SHARP NON-EXISTENCE RESULTS OF PRESCRIBED L^2-NORM SOLUTIONS FOR SOME CLASS OF SCHRÖDINGER-POISSON AND QUASILINEAR EQUATIONS

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ABSTRACT. In this paper we study the existence of minimizers for

$$F(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dxdy - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx$$

on the constraint

$$S(c) = \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} |u|^2 dx = c\},$$

where $c > 0$ is a given parameter. In the range $p \in [3, \frac{10}{3}]$ we explicit a threshold value of $c > 0$ separating existence and non-existence of minimizers. We also derive a non-existence result of critical points of $F(u)$ restricted to $S(c)$ when $c > 0$ is sufficiently small. Finally, as a byproduct of our approaches, we extend some results of [9] where a constrained minimization problem, associated to a quasilinear equation, is considered.

1. INTRODUCTION

The following stationary nonlinear Schrödinger-Poisson equation

$$-\Delta u - \lambda u + (|x|^{-1} * |u|^2)u - |u|^{p-2}u = 0 \text{ in } \mathbb{R}^3, \tag{1.1}$$

where $p \in (2, 6)$ and $\lambda \in \mathbb{R}$ has attracted considerable attention in the recent period. Part of the interest is due to the fact that to a pair $(u(x), \lambda)$ solution of (1.1) corresponds a standing wave $\phi(x) = e^{-i\lambda t}u(x)$ of the evolution equation

$$i\partial_t \phi + \Delta\phi - (|x|^{-1} * |\phi|^2)\phi + |\phi|^{p-2}\phi = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3. \tag{1.2}$$

This class of Schrödinger type equations with a repulsive nonlocal Coulombic potential is obtained by approximation of the Hartree-Fock equation describing a quantum mechanical system of many particles, see for instance [4, 15, 17, 18]. For physical reasons solutions are searched in $H^1(\mathbb{R}^3)$.

A first line of study to (1.1) is to consider $\lambda \in \mathbb{R}$ as a fixed parameter and then to search for a $u \in H^1(\mathbb{R}^3)$ solving (1.1). In that direction, mainly by variational methods, the existence, non-existence and multiplicity of solutions have been

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extensively studied by many authors. See, for example, [1, 2, 10, 11, 13, 19, 20, 22, 23] and the references therein.

In the present paper, motivated by the fact that physicists are often interested in "normalized solutions", we look for solutions in $H^1(\mathbb{R}^3)$ having a prescribed L^2-norm. More precisely, for given $c>0$ we look to

$$(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R} \ \text{ solution of } \ (1.1) \text{ with } \|u_c\|^2_{L^2(\mathbb{R}^3)} = c.$$

In this case, a solution $u_c \in H^1(\mathbb{R}^3)$ of (1.1) can be obtained as a constrained critical point of the functional

$$F(u) := \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}dxdy - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p\,dx$$

on the constraint

$$S(c) := \{u \in H^1(\mathbb{R}^3) : \ \|u\|^2_{L^2(\mathbb{R}^3)} = c, \ c>0\}.$$

The parameter $\lambda_c \in \mathbb{R}$, in this approach, can't be fixed any longer and it will appear as a Lagrange parameter.

It is well known, see for example [19], that for any $p \in (2,6)$, $F(u)$ is a well defined and C^1-functional. We set

$$m(c) := \inf_{u \in S(c)} F(u).$$

It is standard that minimizers of $m(c)$ are exactly critical points of $F(u)$ restricted to $S(c)$, and thus solutions of (1.1). Also it can be checked in many cases that the set of minimizers is orbitally stable under the flow of (1.2). Thus the search of minimizers can provide us some information on the dynamics of (1.2).

By scaling arguments, see Remark 1.1, it is readily seen that for any $c \in (0,\infty)$, $m(c) \in (-\infty, 0]$ if $p \in (2, \frac{10}{3})$ and $m(c) = -\infty$ if $p \in (\frac{10}{3}, 6)$. When $m(c) > -\infty$, the existence of minimizers of $m(c)$ has been studied in [5] [6] [21], see also [11] for a closely related problem. In [21], the authors prove the existence of minimizers when $p = \frac{8}{3}$ and $c \in (0, c_0)$ for a suitable $c_0 > 0$. It is shown in [6] that a minimizer exists if $p \in (2,3)$ and $c>0$ is small enough, and in [5] that when $p \in (3, \frac{10}{3})$, $m(c)$ admits a minimizer for any $c>0$ sufficiently large. In addition, when $p \in (\frac{10}{3}, 6)$, though $m(c) = -\infty$ for all $c>0$, [7] shows that there exists, for $c>0$ small enough, a critical point of $F(u)$ constrained on $S(c)$, at a strictly positive energy level. This critical point is a least energy solution in the sense that it minimizes $F(u)$ on the set of solutions having this L^2-norm. It is proved as well in [7] that it is orbitally unstable.

The first aim of this paper is to establish non-existence results of minimizers and more generally of constrained critical points of $F(u)$ on $S(c)$ in the range $p \in [3, \frac{10}{3}]$. As we shall see our results are sharp in the sense that we explicit a threshold value of $c>0$ separating existence and non-existence of minimizers.

We first present a detailed study of the function $c \to m(c)$ when $p \in [3, \frac{10}{3}]$. This study is, we believe, interesting for itself, but it is also a key to establish the existence or the non-existence of minimizers. Let

$$c_1 = \inf\{c > 0 \ : \ m(c) < 0\}. \tag{1.3}$$

Theorem 1.1. ($\mathbb{I}$) *When $p \in (3, \frac{10}{3})$ we have*

(i) $c_1 \in (0, \infty)$*;*
(ii) $m(c) = 0$*, as* $c \in (0, c_1]$*;*
(iii) $m(c) < 0$ *and is strictly decreasing about* c*, as* $c \in (c_1, \infty)$*.*

($\mathbb{II}$) *When $p = 3$ or $p = \frac{10}{3}$ we have*

(iv) *When* $p = 3$*,* $m(c) = 0$ *for all* $c > 0$*;*
(v) *When* $p = \frac{10}{3}$*, we denote*

$$c_2 = \inf\{c > 0 \ : \ \exists\, u \in S(c) \text{ such that } F(u) \leq 0\}, \tag{1.4}$$

then $c_2 \in (0, \infty)$ *and*

$$\begin{cases} m(c) = 0, & \text{as} \quad c \in (0, c_2); \\ m(c) = -\infty, & \text{as} \quad c \in (c_2, \infty). \end{cases} \tag{1.5}$$

Our result concerning the existence or non-existence of a minimizer is

Theorem 1.2. (i) *When* $p \in (3, \frac{10}{3})$*,* $m(c)$ *has a minimizer if and only if* $c \in [c_1, \infty)$*.*
(ii) *When* $p = 3$ *or* $p = \frac{10}{3}$*,* $m(c)$ *has no minimizer for any* $c > 0$*.*

Remark 1.1. One always has $m(c) \leq 0$ for any $c > 0$. Indeed let $u \in S(c)$ be arbitrary and consider the scaling $u^t(x) = t^{\frac{3}{2}} u(tx)$. We have $u^t \in S(c)$ for any $t > 0$ and also

$$F(u^t) = \frac{t^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{t}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 \, |u(y)|^2}{|x-y|} dxdy - \frac{t^{\frac{3}{2}(p-2)}}{p} \int_{\mathbb{R}^3} |u|^p dx.$$

Thus $F(u^t) \to 0$ as $t \to 0$ and the conclusion follows.

Remark 1.2. In [11, 13] the minimization problem on $S(c)$ for the functional

$$F_{a,b}(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{a}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 \, |u(y)|^2}{|x-y|} dxdy - \frac{b}{p} \int_{\mathbb{R}^3} |u|^p \, dx$$

is considered. When $p = 3$ it is proved that for each $a > 0$, there exists a $b_0 > 0$ such that if $b > b_0$ then a minimizer exists for all $c > 0$ (see Theorem 1.4 of [11]). Theorem 1.2 (ii) implies that when $a = 1$, necessarily $b_0 > 1$.

Remark 1.3. Theorem 1.2 provides a complete answer to the issue of minimizers for $F(u)$ on $S(c)$ when $p \in [3, \frac{10}{3}]$. When $p \in (2, 3)$, this question is still open. In [6] it is proved that a minimizer exists when $c > 0$ is sufficiently small. However even if $m(c) < 0$, for any $c > 0$ and any minimizing sequence is bounded, we still do not know what happen for an arbitrary value of $c > 0$. In trying to develop a minimization process one faces the difficulty to remove the possible dichotomy of the minimizing sequences. Also when $p \in (\frac{10}{3}, 6)$ the existence of a least energy solution is only established for $c > 0$ small (see [7]). In [7] however and even if the result is still to be proved, strong indications are given that there do not exist least energy critical points of $F(u)$ constrained to $S(c)$ when $c > 0$ is large.

In addition to the non-existence results of Theorem 1.2 we also show that, taking eventually $c > 0$ smaller, there are no critical points of $F(u)$ on $S(c)$. Precisely

Theorem 1.3. *When $p \in (3, \frac{10}{3}]$, there exists $\bar{c} > 0$ such that for any $c \in (0, \bar{c})$, there are no critical points of $F(u)$ restricted to $S(c)$. When $p = 3$, for all $c > 0$, $F(u)$ does not admit critical points on the constraint $S(c)$.*

Remark 1.4. Theorem 1.3 is, up to our knowledge, the only result where a non-existence result of small L^2 norm solutions is established for (1.1). Note however that in [12, 19] it was independently proved that when $p \in (2, 3]$ there exists a $\lambda_0 < 0$ such that (1.1) has only trivial solution when $\lambda \in (-\infty, \lambda_0)$.

Another aim of this paper is to clarify and extend some results contained in [9] where a constrained minimization problem associated to a quasilinear equation is considered. Actually in [9] one looks for minimizers of

$$\overline{m}(c) = \inf_{\sigma(c)} \mathcal{E}(u), \tag{1.6}$$

where

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} dx, \tag{1.7}$$

and

$$\sigma(c) = \{u \in H^1(\mathbb{R}^N) \ : \ \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 dx < \infty \text{ with } \|u\|^2_{L^2(\mathbb{R}^N)} = c\}.$$

Here $N \in \mathbb{N}^+$ and we focus on the range $p \in [1 + \frac{4}{N}, 3 + \frac{4}{N}]$. Let

$$c(p, N) = \inf\{c > 0 : \overline{m}(c) < 0\}.$$

Theorem 1.4. (i) *If $p \in [1 + \frac{4}{N}, 3 + \frac{4}{N})$, we have*
  - a) *$c(p, N) \in (0, \infty)$;*
  - b) *$\overline{m}(c) = 0$ if $c \in (0, c(p, N)]$;*
  - c) *$\overline{m}(c) < 0$ if $c \in (c(p, N), \infty)$ and is strictly decreasing about c, as $c \in (c(p, N), \infty)$.*

(ii) *If $p \in [1 + \frac{4}{N}, 3 + \frac{4}{N})$, the mapping $c \longmapsto \overline{m}(c)$ is continuous at each $c > 0$.*

(iii) *If $p = 3 + \frac{4}{N}$, we denote*

$$c_N = \inf\{c > 0 \ : \ \exists\, u \in \sigma(c) \text{ such that } \mathcal{E}(u) \le 0\}, \tag{1.8}$$

then $c_N \in (0,\infty)$ *and*

$$\begin{cases} \overline{m}(c) = 0, & as \quad c \in (0, c_N); \\ \overline{m}(c) = -\infty, & as \quad c \in (c_N, \infty). \end{cases} \tag{1.9}$$

Concerning the existence or non-existence of minimizers we have

Theorem 1.5. (i) *If* $p \in (1+\frac{4}{N}, 3+\frac{4}{N})$*, then* $\overline{m}(c)$ *admits a minimizer if and only if* $c \in [c(p,N),\infty)$.

(ii) *If* $p = 3+\frac{4}{N}$*,* $\overline{m}(c)$ *has no minimizer for all* $c \in (0,\infty)$.

Remark 1.5. We note that in [9] it is proved that when $p \in (1, 1+\frac{4}{N})$, for all $c > 0$, $\overline{m}(c) < 0$ and $\overline{m}(c)$ admits a minimizer. When $p = 1+\frac{4}{N}$, we believe, the conclusion of Theorem 1.5 (i) holds also, though a convinced proof is still open. The obstacle may technically stem from Lemma (4.3). As for $p \in (3+\frac{4}{N},\infty)$, $\overline{m}(c) = -\infty$ for any $c > 0$, for which it's impossible to find a minimizer.

Remark 1.6. We point out that parts of Theorem 1.4 and 1.5 are already contained in Theorem 1.12 of [9]. However, on one hand we provide here additional information. In particular we settle the question of existence for the threshold value $c(p,N)$ which requires a special treatment. On the other hand some statements of Theorem 1.12 are wrong, in particular concerning the case $p = 3+\frac{4}{N}$. There are also some gaps in the proofs of [9]. In particular it is not proved completely that there are no minimizer when $c \in (0, c(p,N))$.

Remark 1.7. In [8], the minimization problem (1.6) is studied and the question of finding explicit bounds on $c(p,N)$ and c_N is addressed by a combination of analytical and numerical arguments in dimension $N = 3$. In particular, when $p = 3+\frac{4}{N}$ a $c_b > 0$ such that $\overline{m}(c) = 0$ if $0 < c \le c_b$ and a $c^b > 0$ such that $\overline{m}(c) = -\infty$ if $c > c^b$ are explicitly given (see Proposition 2.1, points (4) and (5) of [8]). Their values are $c_b \approx 19.73$ and $c^b \approx 85.09$. Theorem 1.4 (iii) complements these results showing that the change from $\overline{m}(c) = 0$ to $\overline{m}(c) = -\infty$ occurs abruptly at the value c_N. We also point out that our results hold for any dimension $N \in \mathbb{N}^+$.

Similarly to Theorem 1.3 we obtain more generally

Theorem 1.6. *Assume that* $p \in [1+\frac{4}{N}, 3+\frac{4}{N}]$ *holds, then there exists* $\hat{c} > 0$ *such that for all* $c \in (0,\hat{c})$*, the functional* $\mathcal{E}(u)$*, restricted to* $\sigma(c)$*, has no critical points.*

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Notations: For convenience we set

$$A(u) := \int_{\mathbb{R}^3} |\nabla u|^2 dx, \quad B(u) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2\,|u(y)|^2}{|x-y|} dxdy$$

$$C(u) := \int_{\mathbb{R}^3} |u|^p dx, \quad D(u) := \int_{\mathbb{R}^3} |u|^2 dx.$$

Then

$$F(u) = \frac{1}{2}A(u) + \frac{1}{4}B(u) - \frac{1}{p}C(u). \tag{1.10}$$

Also we denote by $|| \cdot ||_p$ the standard norm on $L^p(\mathbb{R}^N)$. Throughout the paper we shall denote by $C > 0$ various positive constants which may vary from one line to another and which are not important for the analysis of the problem.

2. Preliminary results

To obtain our non-existence results we use the fact that any critical point of $F(u)$ on $S(c)$ satisfies $Q(u) = 0$ where

$$Q(u) := \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2 \, |u(y)|^2}{|x-y|} dxdy - \frac{3(p-2)}{2p}\int_{\mathbb{R}^3} |u|^p dx.$$

Indeed we have

Lemma 2.1. *If u_0 is a critical point of $F(u)$ on $S(c)$, then $Q(u_0) = 0$.*

Proof. First we denote

$$I_\lambda(u) \; := \; \langle S'_\lambda(u), u\rangle = A(u) - \lambda D(u) + B(u) - C(u), \tag{2.1}$$

$$P_\lambda(u) \; := \; \frac{1}{2}A(u) - \frac{3}{2}\lambda D(u) + \frac{5}{4}B(u) - \frac{3}{p}C(u). \tag{2.2}$$

Here $\lambda \in \mathbb{R}$ is a parameter and $S_\lambda(u)$ is the energy functional corresponding to the equation (1.1), i.e.

$$S_\lambda(u) := \frac{1}{2}A(u) - \frac{\lambda}{2}D(u) + \frac{1}{4}B(u) - \frac{1}{p}C(u). \tag{2.3}$$

Clearly $S_\lambda(u) = F(u) - \frac{\lambda}{2}D(u)$ and simple calculations imply that

$$\frac{3}{2}I_\lambda(u) - P_\lambda(u) = Q(u). \tag{2.4}$$

Now from [10] or Theorem 2.2 of [19], we know that $P_\lambda(u) = 0$ is a Pohozaev identity for the Schrödinger-Poisson equation (1.1). In particular any critical point u of $S_\lambda(u)$ satisfies $P_\lambda(u) = 0$.

On the other hand, since u_0 is a critical point of $F(u)$ restricted to $S(c)$, there exists a Lagrange multiplier $\lambda_0 \in \mathbb{R}$, such that

$$F'(u_0) = \lambda_0 u_0.$$

Thus for any $\phi \in H^1(\mathbb{R}^3)$,

$$\langle S'_{\lambda_0}(u_0), \phi\rangle = \langle F'(u_0) - \lambda_0 u_0, \phi\rangle = 0, \tag{2.5}$$

which shows that u_0 is also a critical point of $S_{\lambda_0}(u)$. Hence

$$P_{\lambda_0}(u_0) = 0, \quad I_{\lambda_0}(u_0) = \langle S'_{\lambda_0}(u_0), u_0\rangle = 0,$$

and $Q(u_0) = 0$ follows from (2.4). □

We now give an estimate on the nonlocal term, which is useful to control the functionals $F(u)$ and $Q(u)$.

Lemma 2.2. *When $p \in [3, 4]$, there exists a constant $C > 0$, depending only on p, such that, for any $u \in S(c)$,*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dxdy \geq -\frac{1}{16\pi} \|\nabla u\|_2^2 + C \frac{\|u\|_p^{\frac{p}{4-p}}}{\|\nabla u\|_2^{\frac{3(p-3)}{4-p}} \|u\|_2^{\frac{p-3}{4-p}}}. \tag{2.6}$$

Proof. Since $p \in [3, 4]$, by interpolation, we have

$$\|u\|_p^p \leq \|u\|_3^{3(4-p)} \|u\|_4^{4(p-3)}. \tag{2.7}$$

In addition, since $(|x|^{-1} * |u|^2) \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ solves the equation

$$-\Delta\Phi = 4\pi|u|^2 \quad \text{in } \mathbb{R}^3, \tag{2.8}$$

on one hand multiplying (2.8) by $(|x|^{-1} * |u|^2) \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ and integrating we get

$$4\pi \int_{\mathbb{R}^3} (|x|^{-1} * |u|^2)|u|^2 dx = \int_{\mathbb{R}^3} |\nabla(|x|^{-1} * |u|^2)|^2 dx. \tag{2.9}$$

On the other hand, multiplying (2.8) by $|u|$ and integrating we get for any $\eta > 0$,

$$\begin{aligned} 4\pi\eta \int_{\mathbb{R}^3} |u|^3 dx &= \eta \int_{\mathbb{R}^3} -\Delta(|x|^{-1} * |u|^2)|u| dx \\ &\leq \eta \int_{\mathbb{R}^3} \nabla(|x|^{-1} * |u|^2) \cdot \nabla|u| dx \\ &\leq \int_{\mathbb{R}^3} |\nabla(|x|^{-1} * |u|^2)|^2 dx + \frac{\eta^2}{4} \int_{\mathbb{R}^3} |\nabla u|^2 dx. \end{aligned} \tag{2.10}$$

Thus, taking $\eta = 1$ in (2.10) it follows from (2.9) and (2.10) that

$$\int_{\mathbb{R}^3} |u|^3 dx \leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dxdy + \frac{1}{16\pi} \|\nabla u\|_2^2. \tag{2.11}$$

Now, using Gagliardo-Nirenberg's inequality, there exists a constant $C > 0$, depending only on p, such that

$$\int_{\mathbb{R}^3} |u|^4 dx \leq C \|\nabla u\|_2^3 \|u\|_2. \tag{2.12}$$

Taking (2.11) and (2.12) into (2.7), we obtain

$$\|u\|_p^p \leq C \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 \, |u(y)|^2}{|x-y|} dxdy + \frac{1}{16\pi} \|\nabla u\|_2^2 \right)^{(4-p)} \|\nabla u\|_2^{3(p-3)} \|u\|_2^{(p-3)},$$

which implies (2.6). □

The estimate (2.6) leads to a lower bound on $Q(u)$.

Lemma 2.3. *When $p \in (3, \frac{10}{3})$, there exists a constant $C > 0$, depending only on p, such that, for any $u \in S(c)$*

$$Q(u) \geq \frac{64\pi - 1}{64\pi} A(u) - C \cdot A(u)^{\frac{3}{2}} \cdot c^{\frac{1}{2}}. \tag{2.13}$$

Proof. By Lemma 2.2 there exists a constant $C > 0$ depending only on p, such that, for any $u \in S(c)$,

$$Q(u) \geq \frac{64\pi - 1}{64\pi} A(u) + C \cdot \frac{C(u)^{\frac{1}{4-p}}}{A(u)^{\frac{3(p-3)}{2(4-p)}} \cdot D(u)^{\frac{p-3}{2(4-p)}}} - \frac{3(p-2)}{2p} C(u). \tag{2.14}$$

To obtain (2.13) from (2.14) we introduce the auxiliary function

$$f_K(x) = \left(\frac{64\pi - 1}{64\pi} \right) K + D \cdot x^{\frac{1}{4-p}} - \frac{3(p-2)}{2p} \cdot x, \quad x > 0$$

with $D = C \cdot \left(K^{\frac{3(p-3)}{2(4-p)}} \cdot c^{\frac{p-3}{2(4-p)}} \right)^{-1}$. Its study will provide us an estimate independent of $C(u)$. Clearly

$$\begin{aligned} f_K'(x) &= D \cdot \frac{1}{4-p} \cdot x^{\frac{p-3}{4-p}} - \frac{3(p-2)}{2p}, \\ f_K''(x) &= D \cdot \frac{1}{4-p} \cdot \frac{p-3}{4-p} \cdot x^{\frac{p-3}{4-p}-1} > 0, \quad \text{for all } x > 0. \end{aligned}$$

Therefore $f_K(x)$ has the unique global minimum at

$$\bar{x} = \left(\frac{3(p-2)(4-p)}{2pD} \right)^{\frac{4-p}{p-3}},$$

and

$$\begin{aligned} f_K(\bar{x}) &= \frac{64\pi - 1}{64\pi} K + D \cdot \left(\frac{3(p-2)(4-p)}{2pD} \right)^{\frac{1}{p-3}} - \frac{3(p-2)}{2p} \cdot \left(\frac{3(p-2)(4-p)}{2pD} \right)^{\frac{4-p}{p-3}} \\ &= \frac{64\pi - 1}{64\pi} K - \left(\frac{3(p-2)(4-p)}{2p} \right)^{\frac{1}{p-3}} \cdot \frac{p-3}{4-p} \cdot D^{\frac{p-4}{p-3}} \\ &= \frac{64\pi - 1}{64\pi} K - \left(\frac{3(p-2)(4-p)}{2p} \right)^{\frac{1}{p-3}} \cdot \frac{p-3}{4-p} \cdot C^{\frac{p-4}{p-3}} \cdot K^{\frac{3}{2}} \cdot c^{\frac{1}{2}}. \end{aligned}$$

Thus $f_K(x) \geq f_K(\bar{x})$ for all $x > 0$. This, together with (2.14) implies (2.13). $\square$

Finally we recall the following results obtained in [5, 6].

Lemma 2.4. *Let $p \in (3, \frac{10}{3})$, then*

(i) *For any $c > 0$ such that $m(c) < 0$, $m(c)$ admits a minimizer.*
(ii) *There exists $d > 0$, such that for all $c \in (d, \infty)$, $m(c) < 0$.*
(iii) *The function $c \mapsto m(c)$ is continuous at each $c > 0$.*

Remark 2.1. Points (i) and (ii) of Lemma 2.4 are proved in [5]. Concerning Point (iii), in [6] the authors prove the continuity of $m(c)$ about $c > 0$ when $p \in (2, 3)$. However inspecting their proof reveals that it also holds for $p \in [3, \frac{10}{3})$.

3. Proofs of the main results

We first give the following non-existence result.

Lemma 3.1. *When $p \in (3, \frac{10}{3})$, there exists a $c_3 > 0$, such that $m(c)$ has no minimizer for all $c \in (0, c_3)$.*

Proof. Let us assume by contradiction that there exist sequences $\{c_n\} \subset \mathbb{R}^+$, with $c_n \to 0$ as $n \to \infty$, and $\{u_n\} \subset S(c_n)$ such that $F(u_n) = m(c_n)$. Then by Lemma 2.1, $Q(u_n) = 0$ for any $n \in \mathbb{N}^+$.

Since $m(c) \leq 0$ for any $c > 0$, see Remark 1.1, we know that $F(u_n) \leq 0$. Thus

$$\begin{aligned} \frac{1}{2}A(u_n) + \frac{1}{4}B(u_n) &\leq \frac{1}{p}C(u_n) \\ &\leq \frac{C}{p}A(u_n)^{\frac{3}{4}(p-2)} \cdot D(u_n)^{\frac{6-p}{4}}, \end{aligned} \tag{3.1}$$

by Gagliardo-Nirenberg's inequality. Since $p \in (3, \frac{10}{3})$, $1 > \frac{3}{4}(p-2)$ and thus (3.1) implies that

$$A(u_n) \to 0, \text{ as } n \to \infty. \tag{3.2}$$

Now due to (3.2) and Lemma 2.3, when $n \in \mathbb{N}^+$ is sufficiently large,

$$\begin{aligned} Q(u_n) &\geq \frac{64\pi - 1}{64\pi}A(u_n) - C \cdot A(u_n)^{\frac{3}{2}} \cdot c_n^{\frac{1}{2}} \\ &\geq \frac{64\pi - 1}{64\pi}A(u_n) - C \cdot A(u_n)^{\frac{3}{2}} > 0. \end{aligned}$$

Obviously this contradicts Lemma 2.1 and this ends the proof. $\square$

The following lemma is crucial to establish a precise threshold between existence and non-existence.

Lemma 3.2. *Assume that $p \in (3, \frac{10}{3})$ holds. For any $c > 0$ such that $m(c) < 0$ or such that $m(c) = 0$ and $m(c)$ has a minimizer we have*

$$m(tc) < tm(c), \text{ for all } t > 1.$$

Proof. By Lemma 2.4 (i) without restriction we can assume that $m(c) \leq 0$ admit a minimizer $u_c \in S(c)$. We set $(u_c)_t(x) = t^2 u_c(tx)$ for $t > 1$. Then $D((u_c)_t) = tD(u_c) = tc$, and since $2p-6 > 0$ in case of $p \in (3, 10/3]$ and $C(u_c) > 0$, we obtain

$$\begin{aligned} m(tc) \leq F((u_c)_t) &= t^3 \cdot \left(\frac{1}{2}A(u_c) + \frac{1}{4}B(u_c) - \frac{t^{2p-6}}{p}C(u_c) \right) \\ &< t^3 \cdot \left(\frac{1}{2}A(u_c) + \frac{1}{4}B(u_c) - \frac{1}{p}C(u_c) \right) \\ &= t^3 \cdot F(u_c) = t^3 m(c). \end{aligned} \tag{3.3}$$

Since $m(c) \leq 0$ and $t > 1$, we conclude from (3.3) that $m(tc) < t^3 m(c) \leq tm(c)$. □

In the case $p = \frac{10}{3}$ we first have

Lemma 3.3. *When $p = \frac{10}{3}$, we have $c_2 \in (0, \infty)$, where c_2 is given by (1.4).*

Proof. First observe that by Gagliardo-Nirenberg's inequality, when $p = \frac{10}{3}$ we have

$$C(u) \leq C \cdot A(u) \cdot c^{\frac{2}{3}}, \quad \text{for all } u \in S(c), \tag{3.4}$$

where $C > 0$ independent of $c > 0$. Thus for any $u \in S(c)$, there holds

$$\begin{aligned} F(u) &\geq \frac{1}{2}A(u) + \frac{1}{4}B(u) - \frac{3}{10}C \cdot A(u) \cdot c^{\frac{2}{3}} \\ &\geq A(u) \left(\frac{1}{2} - \frac{3}{10}C \cdot c^{\frac{2}{3}} \right). \end{aligned} \tag{3.5}$$

Thus $F(u) > 0$, for all $u \in S(c)$ if $c > 0$ is sufficiently small and it proves that $c_2 > 0$.

Now take $u_1 \in S(1)$ arbitrary and consider the scaling

$$u_t(x) = t^2 u_1(tx), \quad \text{for all } t > 0. \tag{3.6}$$

Then $u_t \in S(t)$ and

$$\begin{aligned} F(u_t) &= \frac{t^3}{2}A(u_1) + \frac{t^3}{4}B(u_1) - \frac{3}{10}t^{\frac{11}{3}}C(u_1) \\ &= t^3 \left(\frac{1}{2}A(u_1) + \frac{1}{4}B(u_1) - \frac{3}{10}t^{\frac{2}{3}}C(u_1) \right). \end{aligned} \tag{3.7}$$

This shows that $F(u_t) < 0$ for $t > 0$ large enough and proves that $c_2 < \infty$. □

We can now give the

Proof of Theorem 1.1. First we prove that $c_1 > 0$ by contradiction. If we assume that $c_1 = 0$ then, from the definition of c_1, $m(c) < 0$ for all $c > 0$. Thus Lemma 2.4 (i) implies the existence of a minimizer for any $c > 0$ and this contradicts Lemma 3.1. Additionally Lemma 2.4 (ii) shows that $c_1 < \infty$, thus Point (i) follows. To prove Point (ii) we observe that since $m(c) \leq 0$ for all $c > 0$, from the definition of $c_1 > 0$ it follows that $m(c) = 0$ if $c \in (0, c_1)$. Using the continuity of $c \mapsto m(c)$, see Lemma 2.4 (iii), we obtain that $m(c_1) = 0$ and then Point (ii) holds. Point (iii) is a direct consequence of Lemma 3.2 and of the definition of $c_1 > 0$.

Concerning Point (iv), it is enough to show that if $p = 3$, for any $c > 0$ one has

$$F(u) > 0, \quad \text{for all } u \in S(c). \tag{3.8}$$

Indeed, since $m(c) \leq 0$ for all $c > 0$, (3.8) implies immediately Point (iv). To check (3.8), we use (2.10) with $\eta = 4/3$. From (2.9) and (2.10) we then get

$$\frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}dxdy \geq -\frac{1}{36\pi}||\nabla u||_2^2 + \frac{1}{3}||u||_3^3.$$

Thus when $p = 3$, for any $u \in S(c)$,

$$F(u) \geq \frac{1}{2}||\nabla u||_2^2 - \frac{1}{36\pi}||\nabla u||_2^2 > 0$$

and (3.8) holds.

Finally since, by Lemma 3.3, $c_2 \in (0, \infty)$, to prove Point (v) it is enough to verify (1.5). From the definition of c_2, it follows directly that $m(c) = 0$ for any $c \in (0, c_2)$. Now if $c \in (c_2, \infty)$, we first claim that there exists a $v \in S(c)$ such that $F(v) \leq 0$. Indeed if we assume that $F(u) > 0$ for all $u \in S(c)$ we reach a contradiction as follows. For an arbitrary $\hat{c} \in [c_2, c)$ taking any $u \in S(\hat{c})$ we scale it as in (3.6) where $t = c/\hat{c}$. Then $u_t \in S(c)$ and it follows from (3.7) that $F(u_t) \leq t^3 F(u)$. This implies that $F(u) > 0$ for all $u \in S(\hat{c})$ and since $\hat{c} \in [c_2, c)$ is arbitrary this contradicts the definition of $c_2 > 0$. Hence, for any $c \in (c_2, \infty)$, there exists a $u_0 \in S(c)$ such that $F(u_0) \leq 0$.

Consider now the scaling

$$u^\theta(x) = \theta^{\frac{3}{2}} u_0(\theta x), \quad \text{for all } \theta > 0. \tag{3.9}$$

We have $u^\theta \in S(c)$ for all $\theta > 0$ and

$$\begin{aligned} F(u^\theta) &= \frac{\theta^2}{2}A(u_0) + \frac{\theta}{4}B(u_0) - \frac{10}{3}\theta^2 C(u_0) \\ &= \frac{\theta}{4}B(u_0) - \left(\frac{10}{3}C(u_0) - \frac{1}{2}A(u_0)\right)\cdot\theta^2. \end{aligned} \tag{3.10}$$

Since $F(u_0) \leq 0$, necessarily

$$\frac{10}{3}C(u_0) - \frac{1}{2}A(u_0) > 0.$$

Thus we see from (3.10) that $\lim_{\theta\to\infty} F(u^\theta) = -\infty$ and $m(c) = -\infty$ follows. At this point the proof of the theorem is completed. $\square$

Before giving the proof of Theorem 1.2 we consider the case where $c = c_1$ that requires a special treatment.

Lemma 3.4. *Assume that $p \in (3, \frac{10}{3})$ holds. Then $m(c_1)$ admits a minimizer.*

Proof. Let $k_n := c_1 + 1/n$, for all $n \in \mathbb{N}^+$. We have $k_n \to c_1$ and thus, by Lemma 2.4 (iii), $m(k_n) \to m(c_1) = 0$. Furthermore, by Theorem 1.1 (iii) and Lemma 2.4 (i) we know that for each $n \in \mathbb{N}^+$, $m(k_n) < 0$ and $m(k_n)$ admits a minimizer u_n. Now we claim that the sequence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Indeed, by Gagliardo-Nirenberg's inequality, we have

$$\begin{aligned}\frac{1}{2}A(u_n) + \frac{1}{4}B(u_n) &= \frac{1}{p}C(u_n) + F(u_n) \\ &\leq CA(u_n)^{\frac{3(p-2)}{4}} k_n^{\frac{6-p}{4}} + m(k_n).\end{aligned}$$

This implies that $\{A(u_n)\}$ is bounded, since $m(k_n) \leq 0$ and $1 > 3(p-2)/4$. Thus we conclude that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$.

Now we claim that $C(u_n) \nrightarrow 0$. By contradiction let us assume that $C(u_n) \to 0$ as $n \to \infty$. Since $F(u_n) \to m(c_1) = 0$ it then follows that

$$A(u_n) \to 0 \text{ and } B(u_n) \to 0, \text{ as } n \to \infty. \tag{3.11}$$

Now, similarly to the proof of Lemma 2.3, using (2.6), we can estimate $F(u)$ from below by

$$F(u) \geq \frac{32\pi - 1}{64\pi} A(u) - C \cdot A(u)^{\frac{3}{2}} \cdot c^{\frac{1}{2}}, \quad \text{for all } u \in S(c) \tag{3.12}$$

where $C > 0$ is constant, depending only on p. In particular

$$F(u_n) \geq A(u_n) \left(\frac{32\pi - 1}{64\pi} - C \cdot A(u_n)^{\frac{1}{2}} \cdot k_n^{\frac{1}{2}} \right). \tag{3.13}$$

Taking (3.11) into account, (3.13) implies that $F(u_n) \geq 0$ for $n \in \mathbb{N}^+$ sufficiently large. This contradicts the fact that $F(u_n) = m(k_n) < 0$ for all $n \in \mathbb{N}^+$ and proves the claim.

Now, by Lemma I.1 of [14], we deduce that $\{u_n\}$ does not vanish. Namely that there exists a constant $\delta > 0$ and a sequence $\{x_n\} \subset \mathbb{R}^3$ such that

$$\int_{B(x_n,1)} |u_n|^2 dx \geq \delta > 0,$$

or equivalently

$$\int_{B(0,1)} |u_n(\cdot + x_n)|^2 dx \geq \delta > 0. \tag{3.14}$$

Here $B(0,1)$ denotes the ball centered in 0 with radius $r=1$. Now let $v_n(\cdot)=u_n(\cdot+x_n)$. Clearly $\{v_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and thus there exists $v_0 \in H^1(\mathbb{R}^3)$ such that

$$v_n \rightharpoonup v_0 \;\text{ weakly in } H^1(\mathbb{R}^3) \quad \text{and} \quad v_n \to v_0 \;\text{ in } L^2_{loc}(\mathbb{R}^3).$$

We note that $v_0 \neq 0$, since by (3.14)

$$0<\delta \leq \lim_{n\to\infty} \int_{B(0,1)} |v_n|^2 dx = \int_{B(0,1)} |v_0|^2 dx.$$

Let us prove that v_0 is a minimizer of $m(c_1)$. First we show that $F(v_0)=0$. Clearly

$$\lim_{n\to\infty} \|v_n\|_2^2 = \|v_0\|_2^2 + \lim_{n\to\infty} \|v_n - v_0\|_2^2 = c_1 \tag{3.15}$$

and using Lemma 2.4 (iii) we deduce from (3.15) that

$$\lim_{n\to\infty} F(v_n - v_0) \geq \lim_{n\to\infty} m(||v_n - v_0||_2^2) = m(c_1 - ||v_0||_2^2) = 0. \tag{3.16}$$

Here we make the convention that $m(0)=0$. Now using Lemma 2.2 of [23], we have

$$0 = m(c_1) = \lim_{n\to\infty} F(v_n) = F(v_0) + \lim_{n\to\infty} F(v_n - v_0). \tag{3.17}$$

Since $||v_0||_2^2 \leq c_1$ we have $m(||v_0||_2^2)=0$ and it shows that $F(v_0)<0$ is impossible. From (3.16) and (3.17) we deduce that $F(v_0)=0$ and that v_0 is a minimizer associated to $m(||v_0||_2^2)$. If we assume that $||v_0||_2^2 < c_1$ we get a contradiction with Lemma 3.2 since $m(c_1)=0$. Thus necessarily $\|v_0\|_2^2 = c_1$ and this ends the proof. ◻

Proof of Theorem 1.2. To prove Point (i) we assume by contradiction that there exists $\widetilde{c} \in (0, c_1)$ such that $m(\widetilde{c})$ admits a minimizer. Then from the definition of $c_1>0$ we get that $m(\widetilde{c})=0$ and Lemma 3.2 implies that $m(c)<0$ for any $c>\widetilde{c}$. This contradicts the definition of $c_1>0$. Now when $c>c_1$ the result clearly follows from Theorem 1.1 (iii) and Lemma 2.4 (i). Finally the case $c=c_1$ is considered in Lemma 3.4. For Point (ii), first observe that, because of (3.8), when $p=3$, for any $c>0$, $m(c)$ does not have a minimizer. Then we note that, from the definition of $Q(u)$, it holds, for any $u \in S(c)$,

$$F(u) - \frac{2}{3(p-2)} Q(u) = \frac{3p-10}{6(p-2)} A(u) + \frac{3p-8}{12(p-2)} B(u). \tag{3.18}$$

Taking $p=\frac{10}{3}$ in (3.18) we obtain

$$F(u) - \frac{1}{2} Q(u) = \frac{1}{8} B(u). \tag{3.19}$$

Thus if we assume by contradiction that $m(c)$ has a minimizer $u_c \in S(c)$ for some $c > 0$ we see from Lemma 2.1 and (3.19) that

$$0 \geq m(c) = F(u_c) = \frac{1}{8} B(u_c) > 0.$$

This contradiction ends the proof of Point (ii) and of the theorem. □

Proof of Theorem 1.3. We first consider the case $p \in (3, \frac{10}{3}]$ and we assume by contradiction that there exists sequences $\{c_n\} \subset \mathbb{R}^+$, with $c_n \to 0$, as $n \to \infty$, and $\{u_n\} \subset S(c_n)$ such that $u_n \in S(c_n)$ is a critical point of $F(u)$ restricted to $S(c_n)$. Then since

$$Q(u_n) = A(u_n) + \frac{1}{4} B(u_n) - \frac{3(p-2)}{2p} C(u_n) = 0,$$

we deduce, from Gagliardo-Nirenberg's inequality, that for some $C > 0$,

$$A(u_n) \leq \frac{3(p-2)}{2p} C(u_n) \leq C \cdot A(u_n)^{\frac{3(p-2)}{4}} \cdot c_n^{\frac{6-p}{4}}. \tag{3.20}$$

Thus there holds

$$A(u_n)^{\frac{10-3p}{4}} \leq C \cdot c_n^{\frac{6-p}{4}}$$

and we get that

$$A(u_n) \to 0 \quad \text{as} \quad n \to \infty \tag{3.21}$$

if $p \in (3, \frac{10}{3})$ and directly a contradiction if $p = \frac{10}{3}$. Now when $p \in (3, \frac{10}{3})$ by Lemma 2.3 we know, since $Q(u_n) = 0$, that there exists a constant $C > 0$ such that

$$\frac{64\pi - 1}{64\pi} A(u_n) \leq C \cdot A(u_n)^{\frac{3}{2}} \cdot c_n^{\frac{1}{2}}$$

or equivalently that

$$\frac{64\pi - 1}{64\pi} \leq C \cdot A(u_n)^{\frac{1}{2}} \cdot c_n^{\frac{1}{2}}. \tag{3.22}$$

But (3.22) implies that $A(u_n) \to \infty$ as $n \to \infty$ and this contradicts (3.21).

Now when $p = 3$, it is enough to prove that, for any $c > 0$, there holds

$$Q(u) > 0, \qquad \text{for all } u \in S(c). \tag{3.23}$$

Indeed, if (3.23) holds true, we can conclude the non-existence of minimizers directly from Lemma 2.1. To check (3.23), we use (2.10) with $\eta = 2$. Then, from (2.9) and (2.10), we get

$$\frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dxdy \geq -\frac{1}{16\pi} ||\nabla u||_2^2 + \frac{1}{2} ||u||_3^3.$$

Thus, for any $u \in S(c)$,

$$
\begin{aligned}
Q(u) &= ||\nabla u||_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}dxdy - \frac{1}{2}||u||_3^3 \\
&\geq ||\nabla u||_2^2 - \frac{1}{16\pi}||\nabla u||_2^2 > 0.
\end{aligned}
$$

At this point the proof is completed. □

4. On the quasilinear minimization problem

In the proofs of Theorems 1.4 and 1.5 we only provide the parts which were not established or whose proofs in [9] contains a gap. First we observe

Lemma 4.1. *Assume that* $p \in [1+\frac{4}{N}, 3+\frac{4}{N})$. *If there exists a* $\overline{c} > 0$ *such that* $\overline{m}(\overline{c}) = 0$ *is achieved, then*

$$
\overline{m}(c) < 0, \quad \textit{for all} \ \ c > \overline{c}. \tag{4.1}
$$

Proof. Let $\bar{u} \in \sigma(\overline{c})$ be a minimizer of $\overline{m}(\overline{c})$. Setting $(\bar{u})_t(x) = \bar{u}(t^{-\frac{1}{N}}x)$ for $t > 1$, we have $\|(\bar{u})_t\|_2^2 = t\|\bar{u}\|_2^2 = t\overline{c}$, and

$$
\begin{aligned}
\overline{m}(t\overline{c}) \leq \mathcal{E}((\bar{u})_t) &= t^{1-\frac{2}{N}}\left(\int_{\mathbb{R}^N}\frac{1}{2}|\nabla \bar{u}|^2 + |\bar{u}|^2|\nabla \bar{u}|^2 dx\right) - \frac{t}{p+1}\int_{\mathbb{R}^N}|\bar{u}|^{p+1}dx \\
&= t\left[t^{-\frac{2}{N}}\int_{\mathbb{R}^N}\left(\frac{1}{2}|\nabla \bar{u}|^2 + |\bar{u}|^2|\nabla \bar{u}|^2\right)dx - \frac{1}{p+1}\int_{\mathbb{R}^N}|\bar{u}|^{p+1}dx\right] \qquad (4.2) \\
&< t\mathcal{E}(\bar{u}) = t\overline{m}(\overline{c}).
\end{aligned}
$$

Thus (4.1) follows immediately from (4.2) since $\overline{m}(\overline{c}) = 0$. □

Similarly with Lemma 3.3, we have for c_N given by (1.8).

Lemma 4.2. *Assume that* $p = 3 + \frac{4}{N}$. *Then* $c_N \in (0, \infty)$.

Proof. We know from (4.5) of [9] that when $p \in [1+\frac{4}{N}, 3+\frac{4}{N}]$ there exists a $C > 0$, depending only on p and N, such that

$$
\|u\|_{p+1}^{p+1} \leq C \cdot \|u\|_2^{2(1-\theta)} \cdot \left(\int_{\mathbb{R}^N}|u|^2|\nabla u|^2 dx\right)^{\frac{\theta N}{N-2}}, \quad \text{for all } \in \mathcal{X} \tag{4.3}
$$

where

$$
\theta = \frac{(p-1)(N-2)}{2(N+2)} \quad \text{and} \quad \mathcal{X} = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N}|u|^2|\nabla u|^2 dx < \infty\}.
$$

Letting $p = 3 + \frac{4}{N}$ in (4.3), we obtain that

$$
\|u\|_{4+4/N}^{4+4/N} \leq C \cdot \|u\|_2^{\frac{4}{N}} \cdot \left(\int_{\mathbb{R}^N}|u|^2|\nabla u|^2 dx\right), \quad \text{for all } u \in \mathcal{X}. \tag{4.4}
$$

Thus, for any $u \in \sigma(c)$, there holds

$$\begin{aligned}\mathcal{E}(u) &\geq \frac{1}{2}\|\nabla u\|_2^2 + \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx - C \cdot c^{\frac{2}{N}} \cdot \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx \\ &\geq \left(1 - C \cdot c^{\frac{2}{N}}\right) \cdot \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx\end{aligned}$$

and $\mathcal{E}(u) > 0$ for all $u \in \sigma(c)$ if $c > 0$ is sufficiently small. This proves that $c_N > 0$.

Now take $u_1 \in \sigma(1)$ arbitrary and consider the scaling

$$u_t(x) = u_1(t^{-\frac{1}{N}x}), \quad \text{for all } t > 0. \tag{4.5}$$

We have $u_t \in \sigma(t)$ and

$$\begin{aligned}\mathcal{E}(u_t) &= t^{1-\frac{2}{N}}\left(\frac{1}{2}\|\nabla u_1\|_2^2 + \int_{\mathbb{R}^N} |u_1|^2|\nabla u_1|^2 dx\right) - t \cdot \frac{N}{4(N+1)}\|u_1\|_{4+4/N}^{4+4/N} \\ &= t\left[t^{-\frac{2}{N}}\left(\frac{1}{2}\|\nabla u_1\|_2^2 + \int_{\mathbb{R}^N} |u_1|^2|\nabla u_1|^2 dx\right) - \frac{N}{4(N+1)}\|u_1\|_{4+4/N}^{4+4/N}\right]. \qquad (4.6)\end{aligned}$$

This shows that $\mathcal{E}(u_t) < 0$ for $t > 0$ large and proves that $c_N < \infty$. □

Proof of Theorem 1.4. In Theorem 1.12 of [9], Point (i) was already proved except for the statement that $\overline{m}(c(p,N)) = 0$. But it is a direct consequence of Point (ii) that we shall now prove. Let $c > 0$ be arbitrary but fixed and let $\{c_n\}$ be a sequence such that $c_n \to c$. We need to show that $\overline{m}(c_n) \to \overline{m}(c)$. By the definition of $\overline{m}(c_n)$, for each $n \in \mathbb{N}^+$, there exists a $u_n \in \sigma(c_n)$ such that

$$\mathcal{E}(u_n) \leq \overline{m}(c_n) + \frac{1}{n}. \tag{4.7}$$

It is shown in [9] that $\overline{m}(c) \leq 0$ for any $c > 0$. Thus in particular

$$\mathcal{E}(u_n) \leq \frac{1}{n}. \tag{4.8}$$

Now we claim that the sequences $\{\|\nabla u_n\|_2^2\}$, $\{\int_{\mathbb{R}^N} |u_n|^2|\nabla u_n|^2 dx\}$, $\{\|u_n\|_{p+1}^{p+1}\}$ are bounded. Indeed using (4.8) and (4.3), we have

$$\frac{1}{n} \geq \mathcal{E}(u_n) \geq \int_{\mathbb{R}^N} |u_n|^2|\nabla u_n|^2 dx - \frac{C}{p+1}c_n^{1-\theta}\left(\int_{\mathbb{R}^N} |u_n|^2|\nabla u_n|^2 dx\right)^{\frac{\theta N}{N-2}}. \tag{4.9}$$

Since $\frac{\theta N}{N-2} < 1$ as $p \in [1+\frac{4}{N}, 3+\frac{4}{N})$, we conclude from (4.9) that $\{\int_{\mathbb{R}^N} |u_n|^2|\nabla u_n|^2 dx\}$ is bounded and then from (4.3) that $\{\|u_n\|_{p+1}^{p+1}\}$ is also bounded. At this point the fact that $\{\|\nabla u_n\|_2^2\}$ is bounded follows from the boundedness of $\mathcal{E}(u_n)$. Now we see that

$$\begin{aligned}\overline{m}(c) &\leq \mathcal{E}\left(\sqrt{\frac{c}{c_n}}u_n\right)\\ &= \frac{1}{2}\left(\frac{c}{c_n}\right)\|\nabla u_n\|_2^2 + \left(\frac{c}{c_n}\right)^2\int_{\mathbb{R}^N}|u_n|^2|\nabla u_n|^2dx - \frac{1}{p+1}\left(\frac{c}{c_n}\right)^{\frac{p+1}{2}}\|u_n\|_{p+1}^{p+1}\\ &= \mathcal{E}(u_n)+o(1)\leq \overline{m}(c_n)+o(1).\end{aligned}$$

On the other hand, for a minimizing sequence $\{v_m\}$ of $\overline{m}(c)$, we have

$$\overline{m}(c_n)\leq \mathcal{E}\left(\sqrt{\frac{c_n}{c}}v_m\right) = \mathcal{E}(v_m)+o(1) = \overline{m}(c)+o(1).$$

From these two estimates we deduce that $\lim_{n\to\infty}\overline{m}(c_n)=\overline{m}(c)$.

We now prove Point (iii). Note that the statement in Theorem 1.12 of [9] concerning $p=3+\frac{4}{N}$ was incorrect. We already know, from Lemma 4.2, that $c_N\in(0,\infty)$. Using the definition of c_N, it follows directly that $\overline{m}(c)=0$ for any $c\in(0,c_N)$, since one always has $\overline{m}(c)\leq 0$ for any $c\in(0,\infty)$. Now if $c>c_N$, we proceed as in the proof of Theorem 1.1 (v), namely we observe that there exists a $v\in\sigma(c)$ such that $\mathcal{E}(v)\leq 0$. Indeed if we assume that $\mathcal{E}(u)>0$ for all $u\in\sigma(c)$ we reach a contradiction as follows. For an arbitrary $\hat{c}\in[c_N,c)$ taking any $u\in\sigma(\hat{c})$ we scale it as in (4.5) where $t=c/\hat{c}$. Then $u_t\in\sigma(c)$ and it follows from (4.6) that $\mathcal{E}(u_t)\leq t\mathcal{E}(u)$. This implies that $\mathcal{E}(u)>0$ for all $u\in\sigma(\hat{c})$ and since $\hat{c}\in[c_N,c)$ is arbitrary this contradicts the definition of $c_N>0$.

Hence, for any $c\in(c_N,\infty)$, there exists a $u_0\in\sigma(c)$ such that $\mathcal{E}(u_0)\leq 0$ and we consider the scaling

$$u^\delta(x)=\delta^{\frac{N}{2}}u_0(\delta x),\qquad \text{for all } \delta>0. \tag{4.10}$$

Then $u^\delta\in\sigma(c)$, for all $\delta>0$ and

$$\begin{aligned}\mathcal{E}(u^\delta) &= \frac{\delta^2}{2}\|\nabla u_0\|_2^2+\delta^{N+2}\int_{\mathbb{R}^N}|u_0|^2|\nabla u_0|^2dx-\frac{N}{4(N+1)}\delta^{N+2}\|u_0\|_{4+4/N}^{4+4/N}\\ &= \frac{\delta^2}{2}\|\nabla u_0\|_2^2-\delta^{N+2}\left(\frac{N}{4(N+1)}\|u_0\|_{4+4/N}^{4+4/N}-\int_{\mathbb{R}^N}|u_0|^2|\nabla u_0|^2dx\right). \qquad (4.11)\end{aligned}$$

Since $\mathcal{E}(u_0)\leq 0$, necessarily

$$\frac{N}{4(N+1)}\|u_0\|_{4+4/N}^{4+4/N}-\int_{\mathbb{R}^N}|u_0|^2|\nabla u_0|^2dx>0$$

and thus we see from (4.11) that $\lim_{\delta\to\infty}\mathcal{E}(u^\delta)=-\infty$. It proves that $\overline{m}(c)=-\infty$ for any $c\in(c_N,+\infty)$. □

Before giving the proof of Theorem 1.5 we treat the limit case $c=c(p,N)$.

Lemma 4.3. *Assume that* $p \in (1+\frac{4}{N}, 3+\frac{4}{N})$. *Then* $\overline{m}(c(p,N))$ *admits a minimizer.*

Proof. Let $c_n := c(p,N) + \frac{1}{n}$, for all $n \in \mathbb{N}^+$. Since $\overline{m}(c_n) < 0$ we know by Lemma 4.3 of [9] that $\overline{m}(c_n)$ admits, for all $n \in \mathbb{N}^+$ a minimizer that is Schwartz symmetric. We claim that $\{u_n\}$ is bounded in $\mathcal{X}$, namely that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and $\{\int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx\}$ is bounded. Indeed using (4.3) we have since $\mathcal{E}(u_n) \le 0$, for all $n \in \mathbb{N}^+$,

$$\begin{aligned} \frac{1}{2}\|\nabla u_n\|_2^2 + \int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx &\le \frac{1}{p+1}\int_{\mathbb{R}^N} |u_n|^{p+1} dx \\ &\le \frac{C}{p+1} c_n^{1-\theta} \cdot \left(\int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx\right)^{\frac{\theta N}{N-2}}. \end{aligned} \tag{4.12}$$

Since $p \in [1+\frac{4}{N}, 3+\frac{4}{N})$ we have $\frac{\theta N}{N-2} < 1$ and thus (4.12) implies that both $\{\int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx\}$ and $\{||\nabla u_n||_2^2\}$ are bounded.

Passing to a subsequence we can assume that $u_n \rightharpoonup u_0$ in $\mathcal{X}$. Now from Lemma 4.3 of [9] we have that

$$T(u_0) \le \liminf_{n\to\infty} T(u_n) \quad \text{where} \quad T(u) := \frac{1}{2}\|\nabla u\|_2^2 + \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 dx.$$

Also the fact that $\{u_n\}$ is a sequence of Schwartz symmetric functions readily implies that $u_n \to u_0$ in $L^{p+1}(\mathbb{R}^N)$. Thus, since by Theorem 1.4 (ii), $\lim_{n\to\infty} \mathcal{E}(u_n) = \lim_{n\to\infty} \overline{m}(c_n) = 0$ we obtain that $\mathcal{E}(u_0) \le 0$. Also since $||u_0||_2^2 \le c(p,N)$ necessarily $\mathcal{E}(u_0) = 0$.

In order to show that $||u_0||_2^2 = c(p,N)$ and thus that u_0 is a minimizer of $c(p,N)$ we first show that $u_0 \ne 0$. By contradiction let us assume that $u_0 = 0$. Then using the fact that $u_n \to 0$ in $L^p(\mathbb{R}^N)$ we get from $\mathcal{E}(u_n) \to 0$ that

$$\|\nabla u_n\|_2^2 \to 0 \;\text{ and }\; \int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx \to 0, \quad \text{as } n \to \infty. \tag{4.13}$$

As in the proof of Lemma 3.4 we shall prove that $\mathcal{E}(u_n) \ge 0$ for $n \in \mathbb{N}^+$ sufficiently large and this will contradict the fact that $\mathcal{E}(u_n) = \overline{m}(c_n) < 0$ for $n \in \mathbb{N}^+$. For $p \in (1+\frac{4}{N}, \frac{N+2}{N-2}]$ if $N \ge 3$ and $p \in (1+\frac{4}{N}, +\infty)$ if $N = 1,2$, by Gagliardo-Nirenberg's inequality, we have

$$\int_{\mathbb{R}^N} |u_n|^{p+1} dx \le C\|\nabla u_n\|_2^{\frac{N(p-1)}{2}} \cdot c_n^{\frac{(N+2)-(N-2)p}{4}} \le C\|\nabla u_n\|_2^{\frac{N(p-1)}{2}}. \tag{4.14}$$

Thus

$$\begin{aligned}\mathcal{E}(u_n) &\geq \frac{1}{2}\|\nabla u_n\|_2^2 - C\|\nabla u_n\|_2^{\frac{N(p-1)}{2}} \\ &= \|\nabla u_n\|_2^2\left(\frac{1}{2} - C\|\nabla u_n\|_2^{\frac{Np-(N+4)}{2}}\right).\end{aligned}$$

This, together with (4.13), proves that $\mathcal{E}(u_n) \geq 0$ as $n \in \mathbb{N}^+$ is sufficiently large. For $p \in (\frac{N+2}{N-2}, 3 + \frac{4}{N})$, $N \geq 3$, we know from the proof of Theorem 1.12 of [9] that $\{u_n\}$ it is bounded in $L^q(\mathbb{R}^N)$ for all $q \geq \frac{4N}{N-2}$. Thus by Hölder and Sobolev's inequalities we can write

$$\int_{\mathbb{R}^N} |u_n|^{p+1}dx \leq C(p,N)\|\nabla u_n\|_2^\alpha \cdot \|u_n\|_{(p-1)N}^\beta, \tag{4.15}$$

where

$$\alpha = \frac{2N(p-1) - 2(p+1)}{(p-1)(N-2) - 2} \quad \text{and} \quad \beta = (p-1)\frac{(N-2)(p+1) - 2N}{(p-1)(N-2) - 2}.$$

For more details see, in particular, (4.16) in [9]. Now since $\|u_n\|_{(p-1)N}^\beta$ is bounded we have

$$\begin{aligned}\mathcal{E}(u_n) &\geq \frac{1}{2}\|\nabla u_n\|_2^2 - C(p,N)\|\nabla u_n\|_2^\alpha \\ &= \|\nabla u_n\|_2^2\left(\frac{1}{2} - C(p,N)\|\nabla u_n\|_2^{\alpha-2}\right).\end{aligned}$$

Since $\alpha - 2 > 0$ as $p > 1$, we then deduce using (4.13) that $\mathcal{E}(u_n) \geq 0$ for all $n \in \mathbb{N}^+$ sufficiently large. This proves that $u_0 \neq 0$. Finally if we assume that $\|u_0\|_2^2 < c(p,N)$ we directly get a contradiction from Lemma 4.1 since $\overline{m}(c) = 0$ for all $c \in (0, c(p,N)]$. Thus $\|u_0\|_2^2 = c(p,N)$ and u_0 is a minimizer of $\overline{m}(c(p,N))$. □

Proof of Theorem 1.5. In Theorem 1.12 of [9] it is shown that $\overline{m}(c)$ admits a minimizer if $c \in (c(p,N), \infty)$. By Lemma 4.3 this is also true for $c = c(p,N)$. To complete the proof of Point (i) we need to show that for $c \in (0, c(p,N))$, $\overline{m}(c)$ does not admit a minimizer. But since $\overline{m}(c) = 0$ for $c \in (0, c(p,N)]$ it results directly from Lemma 4.1. To prove Point (ii) we argue by contradiction assuming that there exists a $c > 0$ such that $\overline{m}(c)$ admits a minimizer u_c. Then, by standard arguments, u_c satisfies weakly

$$-\Delta u_c - \lambda_c u_c - u_c\Delta|u_c|^2 = |u_c|^{p-1}u_c, \tag{4.16}$$

where $\lambda_c \in \mathbb{R}$ is the associated Lagrange multiplier. Multiplying (4.16) by u_c and integrating we derive that

$$\int_{\mathbb{R}^N} |\nabla u_c|^2dx + 4\int_{\mathbb{R}^N} |u_c|^2|\nabla u_c|^2dx - \int_{\mathbb{R}^N} |u_c|^{p+1}dx = \lambda_c\|u_c\|_2^2. \tag{4.17}$$

Also, from Lemma 3.1 of [9] we know that u_c satisfies the Pohozaev identity

$$\frac{N-2}{N}\left(\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_c|^2dx+\int_{\mathbb{R}^N}|u_c|^2|\nabla u_c|^2dx\right)=\frac{\lambda_c}{2}\|u_c\|_2^2+\frac{1}{p+1}\|u_c\|_{p+1}^{p+1}. \tag{4.18}$$

It follows from (4.17) and (4.18) that

$$\|\nabla u_c\|_2^2+(N+2)\int_{\mathbb{R}^N}|u_c|^2|\nabla u_c|^2dx-\frac{N(p-1)}{2(p+1)}\|u_c\|_{p+1}^{p+1}=0, \tag{4.19}$$

by which we can rewrite $\mathcal{E}(u_c)$ as

$$\mathcal{E}(u_c)=\frac{Np-(N+4)}{2N(p-1)}\|\nabla u_c\|_2^2+\frac{Np-(3N+4)}{N(p-1)}\int_{\mathbb{R}^N}|u_c|^2|\nabla u_c|^2dx. \tag{4.20}$$

When $p=3+\frac{4}{N}$, (4.20) becomes

$$\mathcal{E}(u_c)=\frac{N}{2N+4}\|\nabla u_c\|_2^2. \tag{4.21}$$

This is clearly a contradiction since by assumption $\mathcal{E}(u_c)=\overline{m}(c)\le 0$ and Point (ii) is established. □

Proof of Theorem 1.6. From the proof of Theorem 1.5, we know that any critical point u_c of $\mathcal{E}(u)$ restricted to $\sigma(c)$ must satisfy (4.19). Denoting

$$\overline{Q}(u)=\|\nabla u\|_2^2+(N+2)\int_{\mathbb{R}^N}|u|^2|\nabla u|^2dx-\frac{N(p-1)}{2(p+1)}\|u\|_{p+1}^{p+1},$$

we thus have $\overline{Q}(u_c)=0$. Now we assume by contradiction that there exist sequence $\{c_n\}\subset\mathbb{R}^+$ with $c_n\to 0$, and $\{u_n\}\subset\sigma(c_n)$ such that u_n is a critical point of $\mathcal{E}(u)$ on $\sigma(c_n)$. Then for each $n\in\mathbb{N}^+$, $\overline{Q}(u_n)=0$ and using (4.3) we obtain

$$\|\nabla u_n\|_2^2+(N+2)\int_{\mathbb{R}^N}|u_n|^2|\nabla u_n|^2dx\le C\cdot c_n^{1-\theta}\cdot\left(\int_{\mathbb{R}^N}|u_n|^2|\nabla u_n|^2dx\right)^{\frac{\theta N}{N-2}}, \tag{4.22}$$

where $\theta=\frac{(p-1)(N-2)}{2(N+2)}$. When $p=3+\frac{4}{N}$ we have $\frac{\theta N}{N-2}=1$, $1-\theta=\frac{4}{N}$ and thus we get immediately a contradiction from (4.22). Now when $p\in[1+\frac{4}{N},3+\frac{4}{N})$, $\frac{\theta N}{N-2}<1$ and we derive from (4.22) that

$$\int_{\mathbb{R}^N}|u_n|^2|\nabla u_n|^2dx\to 0 \text{ and } \|\nabla u_n\|_2^2\to 0 \text{ as } n\to\infty. \tag{4.23}$$

Also when $p \in [1+\frac{4}{N}, \frac{N+2}{N-2}]$ if $N \geq 3$ and $p \in [1+\frac{4}{N}, +\infty)$ if $N = 1, 2$, we obtain from (4.14) that

$$\begin{aligned} \overline{Q}(u_n) &\geq \|\nabla u_n\|_2^2 - C\|\nabla u_n\|_2^{\frac{N(p-1)}{2}} \cdot c_n^{\frac{(N+2)-(N-2)p}{4}} \\ &= \|\nabla u_n\|_2^2 \left(1 - C\|\nabla u_n\|_2^{\frac{Np-(N+4)}{2}} \cdot c_n^{\frac{(N+2)-(N-2)p}{4}}\right). \end{aligned} \tag{4.24}$$

Taking (4.23) into account (4.24) implies that $\overline{Q}(u_n) > 0$ for $n \in \mathbb{N}^+$ large enough and provides a contradiction.

When $p \in (\frac{N+2}{N-2}, 3+\frac{4}{N}), N \geq 3$, using (4.15) and the fact that $\{\|u_n\|_{(p-1)N}^{\beta}\}$ is bounded, we have

$$\overline{Q}(u_n) \geq \|\nabla u_n\|_2^2 - C(p,N)\|\nabla u_n\|_2^{\alpha}.$$

Since $\alpha - 2$ as $p > 1$, using (4.23) we conclude that $\overline{Q}(u_n) > 0$ for $n \in \mathbb{N}^+$ sufficiently large. Here also we have obtained a contradiction and this ends the proof. □

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