Flux-constrained conservation laws: modeling, analysis, many-particles' approximation

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based upon joint works with
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Conference on PDE/Probability Interactions:
Particle Systems, Hyperbolic Conservation Laws

Summary of the talk

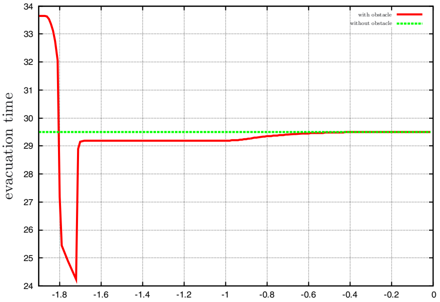
- Scalar LWR model : Kruzhkov entropy solutions and beyond
- Point modification of classical macroscopic road traffic models: modeling of bottlenecks on a road
- Elements of the mathematical theory of point-constrained models (local / nonlocal constraints, well-posedness, numerical schemes)
- Flux constraint versus velocity constraint
- Motivation via *deterministic* many-particles dynamics (FTL)
- Towards extension to systems of conservation laws

Appetitzer:
Non-monotonicity
in LWR-based traffic models
with bottlenecks

Numerical evidence: Braess’ paradox (e.g. pedestrian dynamics)

Braess’ Paradox: [A., Donadello, Razafison, Rosini ’16]

An obstacle is introduced at some distance upstream the bottleneck.
The position of the obstacle is optimized numerically.



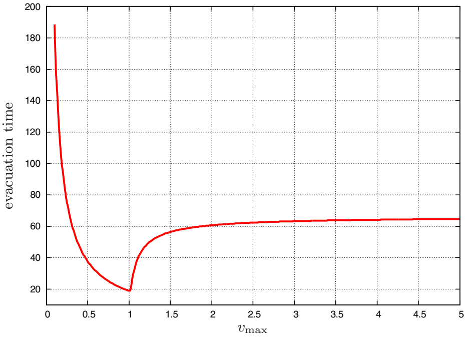
Evacuation time as function of the position of the obstacle

An obstacle may decrease exit densities and evacuation time !

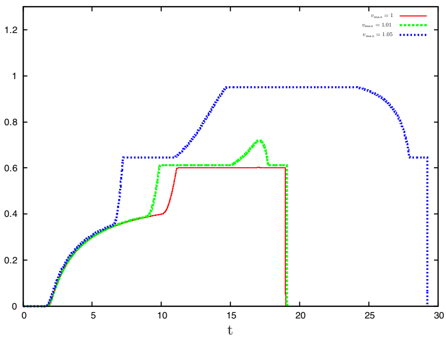
Numerical evidence: Faster Is Slower

Faster is Slower: [A., Donadello, Razafison, Rosini '16]

For same initial densities and constraint functions, we make vary the value of the maximal velocity V_{max} in the LWR model with bottleneck.



Evacuation time as function of V_{max}



Evolution of the traffic density at the exit

Increasing traffic velocity may not accelerate car evacuation !

Playground:
the classical LWR and ARZ models

LWR model

Macroscopic models of trafic

General principle of local conservation of vehicles on a road (modeled by the real line):
$$\rho_t + (v\rho)_x = 0$$
where at every point (t, x) of time-space ($t \geq 0$, $x \in \mathbb{R}$) the unknowns are (ρ, v):

ρ is density of trafic;
v is the traffic speed.

Different models are derived by specifying closure relations $\rho \mapsto v[\rho]$.

Lighthill-Whitham and Richards model (LWR)

The dependence $\rho \mapsto v$ is local, typically, $v = v(\rho) = V_{max}(\rho_{max} - \rho)$.
Outcome: LWR writes as
$$\rho_t + f(\rho)_x = 0,$$
with $f : [0, \rho_{max}] \to \mathbb{R}^+$, $f(\rho) = v(\rho)\rho$.
Typically, it is unimodal (“bell-shaped”), with $f(0) = 0 = f(\rho_{max})$.

Classical mathematical framework for LWR:
Kruzhkov entropy solutions (L^1 contractive, order-preserving solver)

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Shortcomings of LWR

LWR does not exhibit all observed kinds of traffic behavior. Moreover, it fits quite poorly experimental data.

Popular improvement of LWR: the “second-order” ARZ model obtained by enriching the closure relation: $v = v[\rho]$.

ARZ model

Aw-Rascle and Zhang model (ARZ)

The dependence $\rho \mapsto v$ is governed by a PDE,
via the relation $w = v + p(\rho)$. Here w is a “lagrangian marker”.
This means that w is merely transported along the flow: $w_t + vw_x = 0$.

Since $\rho_t + (v\rho)_x = 0$, this can be rewritten in conservative form:

$$(\rho w)_t + (v\rho w)_x = \rho(w_t + vw_x) = 0.$$

Nonlinearity p (“pressure”) is of the kind $p(\rho) = \rho^\gamma$, $\gamma > 0$.

ARZ model writes as $\begin{cases} \rho_t + (v\rho)_x = 0 \\ (\rho w)_t + (v\rho w)_x = 0, \end{cases} \quad w = v + p(\rho).$

Conservative variables are $(\rho, \rho w)$;
the most convenient variables are $(v, w) \in \mathbb{R}^+ \times \mathbb{R}^+$.

Nature of ARZ:
Hyperbolic system of conservation laws (except at vacuum $\rho = 0$).
Admissibility of solutions: via **Riemann solver** (unstable at vacuum).
Except at vacuum, admissibility characterized via entropy inequalities.

Classical entropy solutions of scalar conservation laws

Framework for analysis of LWR. Main aspects of the theory.

Kruzhkov entropy solutions are usually accepted to interpret LWR :

$$\forall k \in \mathbb{R} \;\; |\rho - k|_t + Q(\rho,k)_x \leq 0 \;\text{ in } \mathcal{D}', \quad Q(\rho,k) := \operatorname{sign}(\rho - k)(f(\rho) - f(k)).$$

Entropy solutions $\rightsquigarrow$ order-preserving contractive semigroup on $L^1(\mathbb{R})$:

$$\rho_0 \leq \widehat{\rho}_0 \quad \Longrightarrow \quad \forall t > 0 \;\; \rho(t,\cdot) \leq \widehat{\rho}(t,\cdot),$$
$$\|\rho(t,\cdot) - \widehat{\rho}(t,\cdot)\|_{L^1} \leq \|\rho_0 - \widehat{\rho}_0\|_{L^1}.$$

Admissibility encoded:
shocks in entropy solutions of LWR increase car density ($\rho_- \leq \rho_+$)
and decrease car velocity ($v(\rho_-) \geq v(\rho_+)$) : shock = sudden braking.

Riemann solver:
given a simple jump in initial datum, explicit solution can be given.
Serves to encode admissibility; to produce numerical schemes.

Questioning the vanishing viscosity motivation:
Fluid dynamics models: admissible solution = vanishing viscosity limit.
Not relevant in traffic, even if formally it gives correct result.

Questioning the Keyfitz(Quinn) theorem: [Quinn'71]
"Kruzhkov entropy solutions $\equiv$ the unique L^1-contractive semigroup":
FALSE !!! (True, under the implicit assumption "constants are solutions")

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Framework for analysis of LWR. Numerics. Boundary conditions.

Existence, Numerics : solutions of LWR can be obtained as limits of Finite Volume approximations

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\Big(F(\rho_i, \rho_{i+1}) - F(\rho_{i-1}, \rho_i) \Big),$$

with consistent and monotone numerical flux functions:

$F(k,k) = f(k)$ for all k, $F(\cdot, b)$ is $\nearrow$, $F(a, \cdot)$ is $\searrow$ for all a, b.

Godunov numerical flux F_{God}:
Derived from the Riemann solver for LWR (an explicit formula).

Dirichlet boundary conditions [Bardos,LeRoux,Nédélec'79]
equivalent to: given the formal Dirichlet datum ρ^{Diri} at $x = 0^-$,
the effective trace $\rho(t, 0^-) =: \gamma\rho$ fulfills

$$f(\gamma\rho) = F_{\mathrm{God}}(\gamma\rho, \rho^{Diri}).$$

General dissipative boundary conditions [A., Sbihi'15]
The requirement $(\rho^{Diri}, f(\rho)^{Neum}) \in \beta$ for a maximal monotone graph β
should be interpreted as $(\gamma\rho, f(\gamma\rho)) \in \tilde\beta$ where $\tilde\beta$ is the projection of β
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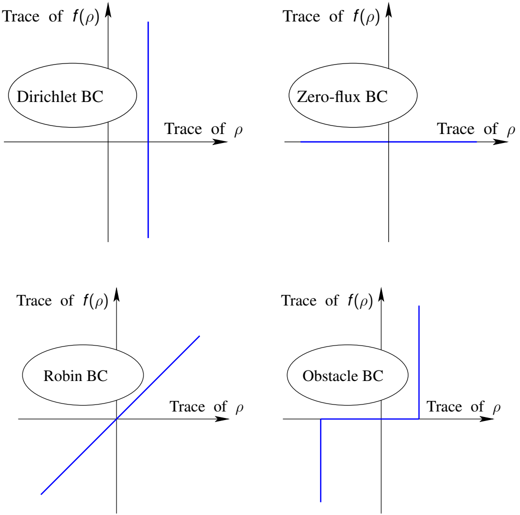
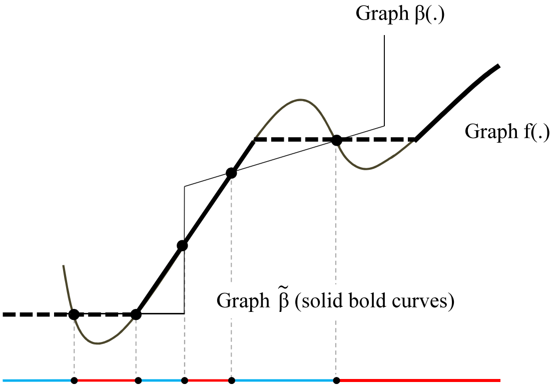
Classical boundary conditions



In all cases, $(\rho, f(\rho)) \in \beta$ for some maximal monotone graph β.

Example of projected graph



The projected graph $\tilde{\beta}$ in terms of F_{God}:

$$\tilde{\beta} = \left\{ (\gamma\rho, \mathcal{F}) \,\middle|\, \exists (\rho^{Diri}, \mathcal{F}^{Neum}) \in \beta \;\text{ s.t. }\; \mathcal{F} = \mathcal{F}^{Neum} = f(\gamma\rho) = F_{\mathrm{God}}(\gamma\rho, \rho^{Diri}) \right\}$$

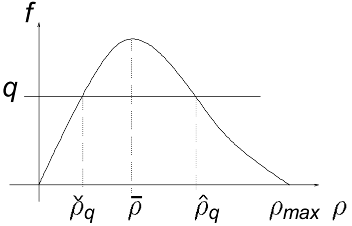
Modeling bottlenecks with point constraints in LWR

Colombo-Goatin model : LWR with point constraint on the flux

[Colombo-Goatin'07] model: in the context of road lights, pay tolls, small-scale construction sites, one may consider the formal model

$$\begin{cases} \rho_t + f(\rho)_x = 0 & \text{(LWR)} \\ f(\,\rho(t,0^\pm)\,) \le q(t) & \text{(point constraint at } x = 0) \end{cases}$$

where the map $t \mapsto q(t)$
(point constraint at $x = 0$, given beforehand)
prescribes the maximal possible value
of the car flow $f(\rho(t,0^-)) \equiv f(\rho(t,0^+))$.



Choice of Riemann solver at $x = 0$:
if the flow at $x = 0$ "wants to be" above q (for unconstrained LWR), then constrained $\rho(t,\cdot)$ jumps from $\rho_- = \hat{\rho}_q$ to $\rho_+ = \check{\rho}_q$, and $f(\rho_\pm) = q$.

Colombo-Goatin model: rigorous notion of solution

Analytical framework:

- Adapted entropy inequalities: standard ones in $\{\pm x > 0\}$, plus

$$|\rho - k(x)|_t + q(\rho, k(x))_x \leq 0 \ \text{ with } \ k(x) = \hat{\rho}_q \mathbf{1}_{x<0} + \check{\rho}_q \mathbf{1}_{x>0}.$$

These inequalities are Riemann-solver compatible, in particular

$\rightsquigarrow$ allow for the non-classical jump from $\hat{\rho}_q$ to $\check{\rho}_q$, at flow level $= q(t)$.

- Weak form of constraint "$f(\rho(t, 0^\pm)) \leq q(t)$":

$\forall \psi \in \mathcal{D}(\mathbb{R}_*^+ \times \mathbb{R}^-)$

$$\int_0^{+\infty} f(\rho(t, 0^-))\psi(t, 0)\, dt = \int_0^{+\infty}\int_{-\infty}^0 (\rho\psi_t + f(\rho)\psi_x)\, dtdx \leq \int_0^{+\infty} q(t)\psi(t, 0)\, dt$$

(weak formulation of LWR + Green-Gauss integration-by-parts used).

$\rightsquigarrow$ forbids classical jumps in ρ at interface with flow level $> q(t)$.

NB: adapted entropy inequalities + weak constraint "pass to the limit"

$\rightsquigarrow$ existence, if consistent approximation procedure + compactness

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Larger framework: general theory of discontinuous-flux SCL.
Theory: [A.,Karlsen,Risebro'11]; application: [A.,Goatin,Seguin'10]

Summary of theoretical results

- given $q(\cdot)$, existence of an admissible solution
- FV scheme with constrained numerical flux at interface $x = 0$

$$F_q(a,b) := \min\{\, F(a,b)\,,\, q\,\}, \quad \text{where } F \text{ is any classical (monotone, consistent with } f) \text{ numerical flux}$$

is convergent. The scheme is structure-preserving:
discrete L^1 contraction + order preservation hold.

- given $q(\cdot)$, L^1 contraction and order preservation ("counterexample" to the Keyfitz(Quinn) Theorem)
- Lipschitz dependence on $q(\cdot)$ w.r.t. the L^1 topology:

$$\|\rho(t,\cdot)-\widehat{\rho}(t,\cdot)\|_{L^1(\mathbb{R})} \le \|\rho_0-\widehat{\rho}_0\|_{L^1(\mathbb{R})} + 2\int_0^t |q(s)-\widehat{q}(s)|\,ds$$

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Modeling capacity drop :

LWR with non-local point constraint

Modeling capacity drop at the exit

Capacity drop and its avatars (Braess Paradox, Faster Is Slower)
Order-preservation: a key feature of LWR (even with point constraint).
Real road traffic / pedestrian flows: non-monotone behavior observed:
high density upstream the exit $\rightsquigarrow$ clogging $\rightsquigarrow$ small densities downstream.

Non-locally defined constraint [A., Donadello, Rosini '14]
One computes a subjective density $\xi(\cdot)$ upstream the exit $x = 0$:

$$\xi(t) = \int_{-\infty}^{0} w(x)\rho(t,x)\,dx \quad \text{where } w \geq 0, \int_{-\infty}^{0} w(x)\,dx = 1.$$

The weight w (assumed Lipschitz & compactly supported on $\mathbb{R}^-$)
and a nonlinearity (constraint function) $p(\cdot)$ define

$$\text{non-local point constraint} \quad q(t) := p(\xi(t)).$$

Capacity drop is modeled by a decreasing $p(\cdot)$.

Rosini et al. model = Colombo-Goatin model + non-local constraint:

$$\begin{cases} \rho_t + f(\rho)_x = 0 \\ f(\,\rho(t, 0^{\pm})\,) \leq q(t), \end{cases} \quad \text{where } q(t) = p\left(\fint \rho(t,\cdot)\,d\mu(\cdot)\right), d\mu(x) = w(x)dx.$$

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Well-posedness for Rosini et al. model

Monotonicity broken: the choice of decreasing $p(.)$ $\rightsquigarrow$ capacity drop.
Permits to reproduce Faster-Is-Slower, Braess paradoxes (cf. "Appetitzer")

Uniqueness and stability of the model: ok if $p(.)$ is Lipschitz continuous.

Technique: Lipschitz dependencies + Gronwall

· Lipschitz dependence w.r.t constraint level q:

$$\|\rho-\widehat{\rho}\|_{L^\infty((0,T);L^1(\mathbb{R}))} \;\leq\; 2\|q-\widehat{q}\|_{L^1(0,T)}$$

· Lipschitz dependence on the subjective density marker ξ:

$$|\xi(t)-\widehat{\xi}(t)| \;\leq\; \|w\|_\infty \|\rho(t,\cdot)-\widehat{\rho}(t,\cdot)\|_{L^1(\mathbb{R})}$$

· Lipschitz dependence of constraint level $q=p(\xi)$ w.r.t. ξ:

$$|q(t)-\widehat{q}(t)| \;\leq\; \|p\|_{Lip}|\xi(t)-\widehat{\xi}(t)|$$

· straightforward application of Gronwall inequality

Existence, numerics: via natural splitting procedures.
Alternative (A.,Donadello,Razafison,Rosini'18]) : use fixed-point techniques.
One can even get well-posedness by local in time use of Banach-Picard.

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Existence, numerics: via natural splitting procedures.
Alternative (A.,Donadello,Razafison,Rosini'18]) : use fixed-point techniques.
One can even get well-posedness by local in time use of Banach-Picard.

LWR with general nonlocal point constraints

Generalizations from [A.,Donadello,Razafison,Rosini'18],[A. Sylla'20?]

Nonlocal constraints: data acquisition, self-organization

· Inertia and memory effects: time non-locality

· Possibility to adapt passing capacity of a paytoll, etc. to observations of traffic: snapshots / video /... Space and time non-locality involved.

· Self-organization: in real (non-panic) flows – but not in Rosini et al. model – there is evidence of *self-organization* at partially clogged exit.

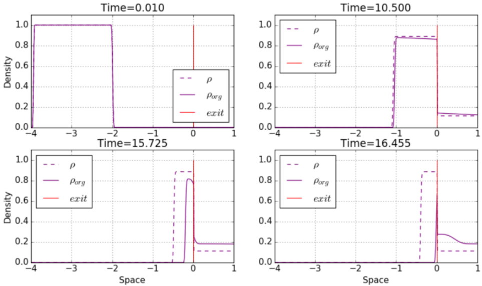

Fig.: Self-organization: basic Rosini et al. model (solid lines) compared to a toy model with switching to a higher level of exit capacity (dashed)

LWR model
with moving bottlenecks

LWR with moving bottleneck

Goal: couple a standard car flow to a "slow vehicle" (bus, truck)

Delle Monache - Goatin model [DelleMonache,Goatin'14],[Liard,Piccoli'19]

- Set the constraint on the flux across the trajectory $x = y(t)$ of the bus/truck: the road maximal capacity is reduced (say, by factor $\frac{m-1}{m}$, $m = \#$lines). Use Colombo-Goatin theory in $(t, \tilde{x})$ referential, $\tilde{x} := x - y(t)$.
- Relate $\dot{y}(t)$ to the instantaneous upstream density $\rho(t, y(t)^+)$
  $\rightsquigarrow$ very delicate analytical and numerical issues

Nonlocal version of the DM-G model [Sylla'20?]

- Relate $\dot{y}(t)$ to averaged value of upstream density $\int \rho(t,x)w(x-y(t))\,dx$
  $\rightsquigarrow$ Gronwall / splitting techniques for analysis and numerics; robust results.

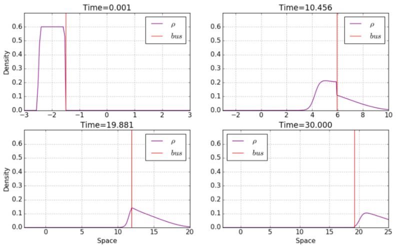

Fig.: An example with moving bottleneck model [Sylla'20?]

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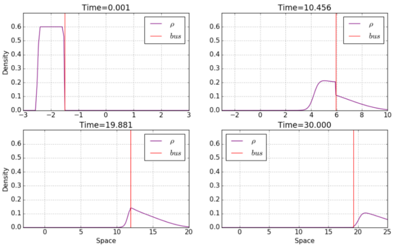

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Attempts to justify
the basic LWR+constraint model

Small-zone constraint: reductions to discontinuous-flux

Original justification: [Colombo,Goatin'07]
Consider "thick point" constraint, under the form

$$\rho_t + f(x,\rho)_x = 0, \quad f(x,\cdot) := f(\cdot)\mathbb{1}_{|x|>a} + \frac{q}{\|f\|_\infty} f(\cdot)\mathbb{1}_{|x|<a}$$

where $2a > 0$ is the constraint thickness.

Discontinuous-flux conservation laws theory can be used (at $x = \pm a$)
$\rightsquigarrow$ sequence of approx. solutions $(\rho_a)_a$; proof of convergence OK.

Modeling consistency questioned:
infinitely many mathematically consistent theories for discont.-flux conserv. laws ! [Adimurthi,Mishra,Gowda'07],[A.,Karlsen,Risebro'11].
Interpretation [A.'15]: wide choice of Interface Coupling Conditions.

The one used to justify LWR+constraint can be traced back to [Towers'00,'01, Bachmann,Seguin'03, Karlsen,Risebro,Towers'03,...]
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Velocity constraint versus flux constraint

Why <u>flux</u> constraint ?

· Flux is the quantity of interest for road engineers
(can be measured, caracterizes the flow)

· Appears as the natural quantity to constrain at a bottleneck...?

Mathematically, flux at a bottleneck is the trace $f(\rho(t, 0^{\pm}))$; while ρ is only L^{∞} in general, thanks to the PDE and Green-Gauss theorem,

· this trace is well defined (in the L^{∞} weak-* sense)

· it is robust (a.e. cv. $\rho_n \to \rho$ implies weak cv. of flux traces)

What about <u>velocity</u> constraint ?

· Is certainly natural (speed limitations on small portions of road)

· Attempts to construct a consistent Riemann solver and numerical experiments with Follow-The-Leader model (wait and see!) lead to

Conjecture: by a kind of boundary layer mechanism,
velocity constraint, once imposed, is relaxed (projected!)
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Dissipative Interface Coupling Conditions (ICC)

Analogy : One assimilates inner interface to a "double boundary"

Interface Coupling Conditions are expressed – like general BC – by

$$\Big((\rho_L, \rho_R)\,,\,(F_L, F_R) \Big) \;\in\; \mathcal{H} \subset \mathbb{R}^2 \times \mathbb{R}^2$$

where $\rho_{L,R}$ are the traces (left and right) of the solution ρ
and $F_{L,R}$ are the normal traces (left and right) of the flux $f(\rho)$.

The ICC is conservative if $\forall \big((\rho_L, \rho_R), (F_L, F_R)\big) \in \mathcal{H},\; F_L + F_R = 0.$

Order-preservation for PDE+ICC $\Longleftrightarrow$ "1-monotonicity" of $\mathcal{H}$

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Trivial example of ICC: Kirchhoff conditions

$$\mathcal{H} := \big\{ \big(\rho, \rho\,;\, F, -F\big) \in \mathbb{R}^2 \times \mathbb{R}^2 \,\big|\, \rho \in \mathbb{R},\; F \in \mathbb{R} \big\}.$$

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ICC mimicking the Colombo-Goatin Riemann solver:

$$\mathcal{H}(t) = \Big\{(\rho,\rho,F,-F) \mid \rho \text{ arbitrary}, \; F \le q_{lim}(t)\Big\} \text{ (the classical part)}$$
$$\bigcup\Big\{(\rho_L,\rho_R,F,-F) \mid \rho_L > \rho_R, \; F = q_{lim}(t)\Big\} \text{ (non-classical jumps)}$$

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Follow-the-leader
and adaptation to discontinuous flux

Follow-the-Leader model (deterministic many-particles approximation)

Microscopic modeling in traffic:

· a very natural approach (few agents, compared to fluid dynamics)
· flexibility for modeling: prescribing mechanistic car interaction laws

Follow-the-leader dynamics (a recent variant):

N cars on the road, ordered from right (position $x_1(t)$) to left (position $x_N(t)$). One sets ℓ_N=length of a car, so that $N\ell_N = const = \int_{\mathbb{R}} \rho_0\, dx$.

· The leader's dynamics is trivial: $\dot{x}_1 = V_{max} = v(0)$

· Followers' dynamics: $\dot{x}_{i+1} = v\left(\frac{\ell_N}{x_i - x_{i+1}}\right), \; i \in [\![1, N-1]\!]$

· Empiric FTL density:

$$\rho^N(t,\cdot) = \sum_{i=1}^{N-1} \frac{\ell_N}{x_i(t) - x_{i+1}(t)} \mathbf{1}_{]x_{i+1}(t), x_i(t)[}(\cdot)$$

· values $x_i(0)$, $i \in [\![1, N]\!]$, computed reciprocally from initial density ρ_0
$\rightsquigarrow$ FTL = deterministic ODE system (triangular $\Rightarrow$ easy to simulate!)

Convergence to LWR: [DiFrancesco,Rosini'15, Holden,Risebro'18]
As $N \to \infty$, ρ_N given by FTL converges a.e. to ρ given by LWR

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Adaptation of FTL to constraint (discon.-flux) setting

Tools of FTL-to-LWR convergence proof:

· Delicate *BV* control of $\rho_n(t,\cdot)$ [DiFrancesco,Fagioli,Rosini'16]
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Thick constraint approximation with flux OR velocity limitation:

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Numerical simulations with FTL (discontinuous flux case)

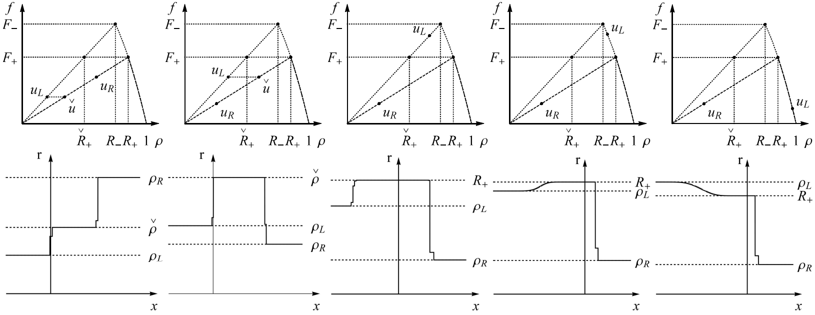

Figure ([A.,Rosini'19]):
FTL numerical approximation of discont.-flux LWR Riemann problems
⇝ the observed behavior is identical to the "optimal flux" Riemann solver

Convergence analysis: (work in progress)
identified difficulty: FTL is NOT order-preserving (counterexamples), unlike Vanishing Viscosity / monotone FV schemes / Wave Front Tracking.

Possibility: Lagrangian discont.-flux LWR ([Wagner'87] theorem,...)

Numerical simulations with FTL (discontinuous flux case)

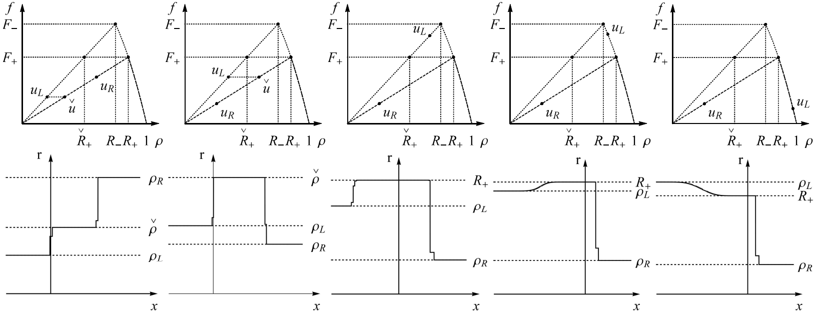

Figure ([A.,Rosini'19]):
FTL numerical approximation of discont.-flux LWR Riemann problems
⇝ the observed behavior is identical to the "optimal flux" Riemann solver

Convergence analysis: (work in progress)
identified difficulty: FTL is NOT order-preserving (counterexamples),
unlike Vanishing Viscosity / monotone FV schemes / Wave Front Tracking.

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ARZ and Phase Transition models

with discontinuous flux,

then with point constraints

Results on ARZ with point constraints, Program on FTL justification

ARZ and Phase Transition (LWR/ARZ) models w/constraint:
[Garavello,Goatin'11, Garavello,Villa'17, A.,Donadello,Rosini'16]
[Goatin'06, Colombo,Marcellini,Rascle'11,..., Marcellini'19]
[Benyahia et al.'17,18, DelSanto et al.'17,18] (Donadello&Rosini team)
$\rightsquigarrow$ plausible Riemann solvers proposed ; Front-Tracking constructions; moving bottlenecks ; specific entropy inequalities ($\rightsquigarrow$ analysis)

Difficulties:

· ad hoc modeling choices (multiple Riemann solvers proposed)
· interpretation of velocity constraint ?

Accurate modeling needed? We have a recipe!

· Trace issues back (Yeah??) to discontinuous-flux ARZ[1]
· Use Follow-the-Leader approximation
  – since it is easily adapted to ARZ ;
  – convergence OK for pure ARZ [DiFrancesco,Fagioli,Rosini'17]
  – FTL is easily implemented as (efficient) numerical method
  – permits to assess numerically "optimal-flux" solvers [Rosini'20?]

[1] Poorly addressed in the community (tools lacking) ! Occasion to learn something ?

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MERCI !

THANK YOU!