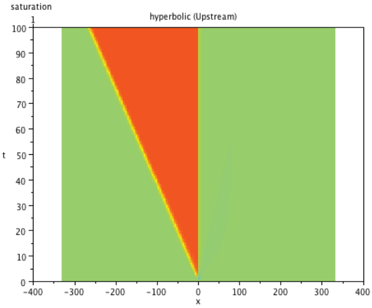
$$\partial_t u + \partial_x \Big(f_L(u) \mathbb{1}_{x<0} + f_R(u) \mathbb{1}_{x>0} \Big) = 0$$



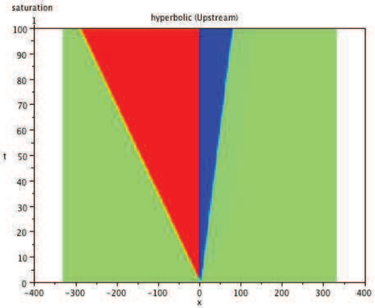
NB: the nonlinearities $\pi_{L,R}$ (capillary pressures) and $\lambda_{L,R}$
enter the model for $\varepsilon > 0$ but don't enter the limit model
$\Rightarrow$ should Interface Coupling keep memory of $\pi_{L,R}$ and $\lambda_{L,R}$?

Different Interface Coupling Conditions lead to different solutions

$\varepsilon = 0$: Simulations for a constant initial condition and given f_L,f_R



(a) Numerical solution for constant datum



(b) Another numerical solution, same datum

Difference : choice of capillary pressure profiles π_L, π_R.

NB: non-classical solutions even if $f_L = f_R$!

NB: order-preserving solver !

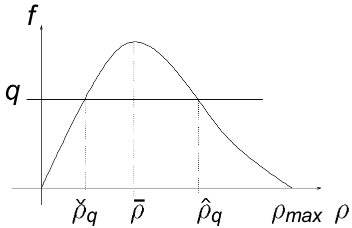
Example from traffic modelling

Colombo-Goatin model : LWR with point constraint on the flux

Colombo-Goatin model: in the context of road lights, pay tolls, small-scale construction sites, one may consider the formal model

$$\begin{cases} \rho_t + f(\rho)_x = 0 & \text{(classical LWR model)} \\ f(\,\rho(t,0^\pm)\,) \le q(t) & \text{(point constraint)} \end{cases}$$

where the map $t \mapsto q(t)$ (constraint at $x = 0$, given beforehand) prescribes the maximal possible value of the car flow $f(\rho(t,0^+)) \equiv f(\rho(t,0^-))$.



Riemann solver at $x = 0$:
if the flow at $x = 0$ "wants to be" above q (for unconstrained LWR), then constrained $\rho(t,\cdot)$ jumps from $\rho_- = \hat{\rho}_q$ to $\rho_+ = \check{\rho}_q$, and $f(\rho_\pm) = q$.

NB : Non-classical shock but order-preserving Riemann solver!

Modeling capacity drop at the exit

Capacity drop and its avatars (Braess Paradox, Faster Is Slower)
Order-preservation: a key feature of LWR. Real road traffic / pedestrian flows: non-monotone behavior observed. Capacity drop (localized at an "exit"): high density upstream the exit $\rightsquigarrow$ clogging $\rightsquigarrow$ small densities downstream.

Non-locally defined constraint [A., Donadello, Rosini '14]
One computes a subjective density $\xi(\cdot)$ upstream the exit $x = 0$:

$$\xi(t) = \int_{-\infty}^{0} w(x)\rho(t,x)\,dx \quad \text{where } w \geq 0, \int_{-\infty}^{0} w(x)\,dx = 1.$$

The weight w (assumed Lipschitz & compactly supported on $\mathbb{R}^-$)
and a nonlinearity (constraint function) $p(\cdot)$ define

$$\text{non-local point constraint} \quad q(t) := p(\xi(t)).$$

Capacity drop is modeled by a positive, non-increasing $p(\cdot)$.

Rosini et al. model = Colombo-Goatin model + non-local constraint:

$$\begin{cases} \rho_t + f(\rho)_x = 0 \\ f(\rho(t,0^\pm)) \leq q(t), \end{cases} \quad \text{where } q(t) = p\left(\fint \rho(t,\cdot)\,d\mu(\cdot) \right), d\mu(x) = w(x)dx.$$

Modeling capacity drop at the exit

Capacity drop and its avatars (Braess Paradox, Faster Is Slower)
Order-preservation: a key feature of LWR. Real road traffic / pedestrian flows: non-monotone behavior observed. Capacity drop (localized at an "exit"): high density upstream the exit $\rightsquigarrow$ clogging $\rightsquigarrow$ small densities downstream.

Non-locally defined constraint [A., Donadello, Rosini '14]
One computes a subjective density $\xi(\cdot)$ upstream the exit $x = 0$:

$$\xi(t) = \int_{-\infty}^{0} w(x)\rho(t,x)\,dx \quad \text{where } w \geq 0, \int_{-\infty}^{0} w(x)\,dx = 1.$$

The weight w (assumed Lipschitz & compactly supported on $\mathbb{R}^-$) and a nonlinearity (constraint function) $p(\cdot)$ define

$$\text{non-local point constraint} \quad q(t) := p(\xi(t)).$$

Capacity drop is modeled by a positive, non-increasing $p(\cdot)$.

Rosini et al. model = Colombo-Goatin model + non-local constraint:

$$\begin{cases} \rho_t + f(\rho)_x = 0 \\ f(\rho(t,0^\pm)) \leq q(t), \end{cases} \quad \text{where } q(t) = p\left(\int \rho(t,\cdot)\,d\mu(\cdot)\right), d\mu(x) = w(x)dx.$$

Modeling capacity drop at the exit

Capacity drop and its avatars (Braess Paradox, Faster Is Slower)
Order-preservation: a key feature of LWR. Real road traffic / pedestrian flows: non-monotone behavior observed. Capacity drop (localized at an "exit"): high density upstream the exit $\rightsquigarrow$ clogging $\rightsquigarrow$ small densities downstream.

Non-locally defined constraint [A., Donadello, Rosini '14]
One computes a subjective density $\xi(\cdot)$ upstream the exit $x = 0$:

$$\xi(t) = \int_{-\infty}^{0} w(x)\rho(t,x)\,dx \quad \text{where } w \geq 0, \int_{-\infty}^{0} w(x)\,dx = 1.$$

The weight w (assumed Lipschitz & compactly supported on $\mathbb{R}^-$) and a nonlinearity (constraint function) $p(\cdot)$ define

$$\text{non-local point constraint} \quad q(t) := p(\xi(t)).$$

Capacity drop is modeled by a positive, non-increasing $p(\cdot)$.

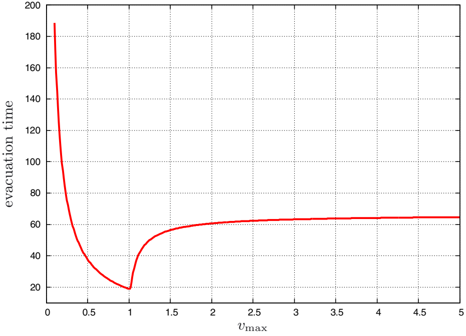
Rosini et al. model = Colombo-Goatin model + non-local constraint:

$$\begin{cases} \rho_t + f(\rho)_x = 0 \\ f(\,\rho(t,0^{\pm})\,) \leq q(t), \end{cases} \quad \text{where } q(t) = p\left(\fint \rho(t,\cdot)\,d\mu(\cdot) \right), d\mu(x) = w(x)dx.$$

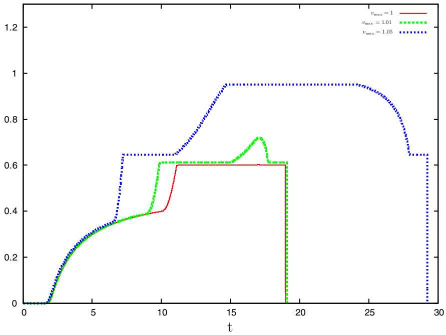
Numerical evidence: Faster Is Slower

Faster is Slower in ADR model: [A.,Donadello,Razafison,Rosini'16]

For same initial densities and constraint functions, we make vary the parameter V_{max} in the fundamental diagram $f : \rho \mapsto V_{max}\rho(1-\rho)$.



Evacuation time as function of V_{max}



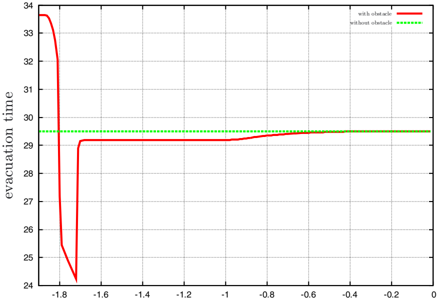
Evolution of the traffic density at the exit

Festina lente: Increasing traffic velocity may slow down evacuation !

Numerical evidence: Braess' paradox

Braess' Paradox in ADR model: [A.,Donadello,Razafison,Rosini'16]

An obstacle (a slow-down zone or a "pre-exit" with 15% higher passing capacity) is introduced at some distance upstream of the exit. The position of the obstacle is optimized numerically.



Evacuation time as function of the position of the obstacle

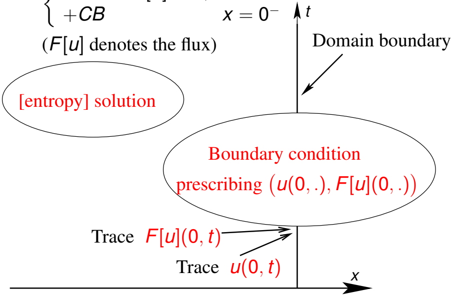
An obstacle may decrease exit densities and evacuation time!

Boundary Conditions revisited

General dissipative boundary conditions

$$\begin{cases} \partial_t u + \partial_x F[u] = 0, & x < 0 \\ +CB & x = 0^- \end{cases}$$

($F[u]$ denotes the flux)



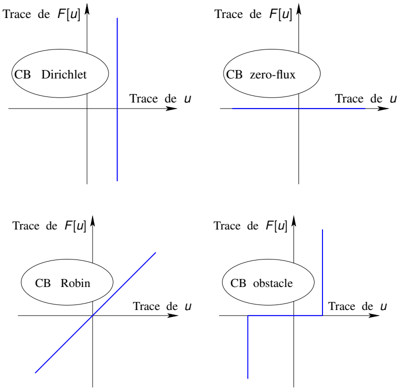
Local "Kato inequality" obtained from the local [entropy] formulation:

$$\int_\Omega |u-\hat{u}|(T,x) - \int_\Omega |u_0-\hat{u}_0| + \int_0^T\int_\Omega \operatorname{sign}(u-\hat{u})(F[u]-F[\hat{u}])\cdot\nabla\xi \leq 0 \qquad \forall \xi \in \mathcal{D}(\Omega)^+$$

Exploit KI near the boundary: test fct. $\xi_n \to \mathbf{1}_\Omega$ with $\nabla\xi_n \rightharpoonup -\delta|_{\partial\Omega}\mathbf{n} \;\Rightarrow$

$$\int_\Omega |u-\hat{u}|(T,x) - \int_\Omega |u_0-\hat{u}_0| \;\leq\; -\int_0^T \gamma_{ad\ hoc}\Big\{\operatorname{sign}(u-\hat{u})(F[u]-F[\hat{u}])\cdot\mathbf{n}\Big\}(t)\,dt$$

Classical boundary conditions



In all these cases, $(u, F[u]) \in \beta$ for some maximal monotone graph β .

Dissipative BC framework: maximal monotone dependence between the solution u and flux $F[u]$ at the boundary.

Boundary dissipation: the RHS of the Kato inequality is ≤ 0 !
Indeed, $\operatorname{sign}(u-\hat{u})\big(F[u]-F[\hat{u}]\big) = \operatorname{sign}(u-\hat{u})\big(\beta(u)-\beta(\hat{u})\big) \geq 0$

Dissipative BC for hyperbolic conservation law. Projection.

Hyperbolic equation $u_t + f(u)_x = 0$ + formal BC $(u, F[u]) \in \beta$:

- Uniqueness is obvious for the formal problem
- Formal problem ill-posed (in general, existence fails)
- Problem with ... $= \varepsilon \partial^2_{xx} u$ is well posed.
  The limit is a local entropy solution verifying
  effective BC $(u, F[u]) \in \tilde{\beta}$ where $\tilde{\beta}$ is a projection of β.
  Problem with effective BC (i.e., $\tilde{\beta}$ is the BC) is well posed.

- One can easily grasp the projection procedure by picturing $\tilde{\beta}$.
  One observes : $\tilde{\beta}$ is the maximal monotone subgraph of $f(.)$
  which is the closest to β !

Example: BLN condition [Bardos,LeRoux,Nédélec'79]
can be reformulated this way [Dubois,LeFloch'88]

- One can describe $\tilde{\beta}$ in terms of the "Godunov numerical flux":

$$\tilde{\beta} = \left\{ (u, \mathcal{F}) \,\middle|\, \exists \tilde{u} \;\; \mathcal{F} = f(u) = \mathrm{God}[u, \tilde{u}] \in \beta(\tilde{u}) \right\}$$

Détails : [Thesis Sbihi'06],[A.,Sbihi'15]

Dissipative BC for hyperbolic conservation law. Projection.

Hyperbolic equation $u_t + f(u)_x = 0$ + formal BC $(u, F[u]) \in \beta$:

- Uniqueness is obvious for the formal problem
- Formal problem ill-posed (in general, existence fails)
- Problem with ... $= \varepsilon\partial^2_{xx} u$ is well posed.
  The limit is a local entropy solution verifying
  effective BC $(u, F[u]) \in \tilde{\beta}$ where $\tilde{\beta}$ is a projection of β.
  Problem with effective BC (i.e., $\tilde{\beta}$ is the BC) is well posed.
- One can easily grasp the projection procedure by picturing $\tilde{\beta}$.
  One observes : $\tilde{\beta}$ is the maximal monotone subgraph of $f(.)$
  which is the closest to β !

Example: BLN condition [Bardos,LeRoux,Nédélec'79]
can be reformulated this way [Dubois,LeFloch'88]

- One can describe $\tilde{\beta}$ in terms of the "Godunov numerical flux":

$$\tilde{\beta} = \left\{ (u, \mathcal{F}) \,\middle|\, \exists \tilde{u} \;\; \mathcal{F} = f(u) = \mathrm{God}[u, \tilde{u}] \in \beta(\tilde{u}) \right\}$$

Détails : [Thesis Sbihi'06],[A.,Sbihi'15]

Dissipative BC for hyperbolic conservation law. Projection.

Hyperbolic equation $u_t + f(u)_x = 0$ + formal BC $(u, F[u]) \in \beta$:

- Uniqueness is obvious for the formal problem
- Formal problem ill-posed (in general, existence fails)
- Problem with ... $= \varepsilon\partial^2_{xx} u$ is well posed.
  The limit is a local entropy solution verifying
  effective BC $(u, F[u]) \in \tilde\beta$ where $\tilde\beta$ is a projection of β.
  Problem with effective BC (i.e., $\tilde\beta$ is the BC) is well posed.

- One can easily grasp the projection procedure by picturing $\tilde\beta$.
  One observes : $\tilde\beta$ is the maximal monotone subgraph of $f(.)$
  which is the closest to β !

Example: BLN condition [Bardos,LeRoux,Nédélec'79]
can be reformulated this way [Dubois,LeFloch'88]

- One can describe $\tilde\beta$ in terms of the "Godunov numerical flux":

$$\tilde\beta = \Big\{ (u, \mathcal{F}) \,\Big|\, \exists \tilde u \;\; \mathcal{F} = f(u) = \mathrm{God}[u, \tilde u] \in \beta(\tilde u) \Big\}$$

Détails : [Thesis Sbihi'06],[A.,Sbihi'15]

Interface Coupling Conditions

Dissipative Interface Coupling Conditions (ICC)

Analogy : One assimilates inner interface to a "double boundary"

Interface Coupling Conditions (ICC) can be expressed, as in the BC case, by

$$\Big((u_L, u_R)\,,\,(F_L, F_R) \Big) \;\in\; \mathcal{H} \subset \mathbb{R}^2 \times \mathbb{R}^2$$

where $u_{L,R}$ are the traces (left and right) of the solution u and $F_{L,R}$ are the normal traces (left and right) of the flux $F[u]$.

The ICC is conservative if $\forall\big((u_L, u_R), (F_L, F_R)\big) \in \mathcal{H},\ F_L + F_R = 0.$

The L^1-dissipativity of the ICC is equivalent to the monotonicity of $\mathcal{H}$ in the sense: $\mathcal{H}$ is called 1-monotone if

$$\forall \Big((u_L, u_R)\,,\,(F_L, F_R) \Big), \Big((\hat{u}_L, \hat{u}_R)\,,\,(\hat{F}_L, \hat{F}_R) \Big) \in \mathcal{H}$$

$$\operatorname{sign}_{max}(u_L - \hat{u}_L)(F_L - \hat{F}_L) + \operatorname{sign}_{max}(u_R - \hat{u}_R)(F_R - \hat{F}_R) \geq 0$$

Principle: The situation of ICC is fully analogous to that of BC!

NB : Idea comes from [Imbert,Monneau'14] (HJeqns on networks).
Natural extension to networks [A.,Coclite,Donadello'17]

Dissipative Interface Coupling Conditions (ICC)

Analogy : One assimilates inner interface to a "double boundary"

Interface Coupling Conditions (ICC) can be expressed, as in the BC case, by

$$\Big((u_L, u_R)\,,\,(F_L, F_R) \Big) \;\in\; \mathcal{H} \subset \mathbb{R}^2 \times \mathbb{R}^2$$

where $u_{L,R}$ are the traces (left and right) of the solution u
and $F_{L,R}$ are the normal traces (left and right) of the flux $F[u]$.

The ICC is conservative if $\forall\big((u_L, u_R), (F_L, F_R)\big) \in \mathcal{H},\ F_L + F_R = 0.$

The L^1-dissipativity of the ICC is equivalent to the monotonicity of $\mathcal{H}$ in the sense: $\mathcal{H}$ is called 1-monotone if

$$\forall \Big((u_L, u_R)\,,\,(F_L, F_R) \Big), \Big((\hat u_L, \hat u_R)\,,\,(\hat F_L, \hat F_R) \Big) \in \mathcal{H}$$

$$\operatorname{sign}_{max}(u_L - \hat u_L)(F_L - \hat F_L) + \operatorname{sign}_{max}(u_R - \hat u_R)(F_R - \hat F_R) \geq 0$$

Principle: The situation of ICC is fully analogous to that of BC!

NB : Idea comes from [Imbert,Monneau'14] (HJeqns on networks).
Natural extension to networks [A.,Coclite,Donadello'17]

The projection procedure for ICC. Link to "$L1D$ germs".

In particular: a formally prescribed ICC is projected: $\mathcal{H} \rightsquigarrow \widetilde{\mathcal{H}}$,

$$\widetilde{\mathcal{H}} := \Big\{ (u_L, u_R\,;\, F_L, F_R) \in \mathbb{R}^2 \times \mathbb{R}^2 \,\Big|\, \exists (\bar{u}_L, \bar{u}_R\,;\, F_L, F_R) \in \mathcal{H}$$
$$F_L = f_L(u_L) = \mathrm{God}_L[u_L, \bar{u}_L],\ -F_R = f_R(u_R) = \mathrm{God}_R[\bar{u}_R, u_R] \Big\}$$

($\mathrm{God}_{L,R}[\cdot,\cdot]$ being the Godunov fluxes associated with $f_{L,R}$).

- $\mathcal{H}$ is conservative $\Rightarrow$ $\widetilde{\mathcal{H}}$ is also conservative
- $\mathcal{H}$ is 1-monotone $\Rightarrow$ $\widetilde{\mathcal{H}}$ is also 1-monotone ; moreover, the domain of $\widetilde{\mathcal{H}}$ is an "L^1D germ" i.e., the projected ICC fits the theory of [A.,Karlsen,Risebro'11]

As for the BC case, $\widetilde{\mathcal{H}}$ should be seen as the effective ICC [A.,'15] .

Trivial example of ICC: Kirchhoff conditions (cf. networks)

$$\mathcal{H} := \{ (u, u\,;\, F, -F) \in \mathbb{R}^2 \times \mathbb{R}^2 \,|\, u \in \mathbb{R},\ F \in \mathbb{R} \}.$$

Example of ICC: "Kedem-Katchalsky membrane condition"

$$\mathcal{H} := \Big\{ (u_L, u_R\,;\, F, -F) \in \mathbb{R}^2 \times \mathbb{R}^2 \,\Big|\, u_{L,R} \in \mathbb{R},\ F = \lambda(u_L - u_R) \Big\}.$$

The projection procedure for ICC. Link to "$L1D$ germs".

In particular: a formally prescribed ICC is projected: $\mathcal{H} \rightsquigarrow \widetilde{\mathcal{H}}$,

$$\widetilde{\mathcal{H}} := \Big\{ (u_L, u_R\,;\, F_L, F_R) \in \mathbb{R}^2 \times \mathbb{R}^2 \,\Big|\, \exists (\bar{u}_L, \bar{u}_R\,;\, F_L, F_R) \in \mathcal{H} \\ F_L = f_L(u_L) = \mathrm{God}_L[u_L, \bar{u}_L],\; -F_R = f_R(u_R) = \mathrm{God}_R[\bar{u}_R, u_R] \Big\}$$

($\mathrm{God}_{L,R}[\cdot,\cdot]$ being the Godunov fluxes associated with $f_{L,R}$).

- $\mathcal{H}$ is conservative $\Rightarrow$ $\widetilde{\mathcal{H}}$ is also conservative
- $\mathcal{H}$ is 1-monotone $\Rightarrow$ $\widetilde{\mathcal{H}}$ is also 1-monotone ;
  moreover, the domain of $\widetilde{\mathcal{H}}$ is an "L^1D germ" i.e.,
  the projected ICC fits the theory of [A.,Karlsen,Risebro'11]

As for the BC case, $\widetilde{\mathcal{H}}$ should be seen as the effective ICC [A.,'15] .

Trivial example of ICC: Kirchhoff conditions (cf. networks)

$$\mathcal{H} := \{ (u, u\,;\, F, -F) \in \mathbb{R}^2 \times \mathbb{R}^2 \,|\, u \in \mathbb{R},\; F \in \mathbb{R} \}.$$

Example of ICC: "Kedem-Katchalsky membrane condition"

$$\mathcal{H} := \Big\{ (u_L, u_R\,;\, F, -F) \in \mathbb{R}^2 \times \mathbb{R}^2 \,\Big|\, u_{L,R} \in \mathbb{R},\; F = \lambda(u_L - u_R) \Big\}.$$

The projection procedure for ICC. Link to "$L1D$ germs".

In particular: a formally prescribed ICC is projected: $\mathcal{H} \rightsquigarrow \widetilde{\mathcal{H}}$,

$$\widetilde{\mathcal{H}} := \Big\{ \,(\,u_L, u_R\,;\; F_L, F_R\,) \in \mathbb{R}^2 \times \mathbb{R}^2 \;\Big|\; \exists(\,\bar u_L, \bar u_R\,;\; F_L, F_R\,) \in \mathcal{H} \\ F_L = f_L(u_L) = \mathrm{God}_L[u_L, \bar u_L],\; -F_R = f_R(u_R) = \mathrm{God}_R[\bar u_R, u_R] \,\Big\}$$

($\mathrm{God}_{L,R}[\cdot,\cdot]$ being the Godunov fluxes associated with $f_{L,R}$).

- $\mathcal{H}$ is conservative $\Rightarrow$ $\widetilde{\mathcal{H}}$ is also conservative
- $\mathcal{H}$ is 1-monotone $\Rightarrow$ $\widetilde{\mathcal{H}}$ is also 1-monotone ; moreover, the domain of $\widetilde{\mathcal{H}}$ is an "L^1D germ" i.e., the projected ICC fits the theory of [A.,Karlsen,Risebro'11]

As for the BC case, $\widetilde{\mathcal{H}}$ should be seen as the effective ICC [A.,'15] .

Trivial example of ICC: Kirchhoff conditions (cf. networks)

$$\mathcal{H} := \big\{ \big(\,u, u\,;\; F, -F\,\big) \in \mathbb{R}^2 \times \mathbb{R}^2 \,\big|\, u \in \mathbb{R},\; F \in \mathbb{R} \big\}.$$

Example of ICC: "Kedem-Katchalsky membrane condition"

$$\mathcal{H} := \Big\{ \Big(\,u_L, u_R\,;\; F, -F\,\Big) \in \mathbb{R}^2 \times \mathbb{R}^2 \,\Big|\, u_{L,R} \in \mathbb{R},\; F = \lambda(u_L - u_R) \Big\}.$$

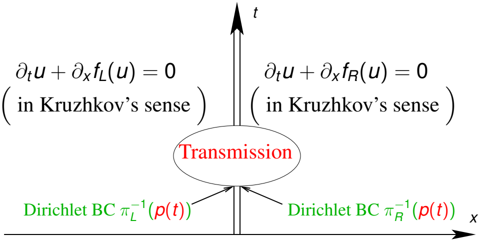
Examples of ICC,
applications and
well-balanced FV schemes

Conservative ICC defined by transmission maps

The example of vanishing capillarity suggests the following ICC:

$$(u_L, u_R\,;\, F_L, F_R) \in \mathcal{H}(\pi_{L,R}) \;\Leftrightarrow\; F_L + F_R = 0, \;\; \pi_L(u_L) = \pi_R(u_R).$$

The interface coupling by transmission map $p \mapsto (\pi_L^{-1}(p), \pi_R^{-1}(p))$:



Transmission: two Dirichlet pbs (in the BLN sense) coupled by

- the Dirichet BC $\pi_{L,R}(p(t))$ ($p(t)$ being additional unknown)
- the conservativity relation
  $\mathrm{God}_L[u(t,0^-), \pi_L^{-1}(p(t))] = \mathrm{God}_R[\pi_R^{-1}(p(t)), u(t,0^+)].$

Well-balanced FV schemes for transmission-map ICC

FV schemes for the transmission-map ICC: [A.,Cancès'12,'14,'15]

the two-point interface flux $F_{int}(\cdot,\cdot)$ is defined by

$$F_{int}(u_-,u_+) = \mathrm{God}_L[u_-,\pi_L^{-1}(p)] = \mathrm{God}_R[\pi_R^{-1}(p),u_+]$$

where $p \in \mathbb{R}$ solves $\mathrm{God}_R[\pi_R^{-1}(p),u_+] - \mathrm{God}_L[u_-,\pi_L^{-1}(p)] = 0.$

Properties of the scheme:

- One implicit unknown p per interface point ;
  the equation on p to be solved is a scalar monotone equation
- The numerical flux F_{int} is monotone and Lipschitz
- The scheme is well balanced
  (it preserves the "germ" steady states) $\Rightarrow$ the scheme converges

  NB: we use Godunov fluxes of $f_{L,R}$...
  but not the Riemann solver at the interface !
- Moreover, $\mathrm{God}_{L,R}$ can be replaced by any classical num. flux !
  The scheme is "asymptotically well-balanced" and convergent.

Well-balanced FV schemes for transmission-map ICC

FV schemes for the transmission-map ICC: [A.,Cancès'12,'14,'15]
the two-point interface flux $F_{int}(\cdot,\cdot)$ is defined by

$$F_{int}(u_-,u_+) = \mathrm{God}_L[u_-,\pi_L^{-1}(p)] = \mathrm{God}_R[\pi_R^{-1}(p),u_+]$$

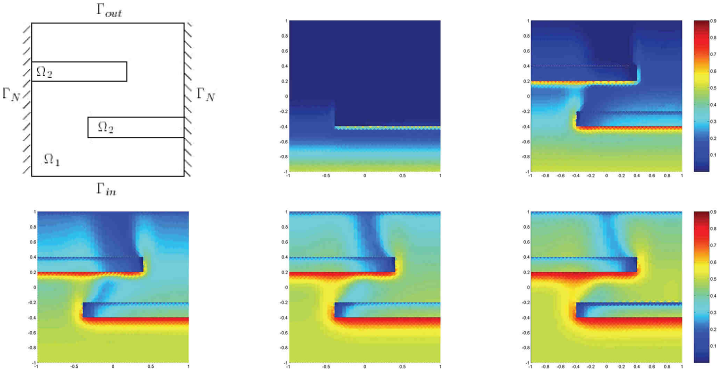
where $p \in \mathbb{R}$ solves $\mathrm{God}_R[\pi_R^{-1}(p),u_+] - \mathrm{God}_L[u_-,\pi_L^{-1}(p)] = 0.$

Properties of the scheme:

- One implicit unknown p per interface point ;
  the equation on p to be solved is a scalar monotone equation
- The numerical flux F_{int} is monotone and Lipschitz
- The scheme is well balanced
  (it preserves the "germ" steady states) $\Rightarrow$ the scheme converges
  NB: we use Godunov fluxes of $f_{L,R}$...
  but not the Riemann solver at the interface !
- Moreover, $\mathrm{God}_{L,R}$ can be replaced by any classical num. flux !
  The scheme is "asymptotically well-balanced" and convergent.

Numerical example in 2D (IMPES scheme)

Combination with 2D IMplicit Pressure - Explicit Saturation Scheme:



The two-rock domain is initially saturated in water. Two barriers (rock Ω_2) have a higher entry pressure. The vertical boundaries are impermeable. Bottom+top : a constant rate of a total flux is prescribed. Saturation $s = 0.5$ imposed on Γ_{in}. Details: [A.,Brenner,Cancès'13] .

Assessing Colombo-Goatin model via ICC

[Colombo,Goatin'07] : LWR model $\partial_t u + \partial_x f(u) = 0$

+ point constraint $f(u)|_{x=0} \leq q_{lim}(t)$.

Models red lights, pay tolls, construction cites,...

- ICC mimicking the Colombo-Goatin Riemann solver

$$\mathcal{H}(t) = \Big\{(\rho,\rho,F,-F) \mid \rho \text{ arbitrary}, \ F \leq q_{lim}(t)\Big\} \ \text{(the classical part)}$$
$$\bigcup \Big\{(\rho_L,\rho_R,F,-F) \mid \rho_L > \rho_R, \ F = q_{lim}(t)\Big\} \ \text{(non-classical jumps)}$$

Not surprisingly, this ICC leads to the Colombo-Goatin coupling.

Finite volume approximation:
an elementary recipe for the interface flux [A.,Goatin,Seguin'10]

$$F_{int}(u_-,u_+) := \min\big\{F(u_-,u_+), q_{lim}\big\} \qquad (F(.,.) \text{ a numerical flux}).$$

Interface flux $F_{int}(.,.)$ inherits monotony and consistency from $F(.,.)$;
it is well-balanced (=preserves exactly the non-classical shock)

NB: Well-balance property $\Rightarrow$ convergence of the scheme!

Assessing Colombo-Goatin model via ICC

[Colombo,Goatin'07] : LWR model $\partial_t u + \partial_x f(u) = 0$

+ point constraint $f(u)|_{x=0} \leq q_{lim}(t)$.

Models red lights, pay tolls, construction cites,...

- ICC mimicking the Colombo-Goatin Riemann solver

$$\mathcal{H}(t) = \Big\{(\rho, \rho, F, -F) \mid \rho \text{ arbitrary}, \ F \leq q_{lim}(t)\Big\} \quad \text{(the classical part)}$$

$$\bigcup \Big\{(\rho_L, \rho_R, F, -F) \mid \rho_L > \rho_R, \ F = q_{lim}(t)\Big\} \quad \text{(non-classical jumps)}$$

Not surprisingly, this ICC leads to the Colombo-Goatin coupling.

Finite volume approximation:
an elementary recipe for the interface flux [A.,Goatin,Seguin'10]

$$F_{int}(u_-, u_+) := \min\big\{F(u_-, u_+), q_{lim}\big\} \qquad (F(.,.) \text{ a numerical flux}).$$

Interface flux $F_{int}(.,.)$ inherits monotony and consistency from $F(.,.)$;
it is well-balanced (=preserves exactly the non-classical shock)

NB: Well-balance property $\Rightarrow$ convergence of the scheme!

Velocity limitation is a flux limitation

Q: how to model localized velocity limitation?
[A.,Donadello,Rosini, in preparation] :

- ICC corresponding to the velocity limitation $v \leq V_{lim}$:

$$\mathcal{H}(t) = \Big\{(\rho,\rho,F,-F) \mid \rho \text{ arbitrary}, \; F \leq V_{lim}\,\rho\Big\} \text{ (classical part)}$$
$$\bigcup\Big\{(\rho_L,\rho_R,F,-F) \mid \rho_L > \rho_R, \; F = V_{lim}\,\rho_L\Big\} \text{ (non-classical part)}$$

Case 1: If the "non-classical part" of $\mathcal{H}$ is empty
($\Longleftrightarrow$ the velocity limitation only conserns low densities).
The projected ICC is trivial (Kirchhoff) $\Longrightarrow$ standard LWR model.

Case 2: Otherwise, the highest allowed flux is $q_{lim} = V_{lim}\,\rho_*$
(ρ_* =the crossing point of $\rho \mapsto f()$ with the straight line $\rho \mapsto V_{lim}\rho$).
Straightforward computing of the projected (effective) ICC
$\Longrightarrow$ flux-limited at level q_{lim} Colombo-Goatin LWR model.

Conclusion (also supported by numerics with FTL [A.,Rosini'20]):
Velocity limitation $\Longleftrightarrow$ *ad hoc* flux limitation (may be trivial)

Velocity limitation is a flux limitation

Q: how to model localized velocity limitation?
[A.,Donadello,Rosini, in preparation] :

- ICC corresponding to the velocity limitation $v \leq V_{lim}$:

$$\mathcal{H}(t) \;=\; \Big\{(\rho,\rho,F,-F) \mid \rho \text{ arbitrary}, \; F \leq V_{lim}\,\rho\Big\} \; \text{(classical part)}$$
$$\bigcup\Big\{(\rho_L,\rho_R,F,-F) \mid \rho_L > \rho_R, \; F = V_{lim}\,\rho_L\Big\} \; \text{(non-classical part)}$$

Case 1: If the "non-classical part" of $\mathcal{H}$ is empty
($\Longleftrightarrow$ the velocity limitation only conserns low densities).
The projected ICC is trivial (Kirchhoff) $\Longrightarrow$ standard LWR model.

Case 2: Otherwise, the highest allowed flux is $q_{lim} = V_{lim}\,\rho_*$
(ρ_* =the crossing point of $\rho \mapsto f()$ with the straight line $\rho \mapsto V_{lim}\rho$).
Straightforward computing of the projected (effective) ICC
$\Longrightarrow$ flux-limited at level q_{lim} Colombo-Goatin LWR model.

Conclusion (also supported by numerics with FTL [A.,Rosini'20]):
Velocity limitation $\Longleftrightarrow$ *ad hoc* flux limitation (may be trivial)

Conclusions:

- in modeling with Discontinuous-Flux SCL,
  identification of Interface Coupling Conditions is essential
- examples where modeling provides simple formal ICC
- the formal ICC is projected according to an explicit procedure
- for specific classes of ICC, successful Finite Volume strategies

Open questions:

- other examples of ICC that appear in practice ?
- non-monotone (non-order-preserving) ICC ?
- extension of transmission strategies to some systems ?

Merci !