

A SOLUTION TO THE PROBLEM OF L^p -MAXIMAL REGULARITY

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ABSTRACT. We give a negative solution to the problem of the L^p -maximal regularity on various classes of Banach spaces including L^q -spaces with $1 < q \neq 2 < +\infty$.

1. INTRODUCTION

In this paper we consider the following abstract Cauchy problem:

$$\begin{cases} u'(t) + B(u(t)) = f(t) & \text{for } 0 \leq t < T \\ u(0) = 0 \end{cases}$$

where $T \in (0, +\infty)$, $-B$ is the infinitesimal generator of a bounded analytic semigroup on a complex Banach space X and u and f are X -valued functions on $[0, T)$. Suppose $1 < p < \infty$. B is said to satisfy L^p -maximal regularity if whenever $f \in L^p([0, T); X)$ then the solution

$$u(t) = \int_0^t e^{-(t-s)B} f(s) ds$$

satisfies $u' \in L^p([0, T); X)$. It is known that B has L^p -maximal regularity for some $1 < p < \infty$ if and only if it has L^p -maximal regularity for every $1 < p < \infty$ [4], [5], [20]. We thus say simply that B satisfies *maximal regularity (MR)*.

The question whether B satisfies maximal regularity has been extensively studied. De Simon [4] showed that B always satisfies (MR) if X is a Hilbert space. We also mention the early work of Grisvard [7] using interpolation spaces between X and the domain $D(B)$. More recently, Dore and Venni [6] showed that B satisfies (MR) if X is an UMD Banach space and B admits bounded imaginary powers with an estimate $\|B^{is}\| \leq Ke^{\theta|s|}$ for some $0 \leq \theta < \pi/2$. The question we address was first asked by H. Brézis (see [2]) and is the following: under which conditions on the Banach space X does every negative generator of a bounded analytic semigroup on X satisfies (MR). Let us say that X has the *the maximal regularity property (MRP)* if B satisfies

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(MR) whenever $-B$ is the generator of a bounded analytic semigroup. The result of De Simon cited above implies that Hilbert spaces satisfy (MRP). Note also that by a theorem of Lotz [14] every strongly continuous semigroup on L^∞ is uniformly continuous and this implies L^∞ has (MRP). On the other hand, Coulhon and Lamberton [2] exhibited counterexamples on $X = L^p(\mathbb{R}; E)$ whenever $1 < p < \infty$ and E is not an UMD Banach space. More recently, Le Merdy [9] found counterexamples on other fundamental spaces such as $L^1(\mathbb{T})$, $C(\mathbb{T})$ and $\mathcal{K}(\ell_2)$. For several years it has been an open question whether the spaces L^p have (MRP) or even whether every UMD-space has (MRP).

In this paper, we provide a fairly complete answer to this question. In fact we will show that (MRP), up to isomorphism, characterizes Hilbert spaces among spaces with an unconditional basis or (more generally) separable Banach lattices. We also extend the Coulhon-Lamberton result cited above by showing that $L^p(\mathbb{R}; E)$ for $1 \leq p < \infty$ can never have (MRP) unless $p = 2$ and E is a Hilbert space.

We would like to mention that our constructions have been initially inspired by a very useful transference principle for maximal regularity proved by C. Le Merdy [9], although we only use a very simple version of it (see our Proposition 2.1).

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2. NOTATION AND BASIC FACTS

We will mainly adopt the notation introduced in [9].

Let X be a complex Banach space, $1 < p < \infty$ and $0 < T < \infty$. We denote by \mathcal{A}_T the differentiation d/dt on $L^p([0, T]; X)$ with domain $D(\mathcal{A}_T) = W_0^{1,p}([0, T]; X) = \{u \in W^{1,p}([0, T]; X) : u(0) = 0\}$. Let now $-B$ be the generator of a bounded analytic semigroup on X . The operator $I_{L^p} \otimes B$ defined on $L^p([0, T]) \otimes D(B)$ is closable and we denote by \mathcal{B} its closure. \mathcal{B} can also be described by $D(\mathcal{B}) = L^p([0, T]; D(B))$ and $(\mathcal{B}u)(t) = B(u(t))$ for u in $D(\mathcal{B})$. We say that B satisfies L^p -maximal regularity (on $[0, T]$) if the operator $\mathcal{A}_T + \mathcal{B}$ with domain $D(\mathcal{A}_T) \cap D(\mathcal{B})$ has a bounded inverse; this is equivalent to the formulation in the introduction, and obviously independent of T . By a result in [3] L^p -maximal regularity is also equivalent to the statement that $\mathcal{A}_T + \mathcal{B}$ is closed, and also to an inequality of the form

$$\|\mathcal{A}_T u\| \leq C \|\mathcal{A}_T u + \mathcal{B}u\|$$

for $u \in D(\mathcal{A}_T) \cap D(\mathcal{B})$. As remarked in the introduction, it is known (see [4], [5], [20]) that this property does not depend on $1 < p < \infty$. Thus we will simply say that B satisfies (MR) and work only on $L^2([0, T]; X)$. Then we say

that a Banach space X has the *maximal regularity property (MRP)*, if every B such that $-B$ generates a bounded analytic semigroup on X satisfies (MR).

We will also use the following terminology. A closed densely defined operator B on a Banach space X is said to be sectorial of type ω , where $0 < \omega < \pi$, if the spectrum $\sigma(B)$ of B is included in $\overline{\Sigma_\omega}$, where $\Sigma_\omega = \{z \in \mathbb{C} : |\text{Arg}(z)| < \omega\}$ and for every $\omega < \theta < \pi$ there exists $C_\theta > 0$ so that for any $\lambda \notin \overline{\Sigma_\theta}$ we have $\|(\lambda - B)^{-1}\| \leq C_\theta |\lambda|^{-1}$. Notice that $-B$ generates a bounded analytic semigroup on X if and only if B is sectorial of type ω , for some $\omega < \pi/2$ (see [21] or [10] for details).

Our next result can be regarded as a form of transference to the circle. For $f \in L^1([0, 2\pi]; X)$ we define the Fourier coefficients $\hat{f}(n)$ in the usual way

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt$$

for $n \in \mathbb{Z}$.

Proposition 2.1. *Let X be a Banach space and let $-B$ be an invertible generator of a bounded analytic semigroup on X . Assume that B satisfies (MR). Then there is a constant C so that for any X -valued trigonometric polynomial*

$$f(t) = \sum_{n=-N}^N \hat{f}(n) e^{int} \text{ we have}$$

$$\left(\int_0^{2\pi} \left\| \sum_{n \in \mathbb{Z}} in(in + B)^{-1} \hat{f}(n) e^{int} \right\|^2 \frac{dt}{2\pi} \right)^{1/2} \leq C \left(\int_0^{2\pi} \|f(t)\|^2 \frac{dt}{2\pi} \right)^{1/2}.$$

Proof. Denote by $(e^{-tB})_{t>0}$ the semigroup generated by $-B$. For any trigonometric polynomial f define $g \in L^2([-2\pi, 2\pi]; X)$ by:

$$\begin{cases} g(s) = (1 - e^{-2\pi B})^{-1} f(s + 2\pi) & \text{for } -2\pi \leq s \leq 0 \\ g(s) = f(s) & \text{for } 0 < s \leq 2\pi \end{cases}$$

Since B has a negative exponential type, the definition of g makes sense. Moreover there is a constant $C > 0$ independent of f so that

$$\|g\|_{L^2([-2\pi, 2\pi]; X)} \leq C \|f\|_{L^2([0, 2\pi]; X)}.$$

Now we solve

$$\begin{cases} u'(t) + B(u(t)) = g(t) & \text{for } -2\pi \leq t < 2\pi \\ u(-2\pi) = 0 \end{cases}$$

Elementary calculations show that $u' = \sum_{n \in \mathbb{Z}} in(in + B)^{-1} \hat{f}(n) e^{int}$ on $(0, 2\pi)$.

Now, the fact that B has (MR) yields the result. \square

Next, is an elementary lemma about (MRP) that we will use extensively:

Lemma 2.2. *Let X be a Banach space and Y be a complemented subspace of X . Assume that X has the (MRP) then Y has the (MRP).*

Proof. Assume $X = Y \oplus Z$ and that B is a sectorial operator of type $< \pi/2$ on Y which fails (MR). Then the operator B' , defined on X by $D(B') = D(B) \oplus Z$ and $B'x = By$ when $x = y + z$ with $(y, z) \in D(B) \times Z$ provides a counterexample to (MR) on X . \square

The operators that we will use will be multipliers associated with various Schauder decompositions. Let us introduce some notation for that purpose. If $F \subset X$, we denote by $[F]$ the closed linear span of F . Let $(E_n)_{n \geq 1}$ be a sequence of closed subspaces of X . Assume that $(E_n)_{n \geq 1}$ is a Schauder decomposition of X and let $(P_n)_{n \geq 1}$ be the associated sequence of projections from X onto E_n . For convenience we will also denote this Schauder decomposition by $(E_n, P_n)_{n \geq 1}$. Notice that the spaces $Z_n = P_n^*(X^*)$ form a Schauder decomposition of the subspace $Z = [\cup_{n=1}^{\infty} Z_n]$ of X^* .

Let now $(b_n)_{n \geq 1}$ be a sequence of complex numbers. We define the (possibly unbounded) operator $M(b_n; E_n)$ with domain $D(M(b_n; E_n)) = \{x \in X \text{ such that } \sum b_n P_n x \text{ converges in } X\}$ by $M(b_n; E_n)x = \sum b_n P_n x$.

The following lemma is elementary (see [1] or [22] for a proof in the case of a Schauder basis).

Lemma 2.3. (i) $M(b_n; E_n)$ is a closed densely defined operator.
(ii) If $b_1 > 0$ and $(b_n)_{n \geq 1}$ is an increasing sequence of reals, then $M(b_n; E_n)$ is invertible and sectorial of type ω for any $\omega \in (0, \pi)$.

3. THE RESULTS

We first establish a necessary condition for spaces with a Schauder decomposition to have the maximal regularity property.

Theorem 3.1. *Let $(E_n, P_n)_{n \geq 1}$ be a Schauder decomposition of the Banach space X . Let $Z_n = P_n^* X^*$ and $Z = [\cup_{n=1}^{\infty} Z_n]$. Assume X has (MRP). Then there is a constant $C > 0$ so that whenever $(u_n)_{n=1}^N$ are such that $u_n \in [E_{2n-1}, E_{2n}]$ and $(u_n^*)_{n=1}^N$ are such that $u_n^* \in [Z_{2n-1}, Z_{2n}]$ then*

$$\left(\int_0^{2\pi} \left\| \sum_{n=1}^N P_{2n} u_n e^{i2^n t} \right\|^2 \frac{dt}{2\pi} \right)^{1/2} \leq C \left(\int_0^{2\pi} \left\| \sum_{n=1}^N u_n e^{i2^n t} \right\|^2 \frac{dt}{2\pi} \right)^{1/2}$$

and

$$\left(\int_0^{2\pi} \left\| \sum_{n=1}^N P_{2n}^* u_n^* e^{i2^n t} \right\|^2 \frac{dt}{2\pi} \right)^{1/2} \leq C \left(\int_0^{2\pi} \left\| \sum_{n=1}^N u_n^* e^{i2^n t} \right\|^2 \frac{dt}{2\pi} \right)^{1/2}.$$

Proof. Let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ be two sequences defined by

$$a_{2n-1} = b_{2n-1} = b_{2n} = 2^{n-1} \quad \text{and} \quad a_{2n} = 2^n.$$

We let $A = M(a_n; E_n)$ and $B = M(b_n; E_n)$. It is easy to see that

$$(i2^n + A)^{-1}u_n = (i2^n + 2^{n-1})^{-1}P_{2n-1}u_n + (i2^n + 2^n)^{-1}P_{2n}u_n$$

while

$$(i2^n + B)^{-1}u_n = (i2^n + 2^{n-1})^{-1}u_n.$$

Hence

$$i2^n(i2^n + B)^{-1}u_n - i2^n(i2^n + A)^{-1}u_n = \frac{i}{(i+1)(2i+1)}P_{2n}u_n.$$

If we assume that X has (MRP) then both A and B satisfy (MR) and so we can apply Proposition 2.1 to each in turn for the polynomial $f(t) = \sum_{n=1}^N u_n e^{i2^n t}$.

Subtracting gives us the first estimate.

The second estimate follows by duality. More precisely the operators $f \mapsto \sum_{n \in \mathbb{Z}} in(in + A)^{-1} \hat{f}(n) e^{int}$ and $f \mapsto \sum_{n \in \mathbb{Z}} in(in + B)^{-1} \hat{f}(n) e^{int}$ can be extended to bounded linear operators on $L^2([0, 2\pi]; X)$. Taking adjoints and restricting to the subspace $L^2([0, 2\pi]; Z)$ one easily obtains similar estimates in the dual. \square

We first examine two important examples.

Corollary 3.2. *c_0 and ℓ_1 fail the (MRP).*

Proof. Denote by $(x_n)_{n \geq 1}$ the canonical basis of c_0 and let $s_n = x_1 + \dots + x_n$. $(s_n)_{n \geq 1}$ is a Schauder basis of c_0 which is usually called the summing basis of c_0 . We now apply Theorem 3.1 with the sequence of projections (P_n) associated with the Schauder basis (s_n) and $u_n = s_{2n} - s_{2n-1}$. Then we obtain that there is $C > 0$ so that for every $N \geq 1$,

$$\left(\int_0^{2\pi} \left\| \sum_{n=1}^N s_{2n} e^{i2^n t} \right\|^2 \frac{dt}{2\pi} \right)^{1/2} \leq C \left(\int_0^{2\pi} \left\| \sum_{n=1}^N (s_{2n} - s_{2n-1}) e^{i2^n t} \right\|^2 \frac{dt}{2\pi} \right)^{1/2}.$$

The right-hand side is equal to C but, considering only the first co-ordinate of the left-hand side with respect to the canonical basis, we have

$$\left(\int_0^{2\pi} \left\| \sum_{n=1}^N s_{2n} e^{i2^n t} \right\|^2 \frac{dt}{2\pi} \right)^{1/2} \geq N^{1/2}.$$

This is a contradiction.

Assume now that ℓ_1 has the (MRP). Let $(v_n)_{n \geq 1}$ be the coordinate functionals associated with the summing basis (s_n) of c_0 . The closed linear space Y spanned in ℓ_1 by the sequence (v_n) is of codimension 1 in ℓ_1 and hence is

isomorphic to ℓ_1 . The bi-orthogonal functionals (v_n^*) in Y^* are equivalent to the summing basis of c_0 . Hence using the same calculation as above and the second inequality of Theorem 3.1 we again get a contradiction. \square

We now explain the consequences of Theorem 3.1 when X admits an unconditional basis.

Theorem 3.3. *A Banach space with an unconditional basis has the (MRP) if and only if it is isomorphic to ℓ_2 .*

Proof. The idea is to show that if X has the (MRP) and an unconditional basis $(x_n)_{n \geq 1}$, then for every permutation π of the integers and for every block basis $(u_j)_{j \geq 1}$ of $(x_{\pi(n)})_{n \geq 1}$ the closed subspace of X spanned by the u_j 's is complemented in X . Once we have shown this the proof is completed by using a theorem of Lindenstrauss and Tzafriri ([11], see also [12] Theorem 2.a.10) which asserts that $(x_n)_{n \geq 1}$ must be equivalent to the canonical basis of c_0 or ℓ_p for some p in $[1, \infty)$. Then, by Corollary 3.2, (e_n) is equivalent to the canonical basis of ℓ_p for some p in $(1, \infty)$. Now, if $1 < p \neq 2 < \infty$, ℓ_p admits an unconditional basis which is not equivalent to any of the canonical bases of c_0 or ℓ_q where $1 \leq q < \infty$. Indeed Pełczyński [17] showed that, for $1 < p < \infty$, ℓ_p is isomorphic to $(\sum_{n \geq 1} \oplus \ell_2^n)_p$.

So assume, as we may, that $(x_n)_{n \geq 1}$ is a normalized 1-unconditional basis of X and that $(u_n)_{n \geq 1}$ is a normalized block basis of $(x_n)_{n \geq 1}$, with

$$\forall n \geq 1, u_n = \sum_{r_{n+1}}^{r_{n+1}} a_j e_j,$$

where $1 = r_1 < r_2 < \dots < r_n < r_{n+1} < \dots$ and $(a_j)_{j \geq 1} \subset \mathbb{C}$. For $n \geq 1$, let $X_n = [x_{r_{n+1}}, \dots, x_{r_{n+1}}]$ and $E_{2n} = [u_n]$. Then (X_n) is an unconditional Schauder decomposition of X with associated projections P_n , say. Now, by the Hahn-Banach theorem there is a norm-one projection $R_n : X_n \rightarrow E_{2n}$. Let $E_{2n-1} = R_n^{-1}(0)$. Then (E_n) is a Schauder decomposition of X with associated projections $Q_{2n-1} = (I - R_n)P_n$ and $Q_{2n} = R_n P_n$. We now apply Theorem 3.1 and exploit the unconditionality of the Schauder decomposition (X_n) . There is a constant C so that if y is in the linear span of the $(x_n)_{n \geq 1}$ then

$$\left\| \sum_{n=1}^{\infty} Q_{2n} y \right\| \leq C \|y\|.$$

This implies that $[u_n]_{n \geq 1}$ is complemented in X .

Clearly the same reasoning can be applied to any permutation of the basis (x_n) so that the proof is complete. \square

Although this will be included in further and more general statements let us point out that this already solves our problem for the spaces $L^p(0, 1)$:

Corollary 3.4. *Let $1 \leq p \leq \infty$. Then $L^p(0, 1)$ has the (MRP) if and only if $p = 2$ or $p = \infty$.*

Proof. For $1 < p < \infty$, the Haar system is known to be an unconditional basis of $L^p(0, 1)$ ([16], see also [13]). So the result follows from the preceding Theorem. The fact that L^1 fails (MRP) was proved by C. Le Merdy in [9]. Notice that L^1 contains a complemented copy of ℓ_1 , so this result can be derived from Lemma 2.2 and Corollary 3.2. \square

We now extend Theorem 3.3 to the case of a space with an unconditional Schauder decomposition.

Theorem 3.5. *Let X be a Banach space with an unconditional decomposition $(F_n, P_n)_{n \geq 1}$. Assume that X has the (MRP).*

Then X is isomorphic to $(\sum_{n=1}^{\infty} \oplus F_n)_{\ell_2}$.

Proof. It suffices to show that if $u_n \in F_n$ with $\|u_n\| = 1$ then $\sum a_n u_n$ converges if and only if $\sum |a_n|^2 < \infty$. As above in the proof of Theorem 3.5, let R_n be a norm-one projection of F_n onto $[u_n]$. Then let $E_{2n} = [u_n]$ and $E_{2n-1} = R_n^{-1}(0)$. Reasoning exactly as in Theorem 3.5 gives that $[u_n]_{n \geq 1}$ is complemented in X . But this subspace has an unconditional basis and so Theorem 3.5 yields that (u_n) is equivalent to the canonical basis of ℓ_2 . \square

Our next theorem completes the important counterexamples obtained by Coulhon and Lambertson [2].

Theorem 3.6. *Suppose X is a Banach space and $1 \leq p < \infty$. Then the Banach space $L^p((0, 1); X)$ has the (MRP) if and only if $p = 2$ and X is isomorphic to a Hilbert space.*

Proof. We first note that $L^p[0, 1]$ is complemented in $L^p((0, 1); X)$ so that if the latter has (MRP) then $p = 2$ by Corollary 3.4. Assume that $L^2((0, 1); X)$ has the (MRP); we will show that X is isomorphic to a Hilbert space (the opposite implication is due to de Simon [4]). By [2], X must have the UMD property. In particular, X does not contain the ℓ_1^n 's uniformly. Hence by Pisier's theorem [18], the space $\text{Rad}(X) = \overline{[\varepsilon_n]_{n \geq 1} \otimes X}$ is complemented in $L^2((0, 1); X)$ (here ε_n is a standard Rademacher function). Therefore, by Lemma 2.2, $\text{Rad}(X)$ has the (MRP). Now, $(E_n)_{n \geq 1} = (\varepsilon_n \otimes X)_{n \geq 1}$ is an unconditional Schauder decomposition of $\text{Rad}(X)$. So, by Theorem 3.5, $\text{Rad}(X)$ must be isomorphic to $(\sum \oplus (\varepsilon_n \otimes X))_{\ell_2}$. Finally, it follows from Kwapien's theorem [8] that X is isomorphic to a Hilbert space. \square

We now extend Theorem 3.3 and Corollary 3.4 to the setting of Banach lattices. All the notions on Banach lattices that we will use can be found in [13] Chapters 1.a and 1.b.

Theorem 3.7. *An order continuous Banach lattice has the (MRP) if and only if it is isomorphic to a Hilbert space.*

Proof. Let X be an order continuous Banach lattice. By [13], Lemma 1.b.13, it is enough to show that every normalized sequence of disjoint elements of X is equivalent to the canonical basis of ℓ_2 . So let $(f_n)_{n \geq 1}$ be such a sequence in X . Then X admits an unconditional Schauder decomposition $(E_n)_{n \geq 1}$ such that the E_n 's are ideals of X and for all $n \geq 1$, $f_n \in E_n$. Now, by Theorem 3.5 X is isomorphic to $(\sum \oplus E_n)_{\ell_2}$ and (f_n) is equivalent to the canonical basis of ℓ_2 . \square

Corollary 3.8. *A separable Banach lattice has the (MRP) if and only if it is isomorphic to a Hilbert space.*

Proof. Let X be a separable Banach lattice which is not order continuous. Then X is not σ -complete (see [13] Proposition 1.a.7) and by a result of P. Meyer-Nieberg ([15], see also [13] Theorem 1.a.5) X contains a subspace isomorphic to c_0 . Since X is separable, it follows from Sobczyk's Theorem [19] that this subspace is complemented in X . So, by Lemma 2.2 and Corollary 3.2, X does not have the (MRP). Then, the preceding Theorem concludes our proof. \square

4. FINAL REMARKS

- 1) One can also consider the problem of the L^p -maximal regularity on the half line $[0, +\infty)$, which has in general a different answer (see [9] for an example). But it follows from Theorem 2.4. in [5] that all our results remain valid in this slightly different setting.
- 2) We do not know if there is a non-Hilbertian subspace of an L^p -space ($1 \leq p < \infty$) with (MRP).
- 3) We do not know if every space with a basis and (MRP) is isomorphic to a Hilbert space.

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