

Orthogonalization of POVMs

ANR ANCG meeting

June 17, 2021

Michael de la Salle

CNRS, ENS de Lyon

based on

arXiv: 2103.14126

(Ulam 60) Stability: "When objects that almost satisfy a property (P) are close to objects satisfying (P) exactly,"

Example: Lim's theorem

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \forall A, B \in M_d(\mathbb{C})_{\text{continuous}}$$

$$\text{if } \|A - B\|_F < \delta$$

$$\text{then } \exists A', B' \in M_d(\mathbb{C})$$

$$\text{s.t. } \|A - A'\| \leq \varepsilon$$

$$\|B - B'\| \leq \varepsilon$$

$$A' B' = B' A'$$

. Voiculescu not true for 3 matrices A, B, C

... (many other examples in group theory)

Today: If φ is Hilbert, a partition of unity (or POVM) is a family $t_1, \dots, t_n \in B(H)$

$$t_i \geq 0 \quad \sum_{i=1}^n t_i = I_H$$

, PVM : if additionally $t_i^2 = t_i \forall i$

Th (Kempe-Vidick 11, Ji-Natarajan-Vidick-Wright-Yuen 20)

$$\text{let } M = M_d(\mathbb{C}) \quad \varphi \text{ state on } M \quad (\varphi(\cdot) = T_\rho(\rho))$$

$$\rho \geq 0 \quad T_\rho(\rho) = 1$$

If $(a_1, \dots, a_m) \subseteq M_d(\mathbb{C})$ a POVM s.t.

$$\varphi(\sum a_i^2) \geq 1 - \varepsilon$$

then $\exists (\rho_1, \dots, \rho_m) \subseteq M_d(\mathbb{C})$ PVM

$$\sum_i \varphi(\|\rho_i - a_i\|^2) \leq 100 \varepsilon^{1/4}$$

Why do they care?

one of the ingredients in MIP = RE problem

a result in theoretical computer science that refutes Connes' embedding

CEP: every $\approx N$ algebra with finite tracial embedding on ultraproduct $\prod^\omega (M_d(\mathbb{C}), \frac{1}{d} T_d)$

Indeed, by

$$\left\{ \begin{array}{l} \text{- Kirchberg} \\ \text{- Fritz} \\ \text{- Junge - Narasimha - Palazuelos - Perez-Garcia} \\ \text{- Schmidt - Werner} \\ \text{- Ozawa} \end{array} \right\} \quad \left\{ \begin{array}{l} \varphi \text{ state on } B(H) \\ \forall i \in [m] \quad \varphi(a_i^2) \text{ PVM in } M \\ \forall i \in [m] \quad \varphi(b_i^2) \text{ PVM in } M \\ \forall i, j \in [m] \quad [\varphi(a_i^2), \varphi(b_j^2)] = 0 \end{array} \right\}$$

$$\text{CEP} \iff \forall m, n$$

$$\left\{ \begin{array}{l} \left(\varphi(a_i^2, b_j^2) \right)_{i, j \in [m]} \mid \varphi \text{ state on } B(H) \\ \forall i, l \in [m] \quad \varphi(a_i^2) \text{ PVM in } M \\ \forall l, j \in [m] \quad \varphi(b_l^2) \text{ PVM in } M \\ \forall i, l, r, j \in [m] \quad [\varphi(a_i^2, b_l^2), \varphi(a_r^2, b_j^2)] = 0 \end{array} \right\}$$

$$\left\{ \text{some with dim } M < \infty \subseteq \mathbb{R}^{m \times m^2} \right\}$$

Question (Vidick - Yuen) Is this true in ∞ dim?

Answer: Yes:

Th (ALS 21) Let (M, φ) von Neumann with a normal state

$$\text{let } (a_1, \dots, a_m) \text{ a POVM in } M \text{ s.t. } \varphi(\rho_i - a_i^2) \geq 1 - \varepsilon$$

then $\exists (\rho_1, \dots, \rho_m)$ PVM in M

$$\text{s.t. } \sum \varphi(\|\rho_i - a_i\|^2) \leq 5\varepsilon$$

RQ: important: $\{\rho_i\} \subseteq \{a_i\}''$

$$\varepsilon^{1/4} \sim \varepsilon$$

works in ∞ dim

Corollary: (M, φ) as above
 $(p_1, \dots, p_n) (q_1, \dots, q_m) \underset{\cong}{\equiv} \text{PVM in } M$

$$\text{If } \sum_{i,j} \varphi(|p_i \cdot q_j - q_j \cdot p_i|^2) \leq \varepsilon$$

then $\exists (p'_1, \dots, p'_m) \text{ PVM in } M$

$$\text{st} \quad \sum_i \varphi(|p_i - p'_i|^2) \leq 10\varepsilon$$

$$- p'_i \cdot q_j = q_j \cdot p'_i \quad \forall i, j$$

$$\begin{aligned} \text{Proof: } & \sum_{i,j} \varphi(|p_i \cdot q_j - q_j \cdot p_i|^2) \\ &= \sum_{i,j} \varphi(p_i \cdot q_j \cdot p_i + q_j \cdot p_i \cdot q_j) - 2\varphi(p_i \cdot q_j \cdot p_i \cdot q_j) \\ &= 2(1 - \varphi(\sum_i \varphi(p_i \cdot a_i))) \\ \text{where } a_i &= \sum_j q_j \cdot p_i \cdot q_j \\ &= \sum_i \varphi(|p_i - a_i|^2) + 1 - \varphi(\sum_i a_i^2) \leq \varepsilon \end{aligned}$$

by Re Th $\exists p'_i \in \{a_i\} \subseteq \{q_j\} \text{ PVM}$

$$\therefore \sum_i \varphi(|a_i - p'_i|^2) \leq 9\varepsilon$$



Proof of Th : assume M is a II_1 factor, τ
 (but φ not nec. a trace!)

Recall $\begin{cases} (\alpha_1 \dots \alpha_m) \\ \varphi(\sum \alpha_i^2) \geq 1-\varepsilon \end{cases}$ POV τ in M

Lemma : $\exists q_1, \dots, q_m$ projections in M

$$\text{or } \begin{cases} \varphi(\sum q_i \alpha_i) \geq 1-\varepsilon \\ q_i \alpha_i = \alpha_i q_i \quad \forall i \\ \sum_i \varphi(q_i) = 1 \end{cases} \quad \boxed{\Delta \sum q_i \neq 1}$$

Proof : Consider

$$C = \left\{ (x_1, \dots, x_m) \in M^m \mid 0 \leq x_i \leq 1, x_i \alpha_i = \alpha_i x_i \quad \forall i, \right. \\ \left. \sum \varphi(x_i) = 1 \right\}$$

. (convex ω^* -closed $\Rightarrow C = \overline{\text{Conv}}(\text{Ext}(C))$)

$$\cdot f: C \rightarrow \mathbb{R}^+ \quad f(x_1, \dots, x_m) = \varphi(\sum x_i \alpha_i)$$

affine ω^* -cont.

$$f(\tilde{\alpha}_1, \dots, \tilde{\alpha}_m) = \varphi(\sum \tilde{\alpha}_i^2) \geq 1-\varepsilon$$

$$\Rightarrow \exists (x_1, \dots, x_m) \in \text{Ext}(C) \quad \text{or} \quad f(x_1, \dots, x_m) \geq 1-\varepsilon$$

Claim : $(x_1, \dots, x_m) \in \text{Ext}(C) \Rightarrow x_i = x_i^2 \quad \forall i$

imaged: otherwise $\exists \delta > 0$ $\wedge p := \chi_{[\delta, 1-\delta]}^{(x_i)} \neq 0$

$$\text{Take } b \in M^{(\sum \alpha_i)}, \quad p b = b_p = b \quad (b \in p M p)$$

$b \notin p M p$

$$x_i = \frac{1}{2} [x_i + \sqrt{x_i^2 + (x_i - \sqrt{b})}] \quad -p \leq \sqrt{b} \leq p \quad \boxed{b \neq 0}$$

The Theorem from functional calculus in $M_n(M)$
 (was II_1 factor by:

$$\forall c \in M \quad p, q \text{ project in } M \quad \text{or} \quad \begin{cases} p n = q = n \\ \varphi(p) = \varphi(q) \end{cases}$$

$$\Rightarrow x = u(n) \quad \text{with} \quad u^* u = q \quad u u^* = p$$