Solutions of Buckley-Leverett equation in heterogeneous medium and their approximation

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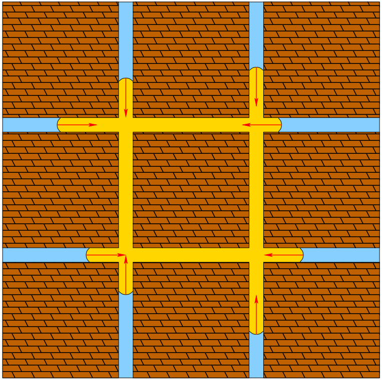
Microscopic motivations

Consider an immiscible incompressible two-phase flow (e.g. oil and water) within a homogeneous porous medium Ω :

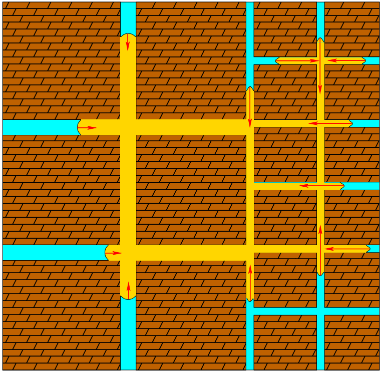


- The interface between the wetting phase (blue) and the non-wetting phase (yellow) is not flat, because of surface tension.
- The capillary force is related to the radius of the pore channels : the smaller is the radius, the greater is the capillary pressure.

$\rightsquigarrow$ In a nearly homogeneous porous medium, capillarity has only a small influence on the relative motion of the phases.

Microscopic motivations

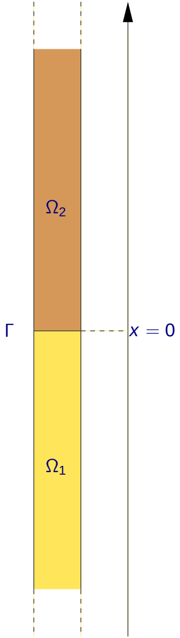
Consider now the same fluid in a heterogeneous porous medium :



- The resulting capillary force can not be neglected at the interface between the two different subdomains.

⇝ In a heterogeneous porous medium, capillarity plays a major role in the phase displacements.

One-dimensional two-rocks' medium



Consider a heterogeneous porous medium $\Omega = \mathbb{R}$, made of two homogeneous subdomains $\Omega_1 = (-\infty, 0)$ and $\Omega_2 = (0, +\infty)$, separated by an interface $\Gamma = \{x = 0\}$.

$$\phi_i \partial_t s + \partial_x \left(f_i(s) - \partial_x \varphi_i(s) \right) = 0, \tag{1}$$

where s denotes the saturation of one phase, q is the total flow rate and

$$f_i(s) = \underbrace{q \frac{\eta_{o,i}(s)}{\eta_{o,i}(s) + \eta_{w,i}(s)}}_{\text{global convection}} + \underbrace{(\rho_w - \rho_o) K_i \frac{\eta_{o,i}(s)\eta_{w,i}(s)}{\eta_{o,i}(s) + \eta_{w,i}(s)}}_{\textit{buoyancy}}$$

$$\varphi_i(s) = \int_0^s \frac{\eta_{o,i}(r)\eta_{w,i}(r)}{\eta_{o,i}(r) + \eta_{w,i}(r)} \pi_i'(r)\, dr.$$

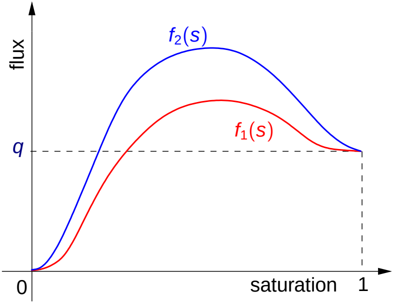
Typical behavior of the flux functions f_i

(A1) The functions f_i are Lipschitz continuous and compatible, i.e.

$$f_1(0) = f_2(0) = 0, \quad f_1(1) = f_2(1) = q.$$

(A2) The functions f_i are bell-shaped, i.e. there exist $\overline{s}_i \in [0, 1]$ s.t.

$$f_i'(s)(\overline{s}_i - s) > 0 \text{ a.e. in } (0, 1).$$



Transmission conditions at the interface

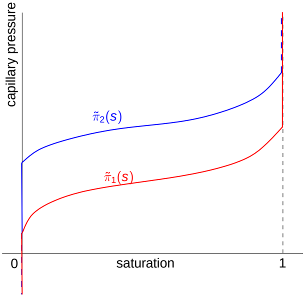
At the interface $\Gamma = \{x = 0\}$, one requires :

- continuity of the oil flux :

$$[f_1(s) - \partial_x \varphi_1(s)]_{|x=0} = [f_2(s) - \partial_x \varphi_2(s)]_{|x=0} \;;$$

- continuity of the capillary pressure [3]

$$\pi_1(s_1) = \pi_2(s_2).$$



Initial condition : One sets

$$s_{|t=0} = s_0 \in L^\infty(\mathbb{R}), \qquad 0 \leq s_0 \leq 1.$$

3. generalization in [Buzzi-Lenziger-Schweizer'09], [Cancès-Gallouët-Porretta'09], [Cancès-Pierre'12]

Existence/Uniqueness of the solution

Theorem ([Cancès 09'])

- *Given $s_0 \in L^\infty(\mathbb{R}; [0,1])$, there exists a unique mild solution s with $0 \le s \le 1$ to the one-dimensional problem.*
- *Let $s_0, \tilde{s}_0$ two initial data, and let $s, \tilde{s}$ their corresponding mild solution, then*

$$\|s(\cdot,t) - \tilde{s}(\cdot,t)\|_{L^1_\phi(\mathbb{R})} \le \|s_0 - \tilde{s}_0\|_{L^1_\phi(\mathbb{R})}, \qquad \forall t > 0.$$

- *The unique mild solution is computable by the mean of an implicit finite volume scheme, which is proved to be convergent.*

IDEA: The flux between two cells is discretized with

- the Godunov scheme for the convection and a usual three point approximation for the diffusion if the edge between the two cells is in Ω_i ;
- the introduction of additional variables at the interface [4], so that the continuity of the flux and of the capillary pressure can be prescribed.

$\rightsquigarrow$ At each time step, one has to solve a nonlinear problem by an iterative method. Each iteration requires the resolution of a (small) non-linear problem. The solution is thus quite **expensive to compute**.

4. [Enchéry-Eymard-Michel'06]

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Numerical results

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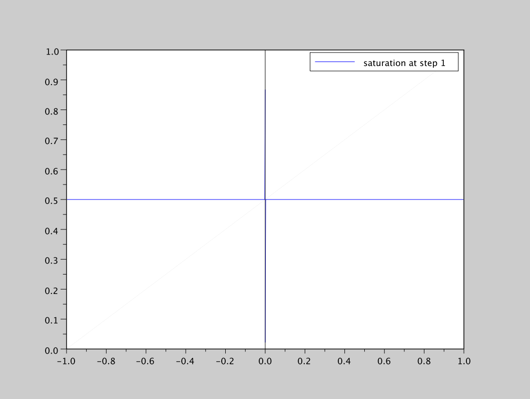

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- The discontinuity of the medium yields a discontinuity of the saturation at the interface.
- Two waves are leaving the interface.

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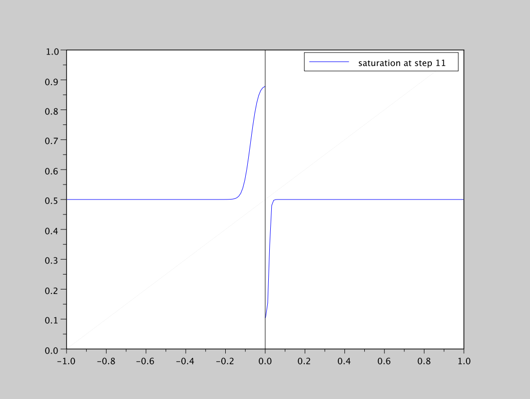

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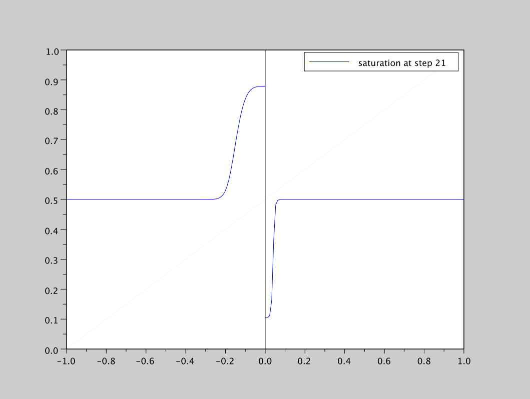

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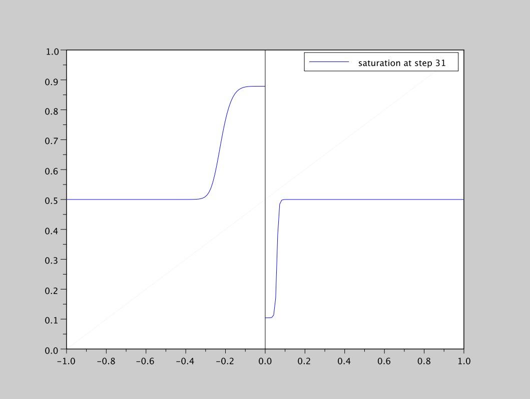

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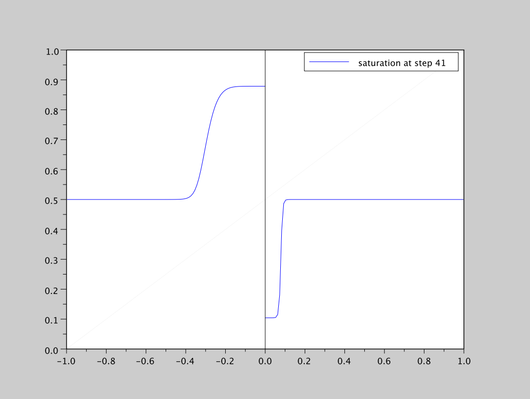

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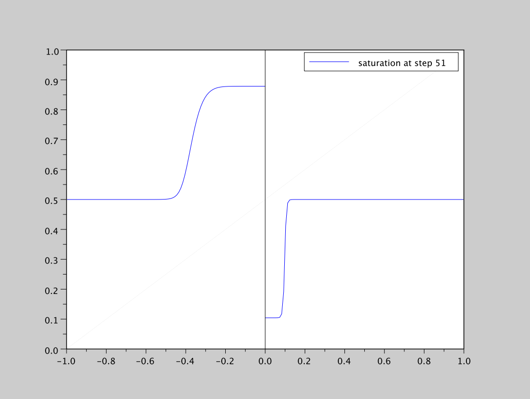

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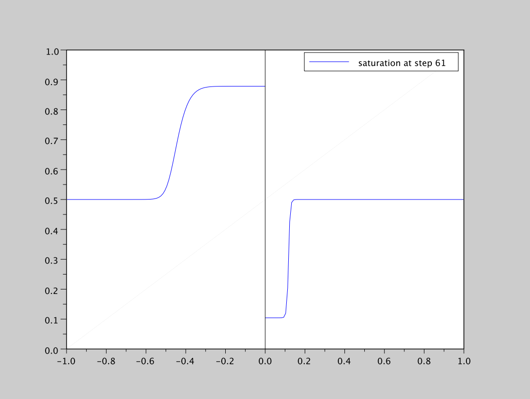

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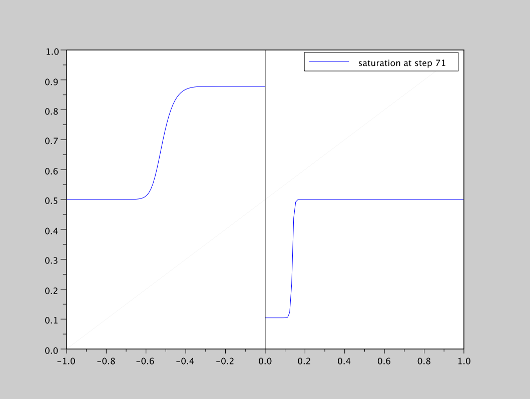

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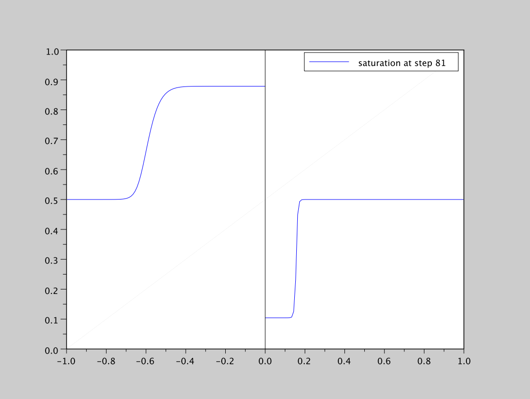

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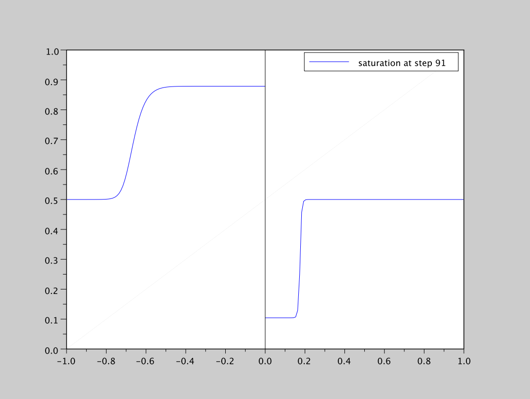

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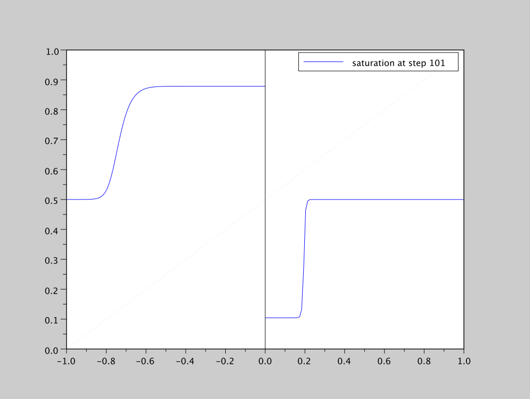

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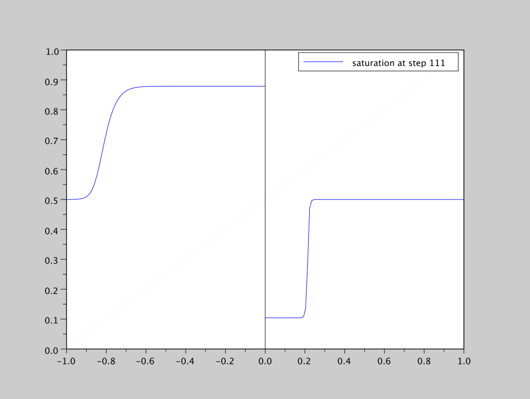

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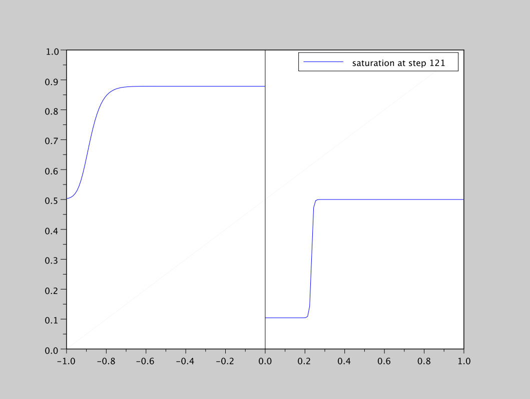

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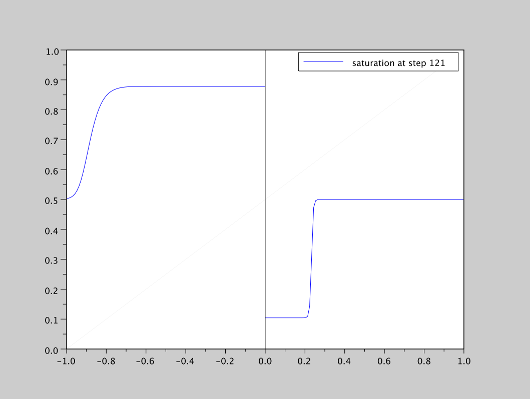

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Neglecting the capillary diffusion

As stressed before, the capillary forces have a small influence within Ω_i, but a large influence at the interface $\Gamma = \{x = 0\}$. By a convenient scaling $\left(t := \frac{t}{\varepsilon}, x := \frac{x}{\varepsilon}\right)$, one gets

$$\begin{aligned}
&\phi_i \partial_t s^\varepsilon + \partial_x f_i(s^\varepsilon) - \varepsilon \partial_{xx} \varphi_i(s^\varepsilon) = 0, && \text{in } \Omega_i \times (0,T),\\
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More precisely, denoting by $\Phi_i(a,b) = \operatorname{sign}(a-b)(f_i(a) - f_i(b))$, the solution s satisfies, for all $\kappa \in [0,1]$, the Kruzhkov entropy inequalities away from the interface[5] :

$$\phi_i \partial_t |s - \kappa| + \Phi_i(s,\kappa) \leq 0 \quad \text{in} \quad \mathcal{D}'(\Omega_i \times (0,T)) \tag{2}$$

$$f_1(s_1) = f_2(s_2) \quad \text{on} \quad \Gamma \times (0,T). \tag{3}$$

But : the connection of capillary pressures may not be inherited at the limit !

5. This means that constants are "evident solutions" of *homogeneous* scalar conservation law and that all other solutions verify a (localized) L^1 contraction property with respect to these constant solutions

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Overcompressive and undercompressive discontinuities

Let u be the Kruzhkov entropy solution of

$$\partial_t u + \partial_x f(u) = 0, \qquad u_{|t=0} = u_0.$$

- Even for smooth initial data u_0, the solution u can become discontinuous after a finite time.
- If the solution is discontinuous, then the Rankine-Hugoniot condition is satisfied :

$$\sigma = \frac{f(u^+) - f(u^-)}{u^+ - u^-}.$$

- The shocks are overcompressive, i.e. they only destroy information, due to the Lax condition :

$$f'(u^-) \geq \sigma \geq f'(u^+).$$

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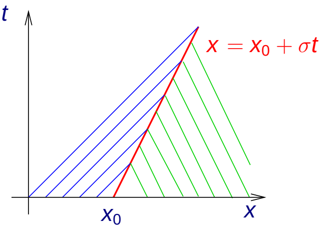


Compressive and undercompressive discontinuities

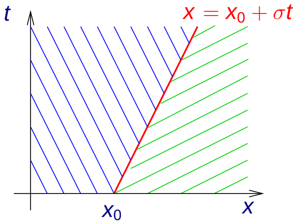
For physical reasons, it can be relevant to allow the generation of information at some $x_0 \in \mathbb{R}$:

- **traffic flow :** lights, change of the road size, ... [Andreianov, Goatin, Seguin '10]
- **fluid mechanics :** small particle in an inviscid fluid [Andreianov, Seguin '12], [Andreianov, Lagoutière, Takahashi, Seguin '10 and '13+ε]
- **porous media flow :** change of the rock type, [Kaasschieter '99], [Adimurthi, Veerappa Gowda with Jaffré, Mishra, '04-'05...], [Andreianov-Cancès '13]

We will have to consider undercompressive shocks ! ! !

One might have (actually, only for one well-chosen couple (u_-, u_+)...)

$$f'(u^-) < \sigma(= 0) < f'(u^+).$$



Scalar conservation laws with discontinuous flux

There exists **infinitely many solutions** [6] to the problem (2)–(3). We have to select the relevant one. To do so, we use the frame of [Bürger-Karlsen-Towers '09], [Andreianov-Karlsen-Risebro '11] based on adapted entropy inequalities [7].

The idea is to recognize admissible solutions of the form $A\mathbf{1}_{\Omega_1} + B\mathbf{1}_{\Omega_2}$ that would replace Kruzhkov constants κ, and use them to define relative (adapted) entropies.

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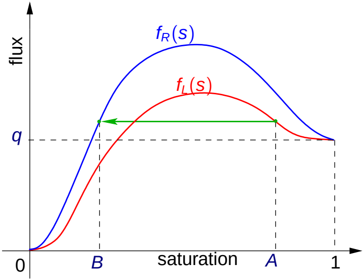

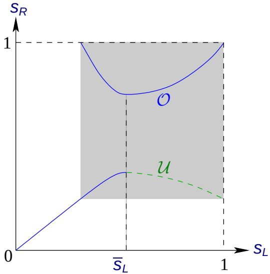

Define the set $\mathcal{U}$ of the under-compressive shocks and by $\mathcal{O}$ the set of the over-compressive (or "regular" shocks) satisfying the conservativity condition :

$$\begin{aligned}
\mathcal{U} &= \left\{(A,B)\in[0,1]^2 \mid f_1(A)=f_2(B), A\geq \overline{s}_1 \text{ and } B\leq \overline{s}_2\right\},\\
\mathcal{O} &= \left\{(A,B)\in[0,1]^2 \mid f_1(A)=f_2(B), A< \overline{s}_1 \text{ or } B> \overline{s}_2\right\}.
\end{aligned}$$

6. [Adimurthi-Mishra-Gowda '05]
7. [Audusse, Perthame '05]

Characterization by the connection

Let us state the claims of [Andreianov-Karlsen-Risebro '11] that are useful here.

- The problem (2)–(3) admits infinitely many solutions, corresponding to infinitely many contraction semi-groups, i.e. such that

$$\|s(t) - \tilde{s}(t)\|_{L^1_\phi} \leq \|s_0 - \tilde{s}_0\|_{L^1_\phi}, \quad \forall t > 0;$$

- There is a bijection between the set $\mathcal{S}$ of the contraction semi-groups and the set $\mathcal{U}$ of the under-compressive shocks

  $\forall (A,B) \in \mathcal{U}$, a unique solution semigroup admits $A\mathbf{1}_{\Omega_1} + B\mathbf{1}_{\Omega_2}$ as a stationary solution.

  $\forall S \in \mathcal{S}$, a unique couple $(A,B) \in \mathcal{U}$ is such that $A\mathbf{1}_{\Omega_1} + B\mathbf{1}_{\Omega_2}$ is a stationary solution.

- Let $(A,B) \in \mathcal{U}$, and let s be some trajectory of the associated semigroup $S_{(A,B)}$, then its traces (s_1, s_2) on $\{x = 0\}$ belong to

$$\mathcal{G}^*_{(A,B)} := \{(s_1, s_2) \mid f_1(s_1) = f_2(s_2) \text{ and } \Phi_2(s_2, B) - \Phi_1(s_1, A) \leq 0\}$$

  In particular, the only solutions $A^*\mathbf{1}_{\Omega_1} + B^*\mathbf{1}_{\Omega_2}$ are such that $(A^*, B^*) \in \mathcal{G}^*_{(A,B)}$.

Heuristically : the "maximal germ" $\mathcal{G}^*_{(A,B)}$ contains all information on the chosen semigroup ; and the "definite germ" $\mathcal{G}_{(A,B)} := \{(A,B)\}$ is the minimal information.

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Define $\bar{F}^{\text{opt}}$ by

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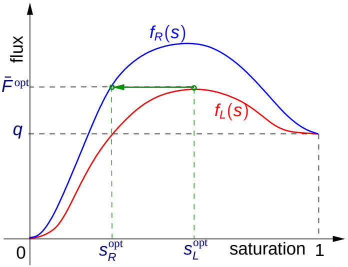

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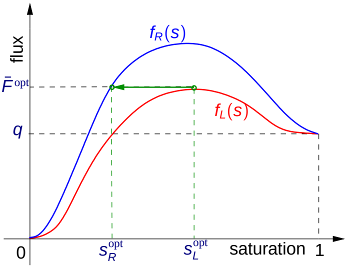

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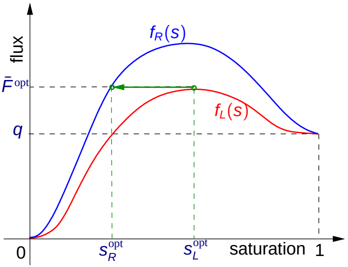

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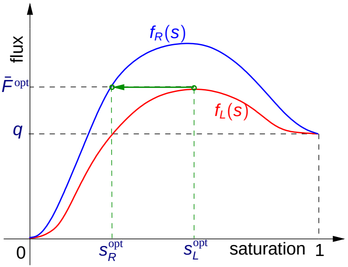

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Determination of the good connection

Due to the bijection between couples $(A,B)\in\mathcal{U}$ and L^1-contractive semigroups of solutions, in the place of the painstaking construction of vanishing capillarity profiles for all possible jumps ([Kaasschieter '99]) we can look only at the jumps with states $(u(0^-),u(0^+))\in\mathcal{U}$. And there is only one such jump !

Theorem ([Andreianov-Karlsen-Risebro'11], [Andreianov-Cancès '13])

Given capillary pressure curves π_1,π_2, $(A,B)\in\mathcal{U}$ is the relevant connection if there exists a steady state $\tilde{s}^\varepsilon(x)$ to the problem with small capillary diffusion such that

$$\tilde{s}^\varepsilon(x)\to A\mathbf{1}_{\Omega_1}(x)+B\mathbf{1}_{\Omega_2}(x)\quad \textit{for a.e. } x\in\mathbb{R} \textit{ as } \varepsilon\to 0.$$

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Denote by $\mathcal{U}$ the set of the under-compressive shocks

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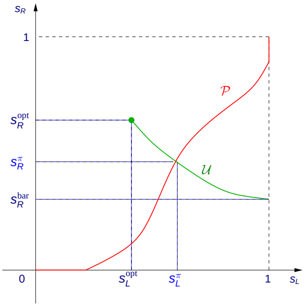

Remark: The vanishing capillarity profile depends (in a continuous way) on the shape of the capillary pressure functions ! ! !

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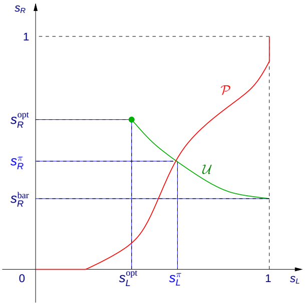

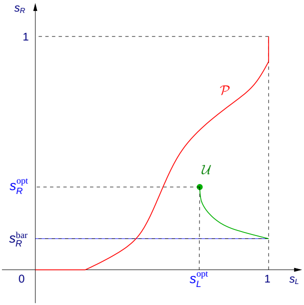

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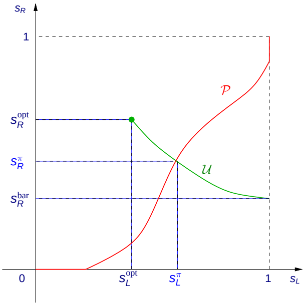

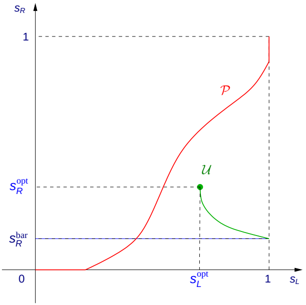

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General principles of finite volume approximation for conservation laws

Finite volumes provide piecewise constant approximation of solution, which is advantageous because shocks may appear sharply. For homogeneous scalar conservation law

$$\partial_t u + \partial_x f(u) = 0$$

the main ingredient of a stable, convergent FV scheme is the numerical flux $u_K, u_L \mapsto F(u_K, u_L)$ needed to approximate $f(u)$ on the boundary $K|L$ between two volumes K, L with values u_K, u_L of the approximated solution u.

The following is required on the map $F(\cdot,\cdot)$:

- consistency : $F(a,a) = f(a)$
- Lipschitz dependence on its arguments (allows for an explicit scheme under a CFL condition)
- monotonicity : $a \mapsto F(a,b)$ is $\nearrow$ and $b \mapsto F(a,b)$ is $\searrow$ (this adds "numerical diffusion", and thus ensures entropy inequalities)

Example:
if $f(\cdot) = f_{\nearrow}(\cdot) + f_{\searrow}(\cdot)$ then $F(a,b) = f_{\nearrow}(a) + f_{\searrow}(b)$ is a good numerical flux.

The only modification to standard schemes we have to apply for our case is :

find a suitable numerical flux $F_{int}(\cdot,\cdot)$ at the interface $\{x = 0\}$.

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- monotonicity : $a \mapsto F(a,b)$ is $\nearrow$ and $b \mapsto F(a,b)$ is $\searrow$
  (this adds "numerical diffusion", and thus ensures entropy inequalities)

EXAMPLE:
if $f(\cdot) = f_\nearrow(\cdot) + f_\searrow(\cdot)$ then $F(a,b) = f_\nearrow(a) + f_\searrow(b)$ is a good numerical flux.

The only modification to standard schemes we have to apply for our case is :

find a suitable numerical flux $F_{int}(\cdot,\cdot)$ at the interface $\{x = 0\}$.

For the optimal connection (A^{opt}, B^{opt}) the answer is well known [Adimurthi, Jaffré, Veerappa Gowda '04] for the most precise numerical flux (Godunov flux) :

$$F^{opt}(a,b) = \min\Big\{ f_1(\min\{\bar{s}_1, a\}), f_2(\max\{\bar{s}_2, b\}) \Big\}.$$

What can be done in the general case ?

General principles of finite volume approximation for conservation laws

Finite volumes provide piecewise constant approximation of solution, which is advantageous because shocks may appear sharply. For homogeneous scalar conservation law

$$\partial_t u + \partial_x f(u) = 0$$

the main ingredient of a stable, convergent FV scheme is the numerical flux $u_K, u_L \mapsto F(u_K, u_L)$ needed to approximate $f(u)$ on the boundary $K|L$ between two volumes K, L with values u_K, u_L of the approximated solution u.

The following is required on the map $F(\cdot,\cdot)$:

- consistency : $F(a,a) = f(a)$
- Lipschitz dependence on its arguments
  (allows for an explicit scheme under a CFL condition)
- monotonicity : $a \mapsto F(a,b)$ is $\nearrow$ and $b \mapsto F(a,b)$ is $\searrow$
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Observation : Flux limitation at the interface

Given $(A,B) \in \mathcal{U}$, we calculate $\mathcal{G}^*_{(A,B)}$ and find that

$$\forall (A^*,B^*) \in \mathcal{G}^*_{(A,B)} \;\; f_1(A^*) = f_2(B^*) \leq \bar{F}_{(A,B)} := f_1(A) = f_2(B)\,;$$

thus (A,B) prescribes the maximal level of flux at the interface.

NB : for this reason, the Buckley-Leverett equation in two-rocks' medium is able to model the oil trapping phenomenon, i.e., the situation where the interface acts as a barrier for the exchange of phases.

IDEA: ([Andreianov, Goatin, Seguin '10], in the context of constrained road traffic)
For numerical approximation, simply apply the flux limitation on the top of the optimal flux of [Adimurthi, Jaffré, Veerappa Gowda '04] :

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And this works !

ALGORITHM:

- calculate the flux constraint value $\bar{F}_{(A,B)}$ from π_1, π_2 ;
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Alternative : introduction of "pressure transmission variables" at the interface

We mimic, at the numerical level, the properties of vanishing capillarity approximation : flux conservation and connection of capillary pressures. To do so, we introduce additional "pressure unknown" p at the interface used to connect everything :

$$F_{int}(a,b) = \text{the common value } F_1(a,\pi_1^{-1}(p)) = F_2(\pi_2^{-1}(p),b),$$

which is actually a nonlinear equation on $p \in \mathbb{R}$. This equation has an (almost) unique solution thanks to monotonicity of $F_1(\cdot,\cdot)$, $F_2(\cdot,\cdot)$.

This means that :
- we connect a to some s_1 by the standard scheme on the left of the interface ;
- we connect pressures at the interface : $\pi_1(s_1) = \pi_2(s_2)$;
- then we connect the resulting s_2 to b by the standard scheme on the right from the interface.

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ALGORITHM:
- introduce the additional interface unknown p and solve the equation for p ;
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Disadvantage : it works slower than flux limitation ;
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The phase-by-phase upstream scheme... does converge to the right solution !

The phase-by-phase upstream scheme (see, e.g., [Brenier, Jaffré '91]) is frequently used in practice of petroleum engineering. Is it compatible with medium heterogeneity ? Interpretations of [Jaffré, Mishra '10],[Tveit, Aavatsmark '12] of the vanishing capillarity analysis of [Kaasschieter '99] led to the following important conclusion :

? ...the phase-by-phase upstream scheme yields a wrong solution... ?

Firstly, we stress that these results rely on the erroneous conclusion that the right vanishing capillarity limit is always the optimal solution.

Secondly, there is no unique extension of this scheme to two-rocks' medium ; we suggest to use the "interface pressure unknown" p in order to obtain the interface flux corresponding to capillary pressure curves π_1, π_2.

Theorem ([Andreianov-Cancès '13+ϵ])

The phase-by-phase upstream scheme corresponding to this choice of interface flux converges under a CFL condition towards the vanishing capillarity solution.

Sketch of the proof: The proof consists in three observations :

- the scheme is L^1 contractive at the discrete level ($\Rightarrow$ semigroups S^h, $h > 0$)
- for each datum (at least, BV), it converges up to a subsequence ($\Rightarrow$ limit semigroup S which inherits L^1 contraction)
- the limits of the scheme are Kruzhkov entropy solutions away from the interface...
- ...and the scheme does approximate the correct (with respect to π_1, π_2) stationary solution $A\mathbf{1}_{\Omega_1} + B\mathbf{1}_{\Omega_2}$, which permits to identify the semigroup S.

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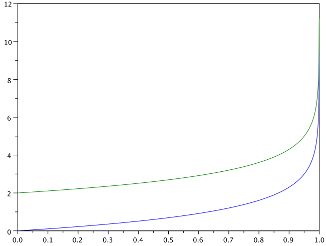
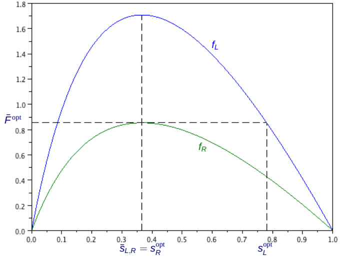
Numerical illustrations and comparison to small capillarity model

NUMERICAL TEST CASE :

$$\pi_i(s) = P_i - \ln(1-s),$$



(a) Capillary pressures



(b) Flux functions f_i

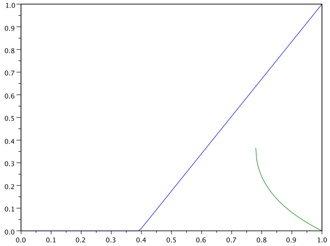
For the initial datum, we will take the constant state $u_0 = 1/2$
(we will see a non-constant solution : this underlines the active role of the interface between the two rocks !)

Numerical illustrations and comparison to small capillarity model

Numerical test case :

$$\pi_i(s) = P_i - \ln(1-s),$$

Test case 1 : $P_1 = 0$, $P_2 = 0.5$



The sets $\mathcal{P}$ and $\mathcal{U}$

Here we should obtain the optimal entropy solution

Numerical illustrations and comparison to small capillarity model

NUMERICAL TEST CASE :

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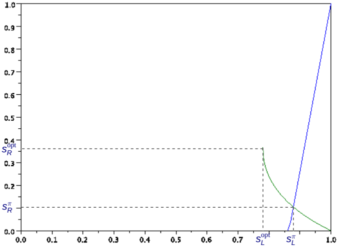
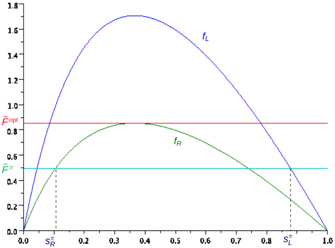
Test case 2 : $P_1 = 0$, $P_2 = 2$



(c)



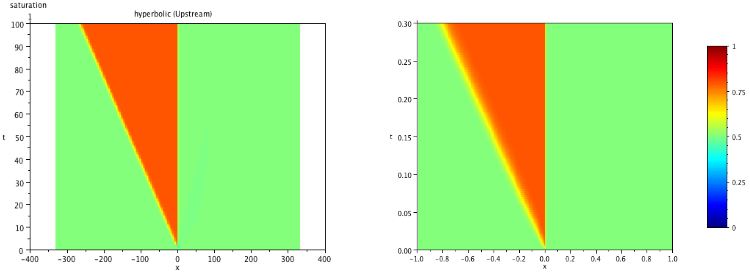
(d) Connection (s_1^{π}, s_2^{π})

Here we should obtain an entropy solution with flux limitation

Numerical results in the optimal case

We compare with the numerical solution corresponding to s^ε for $\varepsilon = 10^{-3}$.

- s is approximated by the explicit Godunov scheme
- s^ε approximated by a convergent asymptotic preserving implicit scheme [9]



(e) Solution s_h to the hyperbolic problem

(f) Solution s_h^ε to the parabolic problem ($\varepsilon = 10^{-3}$)

9. [Cancès '09], [Andreianov-Cancès '13]

Numerical results in the non-optimal case

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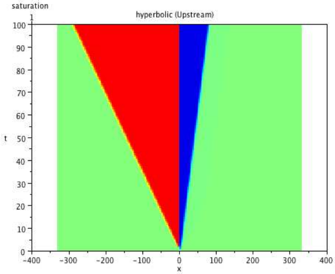
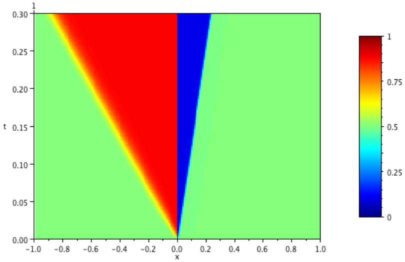


(g) Solution s_h to the hyperbolic problem



(h) Solution s_h^ε to the parabolic problem ($\varepsilon = 10^{-3}$)

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The multi-dimensional model

Our starting point is the model where immiscible incompressible two-phase flow in the homogeneous porous medium Ω_i is governed by the coupling of the degenerate parabolic equation on the saturation s :

$$\phi_i \partial_t s + \nabla \cdot \left(\mathbf{u}_{\mathrm{t}} f_i(s) + \underline{\mathbf{K}}_i \lambda_i(s) (-\varepsilon \nabla \pi_i(s) + \rho \mathbf{g}) \right) = q_o(s), \tag{4}$$

with the uniformly elliptic equation on the total fluid velocity $\mathbf{u}_{\mathrm{t}}$:

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The function P is the so called "global pressure" [Chavent, Jaffré '86].

Coupling : fluxes that appear in the SCL with discontinuous flux depend on $\mathbf{u}_{\mathrm{t}}$.

Transmission conditions on the interface Γ between Ω_1 and Ω_2 :

- conservation of mass

$$\sum_{i=1,2} \mathbf{u}_{\mathrm{t}} \cdot \mathbf{n}_i = 0, \qquad \sum_{i=1,2} \left(\mathbf{u}_{\mathrm{t}} f_i(s) + \underline{\mathbf{K}}_i \lambda_i(s) (-\varepsilon \nabla \pi_i(s) + \rho \mathbf{g}) \right) \cdot \mathbf{n}_i = 0. \tag{6}$$

- "continuity" of the extended phase pressures : $\exists p : \Gamma \times (0,T) \to \overline{\mathbb{R}}$ such that

$$p \in \tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2) \quad \text{and} \quad P_1 - Z_1(p) = P_2 - Z_2(p), \tag{7}$$

where $Z_i(\cdot)$ are suitably defined nonlinearities and s_i, P_i denote the traces of s, P on $\Gamma \times (0,T)$ from the side of Ω_i. Indeed, P is non-physical, it can be discontinuous at the interface ; but physical pressures expressed as $P_i - Z_i(s)$ must be connected.

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A decoupled IMPES scheme

The "decoupled" IMplicit Pressure - Explicit Saturation (IMPES) scheme based on additional interface pressure variables p_σ^n : [Andreianov, Brenner, Cancès '13 ?]

- **Initialization :** the saturation values $(s_K^0)_K$ are given ; the values p_σ^{n+1} of initial capillary pressure are non-physical and must be interpolated :

$$p_\sigma^0 = \text{mean value}\{\pi_K(s_K^0), \pi_L(s_L^0)\}, \quad \forall \sigma = K|L \in \mathcal{E}_\Gamma .$$

- Values $(s_K^n)_K$ and $(p_\sigma^n)_\sigma$ being given, solve the linear system on $(P_K^{n+1})_K, (u_K^{n+1})_K$:

$$\sum_{\sigma\in\mathcal{E}_K} |\sigma| u_{K,\sigma}^{n+1} = |K| q_o(s_K^n) + |K| q_w(s_K^n) \qquad \forall K, \sigma \notin \mathcal{E}_\Gamma \;;$$

$$u_{K,\sigma}^{n+1} = 0 \text{ if } \sigma \subset \partial\Omega, \qquad u_{K,\sigma}^{n+1} = \ldots(P_K^{n+1} - P_L^{n+1}) + \ldots \quad \text{if } \sigma = K|L, \sigma \notin \mathcal{E}_\Gamma,$$

$$u_{K,\sigma}^{n+1} + u_{L,\sigma}^{n+1} = 0, \quad u_{J,\sigma}^{n+1} = \ldots(P_J^{n+1} - P_{J,\sigma}^{n+1}) + \ldots,\ J{=}K,L, \sigma \in \mathcal{E}_\Gamma \;;$$

closed by the pressure connection relation on interface variables $P_{K,\sigma}$:

$$P_{K,\sigma}^{n+1} - Z_K(p_\sigma^n) = P_{L,\sigma}^{n+1} - Z_L(p_\sigma^n), \qquad \forall \sigma \in \mathcal{E}_\Gamma .$$

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A decoupled IMPES scheme

The "decoupled" IMplicit Pressure - Explicit Saturation (IMPES) scheme based on additional interface pressure variables p_σ^n : [Andreianov, Brenner, Cancès '13 ?]

• **Initialization :** the saturation values $(s_K^0)_K$ are given ; the values p_σ^{n+1} of initial capillary pressure are non-physical and must be interpolated :

$$p_\sigma^0 = \text{mean value}\{\pi_K(s_K^0), \pi_L(s_L^0)\}, \quad \forall \sigma = K|L \in \mathcal{E}_\Gamma .$$

• Values $(s_K^n)_K$ and $(p_\sigma^n)_\sigma$ being given, solve the linear system on $(P_K^{n+1})_K, (u_K^{n+1})_K$:

$$\sum_{\sigma \in \mathcal{E}_K} |\sigma| u_{K,\sigma}^{n+1} = |K| q_o(s_K^n) + |K| q_w(s_K^n) \qquad \forall K, \sigma \notin \mathcal{E}_\Gamma ;$$

$$u_{K,\sigma}^{n+1} = 0 \text{ if } \sigma \subset \partial\Omega, \qquad u_{K,\sigma}^{n+1} = \ldots(P_K^{n+1} - P_L^{n+1}) + \ldots \quad \text{if } \sigma = K|L, \sigma \notin \mathcal{E}_\Gamma ,$$

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Numerical results in 2D : a constrained flow

A *2D* test case inspired from [Eymard, Guichard, Herbin, Masson '13 ?] :
The *2D* domain Ω (see Fig. 1, top left), mostly consists of rock Ω_1, is initially saturated in water. The flow is constrained by a presence of two barriers (rock Ω_2) having a higher entry pressure. The vertical boundaries are assumed to be impermeable. At the bottom and the top of Ω we prescribe a constant rate of a total flux. The constant saturation value $s = 0.5$ is imposed on Γ_{in}. Details : [Andreianov, Brenner, Cancès '13 ?].

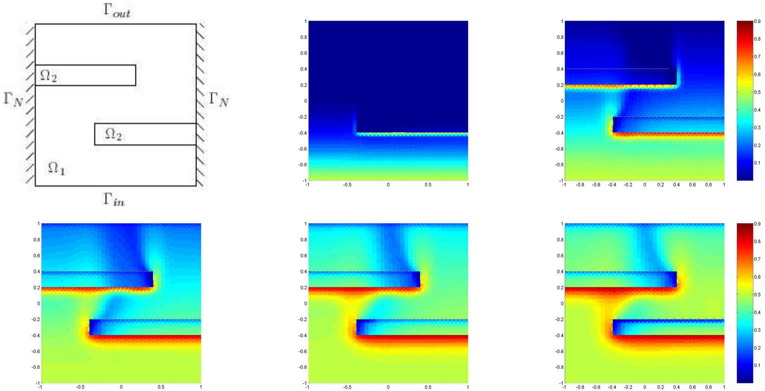

FIGURE: Computational domain and the oil saturation field at time $t = 0.075, 0.2, 0.4, 0.6, 0.8$.

Thank you - Merci - Danke !

DANKE SCHÖN !