EXISTENCE AND INSTABILITY OF STANDING WAVES WITH PRESCRIBED NORM FOR A CLASS OF SCHRÖDINGER-POISSON EQUATIONS

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Abstract. In this paper we study the existence and the instability of standing waves with prescribed L^2-norm for a class of Schrödinger-Poisson-Slater equations in $\mathbb{R}^3$

$$i\psi_t + \Delta\psi - (|x|^{-1} * |\psi|^2)\psi + |\psi|^{p-2}\psi = 0 \tag{0.1}$$

when $p \in (\frac{10}{3}, 6)$. To obtain such solutions we look to critical points of the energy functional

$$F(u) = \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2\,|u(y)|^2}{|x-y|}dxdy - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p\,dx$$

on the constraints given by

$$S(c) = \{u \in H^1(\mathbb{R}^3) : \ \|u\|^2_{L^2(\mathbb{R}^3)} = c, c > 0\}.$$

For the values $p \in (\frac{10}{3}, 6)$ considered, the functional F is unbounded from below on $S(c)$ and the existence of critical points is obtained by a mountain pass argument developed on $S(c)$. We show that critical points exist provided that $c > 0$ is sufficiently small and that when $c > 0$ is not small a non-existence result is expected. Concerning the dynamics we show for initial condition $u_0 \in H^1(\mathbb{R}^3)$ of the associated Cauchy problem with $\|u_0\|_2^2 = c$ that the mountain pass energy level $\gamma(c)$ gives a threshold for global existence. Also the strong instability of standing waves at the mountain pass energy level is proved. Finally we draw a comparison between the Schrödinger-Poisson-Slater equation and the classical nonlinear Schrödinger equation.

1. Introduction

In this paper we prove the existence and the strong instability of standing waves for the following Schrödinger-Poisson-Slater equations:

$$i\partial_t u + \Delta u - (|x|^{-1} * |u|^2)u + |u|^{p-2}u = 0 \ \text{ in } \ \mathbb{R} \times \mathbb{R}^3. \tag{1.1}$$

This class of Schrödinger type equations with a repulsive nonlocal Coulombic potential is obtained by approximation of the Hartree-Fock equation describing a quantum mechanical system of many particles, see for instance [3], [27], [29],

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[30]. We look for standing waves solutions of (1.1). Namely for solutions in the form

$$u(t,x) = e^{-i\lambda t} v(x),$$

where $\lambda \in \mathbb{R}$. Then the function $v(x)$ satisfies the equation

$$-\Delta v - \lambda v + (|x|^{-1} * |v|^2)v - |v|^{p-2}v = 0 \text{ in } \mathbb{R}^3. \tag{1.2}$$

The case where $\lambda \in \mathbb{R}$ is a fixed and assigned parameter has been extensively studied in these last years, see e.g. [1], [14], [22], [23], [31] and the references therein. In this case critical points of the functional defined in $H^1(\mathbb{R}^3)$

$$J(u) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{\lambda}{2}\int_{\mathbb{R}^3} |u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2\,|u(y)|^2}{|x-y|} dxdy - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx$$

give rise to solutions of (1.2). In the present paper, motivated by the fact that physics are often interested in "normalized" solutions, we search for solutions with prescribed L^2-norm. A solution of (1.2) with $\|u\|^2_{L^2(\mathbb{R}^3)} = c$ can be obtained as a constrained critical point of the functional

$$F(u) := \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2\,|u(y)|^2}{|x-y|} dxdy - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p\, dx$$

on the constraint

$$S(c) := \{u \in H^1(\mathbb{R}^3) : \ \|u\|^2_{L^2(\mathbb{R}^3)} = c\}.$$

Note that in this case the frequency can not longer by imposed but instead appears as a Lagrange parameter. As we know, $F(u)$ is a well defined and C^1 functional on $S(c)$ for any $p \in (2,6]$ (see [31] for example). For $p \in (2, \frac{10}{3})$ the functional $F(u)$ is bounded from below and coercive on $S(c)$. The existence of minimizers for $F(u)$ constrained has been studied in the [5], [6], [33]. It has been proved in [33], using techniques introduced in [11], that minimizer exist for $p = \frac{8}{3}$ provided that $c \in (0, c_0)$ for a suitable $c_0 > 0$. In [6] it is proved that minimizers exist provided that $c > 0$ is small and $p \in (2,3)$. In [5] the case $p \in (3, \frac{10}{3})$ is considered and a minimizer is obtained for $c > 0$ large enough.

In this paper we consider the case $p \in (\frac{10}{3}, 6)$. For this range of power the functional $F(u)$ is no more bounded from below on $S(c)$. We shall prove however that it has a mountain pass geometry.

Definition 1.1. Given $c > 0$, we say that $F(u)$ has a mountain pass geometry on $S(c)$ if there exists $K_c > 0$, such that

$$\gamma(c) = \inf_{g \in \Gamma_c} \max_{t \in [0,1]} F(g(t)) > \max\{F(g(0)), F(g(1))\},$$

holds in the set

$$\Gamma_c = \{g \in C([0,1], S(c)), \ g(0) \in A_{K_c}, F(g(1)) < 0\},$$

where $A_{K_c} = \{u \in S(c) : \ \|\nabla u\|^2_{L^2(\mathbb{R}^3)} \leq K_c\}$.

In order to find critical points of $F(u)$ on $S(c)$ we look at the mountain pass level $\gamma(c)$. Our main result concerning the existence of solutions of (1.2) is given by the following

Theorem 1.1. *Let $p \in (\frac{10}{3}, 6)$ and $c > 0$ then $F(u)$ has a mountain pass geometry on $S(c)$. Moreover there exists $c_0 > 0$ such that for any $c \in (0, c_0)$ there exists a couple $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}^-$ solution of (1.2) with $||u_c||_2^2 = c$ and $F(u_c) = \gamma(c)$.*

Let us underline some of the difficulties that arise in the study of the existence of critical points for our functional on $S(c)$. First the mountain pass geometry does not guarantee the existence of a bounded Palais-Smale sequence. To overcome this difficulty we introduce the functional

$$Q(u) := \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 \, |u(y)|^2}{|x-y|} dxdy - \frac{3(p-2)}{2p} \int_{\mathbb{R}^3} |u|^p dx,$$

the set

$$V(c) := \{u \in S(c) \ : \ Q(u) = 0\}$$

and we first prove that

$$\gamma(c) = \inf_{u \in V(c)} F(u). \tag{1.3}$$

We also show that each constrained critical point of $F(u)$ must lie in $V(c)$. At this point taking advantage of the nice "shape" of some sequence of paths $(g_n) \subset \Gamma_c$ such that

$$\max_{t \in [0,1]} F(g_n(t)) \to \gamma(c),$$

we construct a special Palais-Smale sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ at the level $\gamma(c)$ which concentrates around $V(c)$. This localization leads to its boundedness but also provide the information that $Q(u_n) = o(1)$. This last property is crucially used in the study of the compactness of the sequence. Next, since we look for solutions with a prescribed L^2-norm, we must deal with a possible lack of compactness for sequences which does not minimize $F(u)$ on $S(c)$. In our setting it does not seem possible to reduce the problem to the classical vanishing-dichotomy-compactness scenario and to the check of the associated strict subadditivity inequalities, see [28]. To overcome this difficulty we first study the behaviour of the function $c \to \gamma(c)$. The theorem below summarizes its properties.

Theorem 1.2. *Let $p \in (\frac{10}{3}, 6)$ and for any $c > 0$ let $\gamma(c)$ be the mountain pass level. Then*

(i) *$c \to \gamma(c)$ is continuous at each $c > 0$.*
(ii) *$c \to \gamma(c)$ is non-increasing.*

(iii) *There exists* $c_0 > 0$ *such that in* $(0, c_0)$ *the function* $c \to \gamma(c)$ *is strictly decreasing.*
(iv) *There exists* $c_\infty > 0$ *such that for all* $c \geq c_\infty$ *the function* $c \to \gamma(c)$ *is constant.*
(v) $\lim_{c \to 0} \gamma(c) = +\infty$ *and* $\lim_{c \to \infty} \gamma(c) := \gamma(\infty) > 0$.

We show that if $\gamma(c) < \gamma(c_1)$, for all $c_1 \in (0, c)$ then there exists $u_c \in H^1(\mathbb{R}^3)$ such that $||u_c||_2^2 = c$ and $F(u_c) = \gamma(c)$. However we are only able to prove this for $c > 0$ sufficiently small. For the other values of $c > 0$ the information that $c \to \gamma(c)$ is non increasing permits to reduce the problem of convergence to the one of showing that the associated Lagrange multiplier $\lambda_c \in \mathbb{R}$ is non zero. However we do prove that $\lambda_c = 0$ holds for any $c > 0$ is sufficiently large. In view of this property we conjecture that $\gamma(c)$ is not a critical value for $c > 0$ large enough. See Remark 7.1 in that direction.

Remark 1.1. The proof that $c \to \gamma(c)$ is non increasing is not derived through the use of some scaling. Due to the presence of three terms in $F(u)$ which scale differently such an approach seems difficult. Instead we show that if one adds in a suitable way L^2-norm in $\mathbb{R}^3$ then this does not increase the mountain pass level. This approach is reminiscent of the one developed in [25] but here the fact that we deal with a function defined by a mountain pass instead of a global minimum and that $F(u)$ has a nonlocal term makes the proof more delicate.

To show Theorem 1.2 (iv) and that $\gamma(c) \to \gamma(\infty) > 0$ as $c \to \infty$ in (v) we take advantage of some results of [19]. In [19] the equation

$$-\Delta v + (|x|^{-1} * |v|^2)v - |v|^{p-2}v = 0 \quad \text{in} \quad \mathbb{R}^3 \tag{1.4}$$

is considered. Real solutions of (1.4) are searched in the space

$$E := \{u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dxdy < \infty\} \tag{1.5}$$

which contains $H^1(\mathbb{R}^3)$. This space is the natural space when $\lambda = 0$ in (1.2). In [19] it is shown that $F(u)$ defined in E possess a ground state. It is also proved, see Theorem 6.1 of [19], that any real radial solution of (1.4) decreases exponentially at infinity. We extend here this result to any real solution of (1.4). More precisely we prove

Theorem 1.3. *Let* $p \in (3, 6)$ *and* $(u, \lambda) \in E \times \mathbb{R}$ *with* $\lambda \leq 0$ *be a real solution of (1.2). Then there exists constants* $C_1 > 0$, $C_2 > 0$ *and* $R > 0$ *such that*

$$|u(x)| \leq C_1 |x|^{-\frac{3}{4}} e^{-C_2 \sqrt{|x|}}, \ \forall \ |x| > R. \tag{1.6}$$

In particular, $u \in H^1(\mathbb{R}^3)$.

Remark 1.2. Clearly the difficult case here is when $\lambda = 0$ and it correspond to the so-called *zero mass case,* see [8]. This part of Theorem 1.3 was kindly provided to us by L. Dupaigne [15]. We point out that the exponential decay when $\lambda = 0$ is due to the fact that the nonlocal term is sufficienty strong at infinity. Actually we prove that $(|x|^{-1} * |v|^2) \geq C|x|^{-1}$ for some $C > 0$ and $|x|$ large. In contrast we recall that for the equation

$$-\Delta u + V(x)u - |u|^{p-2}u = 0, \quad x \in H^1(\mathbb{R}^3), \tag{1.7}$$

if we assume that $\limsup_{|x|\to\infty} V(x)|x|^{2+\delta} = 0$ for some $\delta > 0$, then positive solutions of (1.7) decay no faster than $|x|^{-1}$. This can be seen by comparing with an explicit subsolution at infinity $|x|^{-1}(1 + |x|^{-\delta})$ of $-\Delta + V$.

Theorem 1.3 is interesting for itself and also it answers a conjecture of [19], see Remark 6.2 there. For our study the information that any solution of (1.4) belongs to $L^2(\mathbb{R}^3)$ is crucial to derive Theorem 1.2 (iv)-(v) and the exponential decay is also used later to prove that our solutions correspond to standing waves unstable by blow-up.

The phenomena described in Theorems 1.1 and 1.2 are also due to the nonlocal term as we can see by comparing 1.1 with the classical nonlinear Schrödinger equation

$$i\psi_t + \Delta\psi + |\psi|^{p-2}\psi = 0 \quad \text{in } \mathbb{R}^3. \tag{1.8}$$

In [24] the existence of standing waves on $S(c)$ when the functional is unbounded from below was considered and a solution obtained for any $c > 0$. Here we show in addition that the mountain pass value $\tilde{\gamma}(c)$ associated to (1.8) is strictly decreasing as a function of $c > 0$ and that $\tilde{\gamma}(c) \to 0$ as $c \to \infty$.

The fact that (1.3) holds and that any constrained critical point of $F(u)$ lies in $V(c)$ implies that the solutions found in Theorem 1.1 can be considered as ground-states within the solutions having the same L^2-norm.

Let us denote the set of minimizers of $F(u)$ on $V(c)$ as

$$\mathcal{M}_c := \{u_c \in V(c) \ : \ F(u_c) = \inf_{u \in V(c)} F(u)\}. \tag{1.9}$$

Theorem 1.4. *Let $p \in (\frac{10}{3}, 6)$ and $c > 0$. For each $u_c \in \mathcal{M}_c$ there exists a $\lambda_c \leq 0$ such that $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ solves (1.2).*

Clearly to prove Theorem 1.4 we need to show that any minimizer of $F(u)$ on $V(c)$ is a critical point of $F(u)$ restricted to $S(c)$, namely that $V(c)$ acts as a natural constraint. As additional properties of elements of $\mathcal{M}_c$ we have :

Lemma 1.1. *Let $p \in (\frac{10}{3}, 6)$ and $c > 0$ be arbitrary. Then*

(i) *If $u_c \in \mathcal{M}_c$ then also $|u_c| \in \mathcal{M}_c$.*
(ii) *Any minimizer $u_c \in \mathcal{M}_c$ has the form $e^{i\theta}|u_c|$ for some $\theta \in \mathbb{S}^1$ and $|u_c(x)| > 0$ a.e. on $\mathbb{R}^3$.*

In view of Lemma 1.1 each elements of $\mathcal{M}_c$ is a real positive function multiply by a constant complex factor.

Remark 1.3. A natural question that arises, as a consequence of Theorem 1.4, is why not search for standing waves solutions of (1.2) with prescribed norm by directly minimizing $F(u)$ on $V(c)$. However starting from an arbitrary minimizing sequence $\{u_n\} \subset V(c)$ and trying to show its convergence seems challenging. Clearly, by definition of $V(c)$, any minimizing sequence is bounded in $H^1(\mathbb{R}^N)$ and thus we can assume that $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$ for some $\bar{u} \in H^1(\mathbb{R}^3)$. Also ruling out the vanishing is not a problem as it can be seen from Lemma 4.1. But to show that dichotomy do not occurs it seems necessary to know that $\bar{u} \in V(||\bar{u}||_2^2)$. For this we use, in Lemma 4.3, the information that $\bar{u} \in H^1(\mathbb{R}^3)$ is solution of (4.2). Then by Lemma 4.2, $Q(\bar{u}) = 0$ and $\overline{u} \in V(||\bar{u}||_2^2)$. For an arbitrary minimizing sequence it does not seems possible to show that the weak limit $\bar{u} \in H^1(\mathbb{R}^3)$ belongs to $V(||\bar{u}||_2^2)$. For this, informations on the derivative of $F(u)$ along the sequence seem necessary and that is why we introduce Palais-Smale sequences to solve our minimization problem.

Concerning the dynamics we first consider the question of global existence of solutions for the Cauchy problem. In the case $p \in (2, \frac{10}{3})$ global existence in time is guaranteed for initial data in $H^1(\mathbb{R}^3)$, see for instance [12]. In the case $p \in (2, \frac{10}{3})$ the standing waves found in [5], [6], [33] by minimization are orbitally stable. This is proved following the approach of Cazenave-Lions [13]. In the case $p \in (\frac{10}{3}, 6)$ the global existence in time of solutions for the Cauchy problem associated to (1.1) does not hold for arbitrary initial condition. However we are able to prove the following global existence result.

Theorem 1.5. *Let $p \in (\frac{10}{3}, 6)$ and $u_0 \in H^1(\mathbb{R}^3, \mathbb{C})$ be an initial condition associated to* (1.1) *with $c = ||u_0||_2^2$. If*

$$Q(u_0) > 0 \text{ and } F(u_0) < \gamma(c),$$

then the solution of (1.1) *with initial condition u_0 exists globally in times.*

In Remark 8.1 we prove that the set

$$\mathcal{O} = \{u_0 \in S(c) \ : \ Q(u_0) > 0 \text{ and } F(u_0) < \gamma(c)\}$$

is not empty.

Next we prove that the standing waves corresponding to elements of $\mathcal{M}_c$ are unstable in the following sense.

Definition 1.2. A standing wave $e^{i\omega t}v(x)$ is strongly unstable if for any $\varepsilon > 0$ there exists $u_0 \in H^1(\mathbb{R}^3, \mathbb{C})$ such that $\|u_0 - v\|_{H^1} < \varepsilon$ and the solution $u(t, \cdot)$ of the equation (1.1) with $u(0, \cdot) = u_0$ blows up in a finite time.

Theorem 1.6. *Let $p \in (\frac{10}{3}, 6)$ and $c > 0$. For each $u_c \in \mathcal{M}_c$ the standing wave $e^{-i\lambda_c t}u_c$ of (1.1), where $\lambda_c \in \mathbb{R}$ is the Lagrange multiplier, is strongly unstable.*

Remark 1.4. The proof of Theorem 1.6 borrows elements of the original approach of Berestycki and Cazenave [7]. The starting point is the variational characterization of $u_c \in \mathcal{M}_c$ and the decay estimates established in Theorem 1.3 proves crucial to use the virial identity.

Remark 1.5. For previous results concerning the instability of standing waves of (1.1) we refer to [23] (see also [22]). In [23], working in the subspace of radially symmetric functions, it is proved that for $\lambda < 0$ fixed and $p \in (\frac{10}{3}, 6)$ the equation (1.2) admits a ground state which is strongly unstable. However when we work in all $H^1(\mathbb{R}^3)$ it is still not known if ground states, or at least one of them, are radially symmetric. In that direction we are only aware of the result of [17] which gives a positive answer when $p \in (2, 3)$ and for $c > 0$ sufficiently small. In this range the critical point is found as a minimizer of $F(u)$ on $S(c)$.

Finally we prove

Theorem 1.7. *Let $p \in (\frac{10}{3}, 6)$. Any ground state of (1.4) is strongly unstable.*

Remark 1.6. In the *zero mass case* there seems to be few results of stability/instability of standing waves. We are only aware of [20] for a stability result.

The paper is organized as follows. In Section 2 we establish the mountain pass geometry of $F(u)$ on $S(c)$. In Section 3 we construct the special bounded Palais-Smale sequence at the level $\gamma(c)$. In Section 4 we show the convergence of the Palais-Smale sequence and we conclude the proof of Theorem 1.1. In Section 5 some parts of Theorem 1.2 are established. In Section 6 we prove Theorem 1.4 and Lemma 1.1. In Section 7 we prove Theorem 1.3 and using elements from [19] we end the proof of Theorem 1.2. Section 8 is devoted to the proof of Theorems 1.5, 1.6 and 1.7. Finally in Section 9 we discuss the nonlinear Schrödinger equation case.

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1.1. **Notations.** In the paper it is understood that all functions, unless otherwise stated, are complex-valued, but for simplicity we write $L^s(\mathbb{R}^3)$, $H^1(\mathbb{R}^3)$...., and for any $1 \leq s < +\infty$, $L^s(\mathbb{R}^3)$ is the usual Lebesgue space endowed with the norm

$$\|u\|_s^s := \int_{\mathbb{R}^3} |u|^s dx,$$

and $H^1(\mathbb{R}^3)$ the usual Sobolev space endowed with the norm

$$\|u\|^2 := \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} |u|^2 dx.$$

Moreover we define, for short, the following quantities

$$A(u) := \int_{\mathbb{R}^3} |\nabla u|^2 dx, \quad B(u) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2 \, |u(y)|^2}{|x-y|} dxdy$$

$$C(u) := -\int_{\mathbb{R}^3} |u|^p dx, \quad D(u) := \int_{\mathbb{R}^3} |u|^2 \, dx.$$

Then

$$Q(u) = A(u) + \frac{1}{4}B(u) + \frac{3(p-2)}{2p}C(u). \tag{1.10}$$

2. The mountain pass geometry on the constraint

In this section, we discuss the Mountain Pass Geometry ("MP Geometry" for short) of the functional $F(u)$ on the L^2-constraint $S(c)$. We show the following:

Theorem 2.1. *When $p \in (\frac{10}{3}, 6)$, for any $c > 0$, $F(u)$ has a MP geometry on the constraint $S(c)$.*

Before proving Theorem 2.1 we establish some lemmas. We first introduce the Cazenave's scaling [12]. For $u \in S(c)$, we set $u^t(x) = t^{\frac{3}{2}} u(tx), t > 0$, then

$$A(u^t) = t^2 A(u) \ , \ D(u^t) = D(u),$$

and

$$B(u^t) = tB(u) \ , \ C(u^t) = t^{\frac{3}{2}(p-2)} C(u).$$

Thus

$$F(u^t) = \frac{t^2}{2}A(u) + \frac{t}{4}B(u) + \frac{t^{\frac{3}{2}(p-2)}}{p}C(u). \tag{2.1}$$

Lemma 2.1. *Let $u \in S(c)$, $c > 0$ be arbitrary but fixed and $p \in (\frac{10}{3}, 6)$, then:*

1. *$A(u^t) \to \infty$ and $F(u^t) \to -\infty$, as $t \to \infty$.*
2. *There exists $k_0 > 0$ such that $Q(u) > 0$ if $||\nabla u||_2 \leq k_0$ and $-C(u) \geq k_0$ if $Q(u) = 0$.*
3. *If $F(u) < 0$ then $Q(u) < 0$.*

Proof. We notice that

$$F(u) - \frac{2}{3(p-2)}Q(u) = \frac{3p-10}{6(p-2)}A(u) + \frac{3p-8}{12(p-2)}B(u). \tag{2.2}$$

Thus (3) holds since the RHS is always positive. Moreover, thanks to Gagliardo-Nirenberg inequality there exists a constant $C(p) > 0$ such that

$$Q(u) \geq A(u) - C(p)A(u)^{\frac{3(p-2)}{4}} D(u)^{\frac{6-p}{4}}.$$

The fact that $\frac{3(p-2)}{4} > 1$ insures that $Q(u) > 0$ for sufficiently small $A(u)$. Also when $Q(u) = 0$

$$-C(u) = \frac{2p}{3(p-2)}[A(u) + \frac{1}{4}B(u)] \geq \frac{2p}{3(p-2)}A(u)$$

and this ends the proof of (2). Finally (1) follows directly from (2.1) and since $A(u^t) = t^2 A(u)$. □

Our next lemma is inspired by Lemma 8.2.5 in [12].

Lemma 2.2. *When $p \in (\frac{10}{3}, 6)$, given $u \in S(c)$ we have:*
(1) There exists a unique $t^\star(u) > 0$, such that $u^{t^\star} \in V(c)$;
(2) The mapping $t \longmapsto F(u^t)$ is concave on $[t^\star, \infty)$;
(3) $t^\star(u) < 1$ if and only if $Q(u) < 0$;
(4) $t^\star(u) = 1$ if and only if $Q(u) = 0$;
(5)

$$Q(u^t) \begin{cases} > 0, \ \forall \ t \ \in \ (0, t^*(u)); \\ < 0, \ \forall \ t \ \in \ (t^*(u), +\infty). \end{cases}$$

(6) $F(u^t) < F(u^{t^\star})$, for any $t > 0$ and $t \neq t^\star$;
(7) $\frac{\partial}{\partial t} F(u^t) = \frac{1}{t} Q(u^t)$, $\forall t > 0$.

Proof. Since

$$F(u^t) = \frac{t^2}{2}A(u) + \frac{t}{4}B(u) + \frac{t^{\frac{3}{2}(p-2)}}{p}C(u)$$

we have that

$$\frac{\partial}{\partial t}F(u^t) = tA(u) + \frac{1}{4}B(u) + \frac{3(p-2)}{2p}t^{\frac{3}{2}(p-2)-1}C(u) = \frac{1}{t}Q(u^t)$$

and this proves (7). Now we denote

$$y(t) = tA(u) + \frac{1}{4}B(u) + \frac{3(p-2)}{2p}t^{\frac{3}{2}(p-2)-1}C(u),$$

and observe that $Q(u^t) = t \cdot y(t)$. After direct calculations, we see that:

$$\begin{aligned} y'(t) &= A(u) + \frac{3(p-2)(3p-8)}{4p}t^{\frac{3p-10}{2}}C(u); \\ y''(y) &= \frac{3(p-2)(3p-8)}{4p} \cdot \frac{3p-10}{2} \cdot t^{\frac{3p-12}{2}}C(u). \end{aligned}$$

¿From the expression of $y'(t)$ we know that $y'(t)$ has a unique zero that we denote $t_0 > 0$. Since $p \in (\frac{10}{3}, 6)$ we see that $y''(t) < 0$ and t_0 is the unique maximum point of $y(t)$. Thus in particular the function $y(t)$ satisfies:
(i) $y(t_0) = \max_{t>0} y(t)$;
(ii) $y(0) = \frac{1}{4}B(u)$;

(iii) $\lim_{t\to+\infty} y(t) = -\infty$;
(iv) $y(t)$ decreases strictly in $[t_0, +\infty)$ and increases strictly in $(0, t_0]$.

Since $B(u) \neq 0$, by the continuity of $y(t)$, we deduce that $y(t)$ has a unique zero $t^\star > 0$. Then $Q(u^{t^*}) = 0$ and point (1) follows. Point (2) (3) and (5) are also easy consequences of (i)-(iv). Since $\frac{\partial}{\partial t}F(u^t)|_{t=t^\star} = 0$, $\frac{\partial^2}{\partial t^2}F(u^t)|_{t=t^\star} = y'(t^*) < 0$ and $t^\star$ is unique we get (4) and (6). □

Proof of Theorem 2.1. We denote

$$\alpha_k := \sup_{u\in C_k} F(u) \quad \text{and} \quad \beta_k := \inf_{u\in C_k} F(u)$$

where

$$C_k := \{u \in S(c) : A(u) = k, k > 0\}.$$

Let us show that there exist $0 < k_1 < k_2$ such that

$$\alpha_k < \beta_{k_2} \text{ for all } k \in (0, k_1] \text{ and } Q(u) > 0 \text{ if } A(u) < k_2. \tag{2.3}$$

Notice that, from Hardy-Littlewood-Sobolev's inequality and Gagliardo-Nirenberg's inequalities, it follows that

$$\begin{aligned} F(u) &\leq \frac{1}{2}A(u) + \frac{1}{4}B(u) \leq \frac{1}{2}A(u) + C(p)\|u\|^4_{L^{\frac{12}{5}}} \\ &\leq \frac{1}{2}A(u) + \widetilde{C}(p)A(u)^{\frac{1}{2}} \cdot D(u)^{\frac{3}{2}}. \end{aligned}$$

In particular $\alpha_{k_1} \to 0^+$ as $k_1 \to 0^+$. On the other hand still by the Gagliardo-Nirenberg inequality we have

$$F(u) \geq \frac{1}{2}A(u) + \frac{1}{p}C(u) \geq \frac{1}{2}A(u) - C(p)A(u)^{\frac{3(p-2)}{4}} \cdot D(u)^{\frac{6-p}{4}}.$$

Thus, since $\frac{3(p-2)}{2} > 1$, $\beta_{k_2} \geq \frac{1}{4}k_2$ for any $k_2 > 0$ small enough. These two observations and Lemma 2.1 (2) prove that (2.3) hold. We now fix a $k_1 > 0$ and a $k_2 > 0$ as in (2.3). Thus for

$$\Gamma_c = \{g \in C([0,1], S(c)),\ g(0) \in A_{k_1}, F(g(1)) < 0\},$$

if $\Gamma_c \neq \emptyset$, then from the definition of $\gamma(c)$, we have $\gamma(c) \geq \beta_{k_2} > 0$ We only need to verify that $\Gamma_c \neq \emptyset$. This fact follows from Lemma 2.1 (1). □

Remark 2.1. As it is clear from the proof of Theorem 2.1 we can assume without restriction that

$$\sup_{u\in A_{K_c}} F(u) < \gamma(c)/2$$

where A_{K_c} is introduced in the Definition 1.1.

Lemma 2.3. *When $p \in (\frac{10}{3}, 6)$, we have*

$$\gamma(c) = \inf_{u\in V(c)} F(u).$$

Proof. Let us argue by contradiction. Suppose there exists $v \in V(c)$ such that $F(v) < \gamma(c)$, and let, for $\lambda > 0$,

$$v^\lambda(x) = \lambda^{3/2} v(\lambda x).$$

Then, since $A(v^\lambda) = \lambda^2 A(v)$ there exists $0 < \lambda_1 < 1$ sufficiently small so that $v^{\lambda_1} \in A_{k_1}$. Also by Lemma 2.1 (1) there exists a $\lambda_2 > 1$ sufficiently large so that $F(v^{\lambda_2}) < 0$. Therefore if we define

$$g(t) = v^{(1-t)\lambda_1 + t\lambda_2}, \quad \text{for } t \in [0,1]$$

we obtain a path in Γ_c. By definition of $\gamma(c)$ and using Lemma 2.2,

$$\gamma(c) \leq \max_{t\in[0,1]} F(g(t)) = F\Big(g(\frac{1-\lambda_1}{\lambda_2-\lambda_1})\Big) = F(v),$$

and thus

$$\gamma(c) \leq \inf_{u \in V(c)} F(u).$$

On other hand thanks to Lemma 2.1 any path in Γ_c crosses $V(c)$ and hence

$$\max_{t\in[0,1]} F(g(t)) \geq \inf_{u\in V(c)} F(u).$$

□

3. Localization of a PS sequence

In this section we prove a localization lemma for a specific Palais-Smale sequence $\{u_n\} \subset S(c)$ for $F(u)$ constrained to $S(c)$. From this localization we deduce that the sequence is bounded and that $Q(u_n) = o(1)$. This last property will be essential later to establish the compactness of the sequence. First we observe that, for any fixed $c > 0$, the set

$$L := \{u \in V(c), F(u) \leq \gamma(c) + 1\}$$

is bounded. This follows directly from the observation that

$$F(u) - \frac{2}{3(p-2)} Q(u) = \frac{3p-10}{6(p-2)} A(u) + \frac{3p-8}{12(p-2)} B(u) \tag{3.1}$$

and the fact that $\frac{3p-10}{6(p-2)} > 0$, $\frac{3p-8}{12(p-2)} > 0$ if $p \in (\frac{10}{3}, 6)$.

Let $R_0 > 0$ be such that $L \subset B(0, R_0)$ where $B(0,R_0) := \{u \in H^1(\mathbb{R}^3), ||u|| \leq R_0\}$.

The crucial localization result is the following.

Lemma 3.1. *Let* $p \in (\frac{10}{3}, 6)$ *and*

$$K_\mu := \Big\{u \in S(c) \ s.t. \ |F(u) - \gamma(c)| \leq \mu, \ dist(u, V(c)) \leq 2\mu, \ ||F'|_{S(c)}(u)||_{H^{-1}} \leq 2\mu \Big\},$$

then for any $\mu > 0$, *the set* $K_\mu \bigcap B(0, 3R_0)$ *is not empty.*

In order to prove Lemma 3.1 we need to develop a deformation argument on $S(c)$. Following [9] we recall that, for any $c>0$, $S(c)$ is a submanifold of $H^1(\mathbb{R}^3)$ with codimension 1 and the tangent space at a point $\bar{u} \in S(c)$ is defined as

$$T_{\bar{u}}S(c) = \{v \in H^1(\mathbb{R}^3) \text{ s.t. } (\bar{u}, v)_2 = 0\}.$$

The restriction $F_{|_{S(c)}} : S(c) \to \mathbb{R}$ is a C^1 functional on $S(c)$ and for any $\bar{u} \in S(c)$ and any $v \in T_{\bar{u}}S(c)$

$$\langle F'_{|_{S(c)}}(\bar{u}), v\rangle = \langle F'(\bar{u}), v\rangle.$$

We use the notation $||dF_{|_{S(c)}}(\bar{u})||$ to indicate the norm in the cotangent space $T_{\bar{u}}S(c)'$, i.e the dual norm induced by the norm of $T_{\bar{u}}S(c)$, i.e

$$||dF_{|_{S(c)}}(\bar{u})|| := \sup_{||v|| \leq 1,\ v \in T_{\bar{u}}S(c)} |\langle dF(\bar{u}), v\rangle|.$$

Let $\tilde{S}(c) := \{u \in S(c) \text{ s.t. } dF|_{S(c)}(u) \neq 0\}$. We know from [9] that there exists a locally Lipschitz pseudo gradient vector field $Y \in \mathcal{C}^1(\tilde{S}(c), T(S(c))$ (here $T(S(c))$ is the tangent bundle) such that

$$\|Y(u)\| \leq 2\, ||dF_{|_{S(c)}}(u)||, \tag{3.2}$$

and

$$\langle F'_{|_{S(c)}}(\bar{u}), Y(u)\rangle \geq ||dF_{|_{S(c)}}(u)||^2, \tag{3.3}$$

for any $u \in \tilde{S}(c)$. Note that $||Y(u)|| \neq 0$ for $u \in \tilde{S}(c)$ thanks to (3.3). Now for an arbitrary but fixed $\mu > 0$ we consider the sets

$$\tilde{N}_\mu := \{u \in S(c) \text{ s.t. } |F(u) - \gamma(c)| \leq \mu,\ dist(u, V(c)) \leq 2\mu,\ ||Y(u)|| \geq 2\mu\}$$

$$N_\mu := \{u \in S(c) \text{ s.t. } |F(u) - \gamma(c)| < 2\mu\}$$

where, for a subset $\mathcal{A}$ of $S(c)$, $dist(x, \mathcal{A}) := \inf\{||x-y|| : y \in \mathcal{A}\}$. Assuming that $\tilde{N}_\mu$ is non empty there exists a locally Lipschitz function $g : S(c) \to [0,1]$ such that

$$g = \begin{cases} 1 \text{ on } \tilde{N}_\mu \\ 0 \text{ on } N_\mu^c. \end{cases}$$

We also define on $S(c)$ the vector field W by

$$W(u) = \begin{cases} -g(u)\frac{Y(u)}{||Y(u)||} \text{ if } u \in \tilde{S}(c) \\ 0 \text{ if } u \in S(c)\backslash\tilde{S}(c) \end{cases} \tag{3.4}$$

and the pseudo gradient flow

$$\begin{cases} \frac{d}{dt}\eta(t,u) = W(\eta(t,u)) \\ \eta(0,u) = u. \end{cases} \tag{3.5}$$

The existence of a unique solution $\eta(t,\cdot)$ of (3.5) defined for all $t \in \mathbb{R}$ follows from standard arguments and we refer to Lemma 5 in [9] for this. Let us recall some of its basic properties that will be useful to us

- $\eta(t,\cdot)$ is a homeomorphism of $S(c)$;
- $\eta(t,u)=u$ for all $t\in\mathbb{R}$ if $|F(u)-\gamma(c)|\geq 2\mu$;
- $\frac{d}{dt}F(\eta(t,u))=\langle dF(\eta(t,u)),W(\eta(t,u))\rangle\leq 0$ for all $t\in\mathbb{R}$ and $u\in S(c)$.

Proof of Lemma 3.1. : Let us define, for $\mu>0$,

$$\Lambda_\mu=\{u\in S(c)\text{ s.t. }|F(u)-\gamma(c)|\leq\mu,\ dist(u,V(c))\leq 2\mu\}\,.$$

In order to prove Lemma 3.1 we argue by contradiction assuming that there exists $\bar{\mu}\in(0,\gamma(c)/4)$ such that

$$u\in\Lambda_{\bar{\mu}}\cap B(0,3R_0)\implies ||F'|_{S(c)}(u)||_{H^{-1}}>2\bar{\mu}. \tag{3.6}$$

Then it follows from (3.3) that

$$u\in\Lambda_{\bar{\mu}}\cap B(0,3R_0)\implies u\in\tilde{N}_{\bar{\mu}}. \tag{3.7}$$

Also notice that, since by (3.5),

$$||\frac{d}{dt}\eta(t,u)||\leq 1,\quad\forall t\geq 0,\,\forall u\in S(c),$$

there exists $s_0>0$ depending on $\bar{\mu}>0$ such that, for all $s\in(0,s_0)$,

$$u\in\Lambda_{\frac{\bar{\mu}}{2}}\cap B(0,2R_0)\implies\eta(s,u)\in B(0,3R_0)\text{ and }dist(\eta(s,u),V(c))\leq 2\bar{\mu}. \tag{3.8}$$

We claim that, taking $\varepsilon>0$ sufficiently small, we can construct a path $g_\varepsilon(t)\in\Gamma_c$ such that

$$\max_{t\in[0,1]}F(g_\varepsilon(t)\leq\gamma(c)+\varepsilon$$

and

$$F(g_\varepsilon(t))\geq\gamma(c)\implies g_\varepsilon(t)\in\Lambda_{\frac{\bar{\mu}}{2}}\cap B(0,2R_0). \tag{3.9}$$

Indeed, for $\varepsilon>0$ small, let $u\in V(c)$ be such that $F(u)\leq\gamma(c)+\varepsilon$ and consider the path defined in Lemma 2.3 by

$$g_\varepsilon(t)=u^{(1-t)\lambda_1+t\lambda_2},\quad\text{for }t\in[0,1]. \tag{3.10}$$

Clearly

$$\max_{t\in[0,1]}F(g_\varepsilon(t))\leq\gamma(c)+\varepsilon.$$

Also for $t_\varepsilon^*>0$ such that $(1-t_\varepsilon^*)\lambda_1+t_\varepsilon^*\lambda_2=1$ we have, since $g_\varepsilon(t_\varepsilon^*)\in V(c)$, that

$$\frac{d^2}{d^2s}F(g_\varepsilon(s))_{|t_\varepsilon^*}=-\frac{1}{4}B(u)-\frac{3}{2p}(p-2)(5-\frac{3}{2}p)C(u)\leq -Ck_0<0 \tag{3.11}$$

where $k_0>0$ is given in Lemma 2.1 (2). The estimate (3.11) is uniform with respect to the choice of $\varepsilon>0$ and of $u\in V(c)$. Thus, by Taylor's formula, it is readily seen that

$$\{t\in[0,1]:F(g_\varepsilon(t))\geq\gamma(c)\}\subset[t_\varepsilon^*-\alpha_\varepsilon,t_\varepsilon^*+\alpha_\varepsilon]$$

for some $\alpha_\varepsilon > 0$ with $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$. The claim (3.10) follows for continuity arguments.

We fix a $\varepsilon \in (0, \frac{1}{4}\bar{\mu}s_0)$ such that (3.9) hold. Applying the pseudo gradient flow, constructed with $\bar{\mu} > 0$, on $g_\varepsilon(t)$ we see that $\eta(s, g_\varepsilon(\cdot)) \in \Gamma_c$ for all $s > 0$. Indeed $\eta(s,u) = u$ for all $s > 0$ if $|F(u) - \gamma(c)| \geq 2\bar{\mu}$ and we conclude by Remark 2.1.

We claim that taking $s^* := \dfrac{4\varepsilon}{\bar{\mu}} < s_0$

$$\max_{t\in[0,1]} F(\eta(s^*, g_\varepsilon(t))) < \gamma(c). \tag{3.12}$$

If (3.12) hold we have a contradiction with the definition of $\gamma(c)$ and thus the lemma is proved. To prove (3.12) for simplicity we set $w = g_\varepsilon(t)$ where $t \in [0,1]$. If $F(w) < \gamma(c)$ there is nothing to prove since then $F(\eta(s^*, w) \leq F(w) < \gamma(c)$ for any $s > 0$. If $F(w) \geq \gamma(c)$ we assume by contradiction that $F(\eta(s,w)) \geq \gamma(c)$ for all $s \in [0, s^*]$. Then by (3.8) and (3.9), $\eta(s,w) \in \Lambda_{\bar{\mu}} \cap B(0, 3R_0)$ for all $s \in [0, s^*]$. In particular $||Y(\eta(s,w))|| \geq 2\bar{\mu}$ and $g(\eta(s,w)) = 1$ for all $s \in [0, s^*]$. Thus

$$\frac{d}{ds}F(\eta(s,w)) = \langle dF(\eta(s,w)), -\frac{Y(\eta(t,u))}{||Y(\eta(t,u))||}\rangle.$$

By integration, and since $s^* = \dfrac{4\varepsilon}{\bar{\mu}}$, we get

$$F(\eta(s^*, w)) \leq F(w) - \bar{\mu}s^* \leq (\gamma(c) + \varepsilon) - 2\varepsilon < \gamma(c) - \varepsilon.$$

This proves the claim (3.12) and the lemma. □

Lemma 3.2. *Let $p \in (\frac{10}{3}, 6)$, then there exists a sequence $\{u_n\} \subset S(c)$ and a constant $\alpha > 0$ fulfilling*

$$Q(u_n) = o(1), \quad F(u_n) = \gamma(c) + o(1),$$
$$||F'|_{S(c)}(u_n)||_{H^{-1}} = o(1), \ ||u_n|| \leq \alpha.$$

Proof. First let us consider $\{u_n\} \subset S(c)$ such that $\{u_n\} \subset B(0, 3R_0)$,

$$dist(u_n, V(c)) = o(1), \ |F(u_n) - \gamma(c)| = o(1), \ ||F'|_{S(c)}(u_n)||_{H^{-1}} = o(1).$$

Such sequence exists thanks to Lemma 3.1. To prove the lemma we just have to show that $Q(u_n) = o(1)$. It is readily checked that $||dQ(\cdot)||_{H^{-1}}$ is bounded on any bounded set of $H^1(\mathbb{R}^3)$ and thus in particular on $B(0, 3R_0)$. Now, for any $n \in \mathbb{N}$ and any $w \in V(c)$ we can write

$$Q(u_n) = Q(w) + dQ(au_n + (1-a)w)(u_n - w)$$

where $a \in [0,1]$. Thus since $Q(w) = 0$ we have

$$|Q(u_n)| \leq \max_{u\in B(0,3R_0)} ||dQ||_{H^{-1}}||u_n - w||. \tag{3.13}$$

Finally choosing $\{w_m\} \subset V(c)$ such that

$$||u_n - w_m|| \to dist(u_n, V(c)) \text{ as } m \to \infty,$$

since $dist(u_n, V(c)) \to 0$ we obtain from (3.13) that $Q(u_n) = o(1)$. □

4. Compactness of our Palais-Smale sequence

Proposition 4.1. *Let $\{v_n\} \subset S(c)$ be a bounded Palais-Smale for $F(u)$ restricted to $S(c)$ such that $F(v_n) \to \gamma(c)$. Then there is a sequence $\{\lambda_n\} \subset \mathbb{R}$, such that, up to a subsequence:*
(1) $v_n \rightharpoonup v_c$ weakly in $H^1(\mathbb{R}^3)$;
(2) $\lambda_n \to \lambda_c$ in $\mathbb{R}$;
(3) $-\Delta v_n - \lambda_n v_n + (|x|^{-1} * |v_n|^2)v_n - |v_n|^{p-2}v_n \to 0$ in $H^{-1}(\mathbb{R}^3)$;
(4) $-\Delta v_n - \lambda_c v_n + (|x|^{-1} * |v_n|^2)v_n - |v_n|^{p-2}v_n \to 0$ in $H^{-1}(\mathbb{R}^3)$;
(5) $-\Delta v_c - \lambda_c v_c + (|x|^{-1} * |v_c|^2)v_c - |v_c|^{p-2}v_c = 0$ in $H^{-1}(\mathbb{R}^3)$.

Proof. Point (1) is trivial. Since $\{v_n\} \subset H^1(\mathbb{R}^3)$ is bounded, following Berestycki and Lions (see Lemma 3 in [9]), we know that:

$$\begin{aligned} F'|_{S(c)}(v_n) \quad &\longrightarrow \quad 0 \ \text{ in } H^{-1}(\mathbb{R}^3) \\ &\iff \quad F'(v_n) - \langle F'(v_n), v_n\rangle v_n \longrightarrow 0 \text{ in } H^{-1}(\mathbb{R}^3). \end{aligned}$$

Thus, for any $w \in H^1(\mathbb{R}^3)$,

$$\begin{aligned} \langle F'(v_n) - \langle F'(v_n), v_n\rangle v_n, w\rangle = &\int_{\mathbb{R}^3} \nabla v_n \nabla w dx + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v_n(x)|^2}{|x-y|} v_n(y)w(y)dxdy \\ &- \int_{\mathbb{R}^3} |v_n|^{p-2}v_n w dx - \lambda_n \int_{\mathbb{R}^3} v_n(x)w(x)dx, \end{aligned}$$

with

$$\lambda_n = \frac{1}{\|v_n\|_2}\Big\{\|\nabla v_n\|_2^2 + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v_n(x)|^2 v_n(x)^2}{|x-y|}dxdy - \|v_n\|_p^p\Big\}. \tag{4.1}$$

Thus we obtain (3) with $\{\lambda_n\} \subset \mathbb{R}$ defined by (4.1). If (2) holds then (4) follows immediately from (3). To prove (2), it is enough to verify that $\{\lambda_n\} \subset \mathbb{R}$ is bounded. But since $\{v_n\} \subset H^1(\mathbb{R}^N)$ is bounded, by the Hardy-Littlewood-Sobolev inequality and Gagliardo-Nirenberg inequality, it is easy to see that all terms in the RHS of (4.1) are bounded. Finally we refer to Lemma 2.2 in [35] for a proof of (5). □

Lemma 4.1. *Let $p \in (\frac{10}{3}, 6)$ and $\{u_n\} \subset S(c)$ be a bounded sequence such that*

$$Q(u_n) = o(1) \quad \text{and} \quad F(u_n) \to \gamma(c) \text{ with } \gamma(c) > 0,$$

then, up to a subsequence and up to translation $u_n \rightharpoonup \bar{u} \neq 0$.

Proof. If the lemma does not hold it means by standard arguments that $\{u_n\} \subset S(c)$ is vanishing and thus that $C(u_n) = o(1)$ (see [28]). Thus let us argue by contradiction assuming that $C(u_n) = o(1)$, i.e. that, since $Q(u_n) = o(1)$, $A(u_n) + \frac{1}{4}B(u_n) = o(1)$. Now from (3.1) we immediately deduce that $F(u_n) = o(1)$ and this contradicts the assumption that $F(u_n) \to \gamma(c) > 0$. $\square$

Lemma 4.2. *Let $p \in (\frac{10}{3}, 6)$, $\lambda \in \mathbb{R}$. If $v \in H^1(\mathbb{R}^3)$ is a weak solution of*

$$-\Delta v + \left(|x|^{-1} * |v|^2\right) v - |v|^{p-2} v = \lambda v \tag{4.2}$$

then $Q(v) = 0$. Moreover if $\lambda \geq 0$, there exists a constant $c_0 > 0$ independent on $\lambda \in \mathbb{R}$ such that the only solution of (4.2) fulfilling $||v||_2^2 \leq c_0$ is the null function.

Proof. The following Pohozaev type identity holds for $v \in H^1(\mathbb{R}^3)$ weak solution of (4.2), see [14],

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{5}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v(x)|^2 |v(y)|^2}{|x-y|} dxdy - \frac{3}{p}\int_{\mathbb{R}^3} |v|^p\, dx = \frac{3\lambda}{2}\int_{\mathbb{R}^3} |v|^2\, dx.$$

By multiplying (4.2) by v and integrating we derive a second identity

$$\int_{\mathbb{R}^3} |\nabla v|^2 dx + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v(x)|^2 |v(y)|^2}{|x-y|} dxdy - \int_{\mathbb{R}^3} |v|^p\, dx = \lambda \int_{\mathbb{R}^3} |v|^2\, dx.$$

With simple calculus we obtain the following relations

$$\begin{aligned} &A(v) + \frac{1}{4}B(v) + 3\left(\frac{p-2}{2p}\right) C(v) = 0, \\ &(\frac{p-6}{3p-6})A(v) + (\frac{5p-12}{3p-6})\frac{B(v)}{2} = \lambda D(v). \end{aligned} \tag{4.3}$$

The first relation of (4.3) is $Q(v) = 0$. This identity together with the Gagliardo-Nirenberg inequality assures the existence of a constant $C(p)$ such that

$$A(v) - C(p)A(v)^{\frac{3(p-2)}{4}} D(v)^{\frac{6-p}{4}} \leq A(v) + 3\left(\frac{p-2}{2p}\right) C(v) \leq 0, \tag{4.4}$$

i.e

$$A(v)^{\frac{10-3p}{4}} \leq C(p) D(v)^{\frac{6-p}{4}}. \tag{4.5}$$

Now we recall that by the Hardy-Littlehood-Sobolev inequality and the Gagliardo-Nirenberg inequality we have

$$B(v) \leq C A(v)^{\frac{1}{2}} D(v)^{\frac{3}{2}}, \tag{4.6}$$

then, from the second relation of (4.3) we obtain

$$\lambda D(v) \leq (\frac{p-6}{3p-6})A(v) + \tilde{C}(p) A(v)^{\frac{1}{2}} D(v)^{\frac{3}{2}}. \tag{4.7}$$

Notice that (4.5) tells us that, for any solution u of (4.2) with small L^2-norm, $A(u)$ must be large. This fact assures that the left hand side of (4.7) cannot be non negative when $D(v)$ is sufficiently small. □

Lemma 4.3. *Let $p \in (\frac{10}{3}, 6)$. Assume that the bounded Palais-Smale sequence $\{u_n\} \subset S(c)$ given by Lemma 3.2 is weakly convergent, up to translations, to the nonzero function $\bar{u}$. Moreover assume that*

$$\forall c_1 \in (0, c), \quad \gamma(c_1) > \gamma(c). \tag{4.8}$$

Then $\|u_n - \bar{u}\| \to 0$. In particular it follows that $\bar{u} \in S(c)$ and $F(\bar{u}) = \gamma(c)$.

Proof. Let $T(u) := \frac{1}{4}B(u) + \frac{1}{p}C(u)$ such that

$$F(u) := \frac{1}{2}\|\nabla u\|_2^2 + T(u). \tag{4.9}$$

In [5] or [35] it is shown that the nonlinear term T fulfills the following splitting properties of Brezis-Lieb type (see [10]),

$$T(u_n - \bar{u}) + T(\bar{u}) = T(u_n) + o(1). \tag{4.10}$$

We argue by contradiction and assume that $c_1 = ||\bar{u}||_2^2 < c$. By Proposition 4.1 (5) and Lemma 4.2 we have $Q(\bar{u}) = 0$ and thus $\bar{u} \in V(c_1)$. Now since $u_n - \bar{u} \rightharpoonup 0$,

$$\|\nabla(u_n - \bar{u})\|_2^2 + \|\nabla \bar{u}\|_2^2 = \|\nabla u_n\|_2^2 + o(1). \tag{4.11}$$

Also since $\{u_n\} \subset S(c)$ is a sequence at the level $\gamma(c)$ we get

$$\frac{1}{2}\|\nabla u_n\|_2^2 + T(u_n) = \gamma(c) + o(1). \tag{4.12}$$

Combining (4.10)- (4.12) we deduce that

$$\frac{1}{2}\|\nabla(u_n - \bar{u})\|_2^2 + \frac{1}{2}\|\nabla \bar{u}\|_2^2 + T(u_n - \bar{u}) + T(\bar{u}) = \gamma(c) + o(1). \tag{4.13}$$

At this point, using that $\bar{u} \in V(c_1)$ and Lemma 2.3 we get from (4.13) that

$$F(u_n - \bar{u}) + \gamma(c_1) \leq \gamma(c) + o(1). \tag{4.14}$$

On the other hand,

$$F(u_n - \bar{u}) - \frac{2}{3(p-2)}Q(u_n - \bar{u}) = \frac{3p-10}{6(p-2)}A(u_n - \bar{u}) + \frac{3p-8}{12(p-2)}B(u_n - \bar{u}) \tag{4.15}$$

and

$$Q(u_n - \bar{u}) = Q(u_n - \bar{u}) + Q(\bar{u}) = Q(u_n) + o(1) = o(1). \tag{4.16}$$

¿From (4.15) and (2.2) we deduce that $F(u_n - \bar{u}) \geq o(1)$. But then from (4.14) we obtain a contradiction with (4.8). This contradiction proves that $\|\bar{u}\|_2^2 = c$ and $F(\bar{u}) \geq \gamma(c)$. Now still by (4.14) we get $F(u_n - \bar{u}) \leq o(1)$ and thanks to (4.15) and (4.16) $A(u_n - \bar{u}) = o(1)$. i.e $\|\nabla(u_n - \bar{u})\|_2 = o(1)$. □

Lemma 4.4. *Let $p \in (\frac{10}{3}, 6)$. Assume that the bounded Palais-Smale sequence $\{u_n\} \subset S(c)$ given by Lemma 3.2 is weakly convergent, up to translations, to the nonzero function $\bar{u}$. Moreover assume that*

$$\forall c_1 \in (0, c), \ \gamma(c_1) \geq \gamma(c) \tag{4.17}$$

and that the Lagrange multiplier given by Proposition 4.1 fulfills

$$\lambda_c \neq 0.$$

Then $\|u_n - \bar{u}\| \to 0$. In particular it follows that $\bar{u} \in S(c)$ and $F(\bar{u}) = \gamma(c)$.

Proof. Let us argue as in Lemma 4.3. We obtain again

$$F((u_n - \bar{u})) + \gamma(c_1) \leq \gamma(c) + o(1),$$

$$F(u_n - \bar{u}) - \frac{2}{3(p-2)} Q(u_n - \bar{u}) = \frac{3p-10}{6(p-2)} A(u_n - \bar{u}) + \frac{3p-8}{12(p-2)} B(u_n - \bar{u})$$

and

$$Q(u_n - \bar{u}) = Q(u_n - \bar{u}) + Q(\bar{u}) = Q(u_n) + o(1) = o(1).$$

Thanks to (4.17) we conclude that

$$\frac{3p-10}{6(p-2)} A(u_n - \bar{u}) + \frac{3p-8}{12(p-2)} B(u_n - \bar{u}) = o(1).$$

Then

$$A(u_n - \bar{u}) = o(1), B(u_n - \bar{u}) = o(1) \text{ and also } C(u_n - \bar{u}) = o(1), \tag{4.18}$$

since $Q(u_n - \bar{u}) = o(1)$. Now we use (5) of Proposition 4.1, i.e

$$A(u_n) - \lambda_c D(u_n) + B(u_n) + C(u_n) = A(\bar{u}) - \lambda_c D(\bar{u}) + B(\bar{u}) + C(\bar{u}) + o(1).$$

Thanks to the splitting properties of $A(u), B(u), C(u)$ and to (4.18) we get

$$-\lambda_c D(u_n) = -\lambda_c D(\bar{u}) + o(1),$$

which implies $D(u_n - \bar{u}) = o(1)$, i.e $||u_n - \bar{u}||_2 = o(1)$. From this point we conclude as in the proof of Lemma 4.3. □

Admitting for the moment that $c \to \gamma(c)$ is non-increasing (we shall prove it in the next section) we can now complete the proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemmas 3.2 and 4.1 there exists a bounded Palais-Smale sequence $\{u_n\} \subset S(c)$ such that, up to translation, $u_n \rightharpoonup u_c \neq 0$. Thus, by Proposition 4.1 there exists a $\lambda_c \in \mathbb{R}$ such that $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \backslash \{0\} \times \mathbb{R}$ solves (1.2). Now by Lemma 4.2 there exists a $c_0 > 0$ such that $\lambda_c < 0$ if $c \in (0, c_0)$. Also we know from Theorem 1.2 (ii) that (4.17) holds. At this point the proof follows from Lemma 4.4. □

5. The behaviour of $c \to \gamma(c)$

In this section we give the proof of Theorem 1.2. Let us denote

$$\gamma_1(c) = \inf_{u \in S(c)} \max_{t>0} F(u^t), \tag{5.1}$$

and

$$\gamma_2(c) = \inf_{u \in V(c)} F(u). \tag{5.2}$$

Lemma 5.1. *For $p \in (\frac{10}{3}, 6)$, we have:*

$$\gamma(c) = \gamma_1(c) = \gamma_2(c).$$

Proof. When $p \in (\frac{10}{3}, 6)$, from Lemma 2.3, we know that $\gamma(c) = \gamma_2(c)$. In addition, by Lemma 2.2, it is clear that for any $u \in S(c)$, there exists a unique $t_0 > 0$, such that $u^{t_0} \in V(c)$ and $\max_{t>0} F(u^t) = F(u^{t_0}) \geq \gamma_2(c)$, thus we get $\gamma_1(c) \geq \gamma_2(c)$. Meanwhile, for any $u \in V(c)$, $\max_{t>0} F(u_t) = F(u)$ and this readily implies that $\gamma_1(c) \leq \gamma_2(c)$. Thus we conclude that $\gamma_1(c) = \gamma_2(c)$. □

Lemma 5.2. *We denote*

$$f(a,b,c) = \max_{t>0} \left\{ a \cdot t^2 + b \cdot t - c \cdot t^{\frac{3}{2}(p-2)} \right\},$$

where $p \in (\frac{10}{3}, 6)$ and $a > 0, b \geq 0, c > 0$ which are totally independent of t. Then the function: $(a,b,c) \longmapsto f(a,b,c)$ is continuous in $\mathbb{R}^+ \times \mathbb{R}^c_- \times \mathbb{R}^+$ (here we denote $\mathbb{R}^c_-$ the non negative real number set).

Proof. Let $g(a,b,c,t) = a \cdot t^2 + b \cdot t - c \cdot t^{\frac{3}{2}(p-2)}$, then

$$\partial_t g(a,b,c,t) = 2a \cdot t + b - \frac{3}{2}(p-2) \cdot c \cdot t^{\frac{3p-8}{2}},$$

$$\partial^2_{tt} g(a,b,c,t) = 2a - \frac{3p-6}{2} \cdot \frac{3p-8}{2} \cdot c \cdot t^{\frac{3p-10}{2}}.$$

It's not difficult to see that for any (a_0, b_0, c_0) with $a_0 > 0, b_0 \geq 0, c_0 > 0$, there exists a unique $t_1 > 0$, such that $\partial_t g(a_0, b_0, c_0, t_1) = 0$ and $\partial^2_{tt} g(a_0, b_0, c_0, t_1) < 0$, thus $f(a_0, b_0, c_0) = g(a_0, b_0, c_0, t_1)$. Then applying the Implicit Function Theorem to the function $\partial_t g(a,b,c,t)$, we deduce the existence of a continuous function $t = t(a,b,c)$ in some neighborhood O of (a_0, b_0, c_0) that satisfies $\partial_t g(a,b,c,t(a,b,c)) = 0$, $\partial^2_{tt} g(a,b,c,t(a,b,c)) < 0$. Thus $f(a,b,c) = g(a,b,c,t(a,b,c))$ in O. Now since the function $g(a,b,c,t)$ is continuous in (a,b,c,t), it follows that $f(a,b,c)$ is continuous in (a_0, b_0, c_0). The point (a_0, b_0, c_0) being arbitrary this concludes the proof. □

Lemma 5.3. *When $p \in (\frac{10}{3}, 6)$, the function $c \mapsto \gamma(c)$ is non increasing for $c > 0$.*

Proof. To show that $c \mapsto \gamma(c)$ is non increasing, it is enough to verify that: for any $c_1 < c_2$ and $\varepsilon > 0$ arbitrary, we have

$$\gamma(c_2) \leq \gamma(c_1) + \varepsilon. \tag{5.3}$$

By definition of $\gamma_2(c_1)$, there exists $u_1 \in V(c_1)$ such that $F(u_1) \leq \gamma_2(c_1) + \frac{\varepsilon}{2}$. Thus by Lemma 5.1, we have

$$F(u_1) \leq \gamma(c_1) + \frac{\varepsilon}{2} \tag{5.4}$$

and also

$$F(u_1) = \max_{t>0} F(u_1^t). \tag{5.5}$$

We truncate u_1 into a function with compact support $\widetilde{u}_1$ as follows. Let $\eta \in C_0^\infty(\mathbb{R}^3)$ be radial and such that

$$\eta(x) = \begin{cases} 1, & |x| \leq 1, \\ \in [0,1], & 1 < |x| < 2, \\ 0, & |x| \geq 2. \end{cases}$$

For any small $\delta > 0$, let

$$\widetilde{u}_1(x) = \eta(\delta x) \cdot u_1(x). \tag{5.6}$$

It is standard to show that $\widetilde{u}_1(x) \to u_1(x)$ in $H^1(\mathbb{R}^3)$ as $\delta \to 0$. Then, by continuity, we have, as $\delta \to 0$,

$$A(\widetilde{u}_1) \to A(u_1),\ B(\widetilde{u}_1) \to B(u_1) \text{ and } C(\widetilde{u}_1) \to C(u_1). \tag{5.7}$$

At this point applying Lemma 5.2, we deduce that there exists $\delta > 0$ small enough, such that

$$\begin{aligned} \max_{t>0} F(\widetilde{u}_1^t) &= \max_{t>0} \Big\{ \frac{t^2}{2} A(\widetilde{u}_1) + tB(\widetilde{u}_1) + t^{\frac{3}{2}(p-2)} C(\widetilde{u}_1) \Big\} \\ &\leq \max_{t>0} \Big\{ \frac{t^2}{2} A(u_1) + tB(u_1) + t^{\frac{3}{2}(p-2)} C(u_1) \Big\} + \frac{\varepsilon}{4} \\ &= \max_{t>0} F(u_1^t) + \frac{\varepsilon}{4}. \end{aligned} \tag{5.8}$$

Now let $v(x) \in C_0^\infty(\mathbb{R}^3)$ be radial and such that *supp* $v \subset B_{2R_\delta+1} \backslash B_{2R_\delta}$. Here *supp* v denotes the support of v and $R_\delta = \frac{2}{\delta}$. Then we define

$$v_0 = (c_2 - \|\widetilde{u}_1\|_2^2)/\|v\|_2^2 \cdot v$$

for which we have $\|v_0\|_2^2 = c_2 - \|\widetilde{u}_1\|_2^2$. Finally letting $v_0^\lambda = \lambda^{\frac{3}{2}} v_0(\lambda x)$, for $\lambda \in (0,1)$, we have $\|v_0^\lambda\|_2^2 = \|v_0\|_2^2$ and

$$A(v_0^\lambda) = \lambda^2 \cdot A(v_0),\ B(v_0^\lambda) = \lambda \cdot B(v_0) \text{ and } C(v_0^\lambda) = \lambda^{\frac{3}{2}(p-2)} \cdot C(v_0). \tag{5.9}$$

Now for any $\lambda \in (0,1)$ we define $w_\lambda = \widetilde{u}_1 + v_0^\lambda$. We observe that

$$dist\{supp\ \widetilde{u}_1, supp\ v_0^\lambda\} \geq \frac{2R_\delta}{\lambda} - R_\delta = \frac{2}{\delta}(\frac{2}{\lambda} - 1). \tag{5.10}$$

Thus $\|w_\lambda\|_2^2 = \|\widetilde{u}_1\|_2^2 + \|v_0^\lambda\|_2^2$ and $w_\lambda \in S(c_2)$. Also

$$A(w_\lambda) = A(\widetilde{u}_1) + A(v_0^\lambda) \text{ and } C(w_\lambda) = C(\widetilde{u}_1) + C(v_0^\lambda). \tag{5.11}$$

We claim that, for any $\lambda \in (0,1)$,

$$\Big|B(w_\lambda) - B(\widetilde{u}_1) - B(v_0^\lambda)\Big| \leq \lambda\, \|\widetilde{u}_1\|_2^2 \cdot \|v_0^\lambda\|_2^2. \tag{5.12}$$

Indeed, from (5.10),

$$\left(\widetilde{u}_1 + v_0^\lambda\right)^2(x) = \widetilde{u}_1^2(x) + \left(v_0^\lambda\right)^2(x),\ \left(\widetilde{u}_1 + v_0^\lambda\right)^2(y) = \widetilde{u}_1^2(y) + \left(v_0^\lambda\right)^2(y).$$

Thus

$$\begin{aligned} B(w_\lambda) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\left(\widetilde{u}_1 + v_0^\lambda\right)^2(x)\cdot\left(\widetilde{u}_1 + v_0^\lambda\right)^2(y)}{|x-y|}dxdy \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\widetilde{u}_1^2(x)\cdot\widetilde{u}_1^2(y)}{|x-y|}dxdy + 2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\widetilde{u}_1^2(x)\cdot\left(v_0^\lambda\right)^2(y)}{|x-y|}dxdy \\ &+ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\left(v_0^\lambda\right)^2(x)\cdot\left(v_0^\lambda\right)^2(y)}{|x-y|}dxdy \\ &= B(\widetilde{u}_1) + B(v_0^\lambda) + 2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\widetilde{u}_1^2(x)\cdot\left(v_0^\lambda\right)^2(y)}{|x-y|}dxdy \end{aligned}$$

with

$$\begin{aligned} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\widetilde{u}_1^2(x)\cdot\left(v_0^\lambda\right)^2(y)}{|x-y|}dxdy &= \int_{supp\ \widetilde{u}_1}\int_{supp\ v_0^\lambda} \frac{\widetilde{u}_1^2(x)\cdot\left(v_0^\lambda\right)^2(y)}{|x-y|}dxdy \\ &\leq \frac{\delta\lambda}{2(2-\lambda)}\int_{supp\ \widetilde{u}_1}\int_{supp\ v_0^\lambda} \widetilde{u}_1^2(x)\cdot\left(v_0^\lambda\right)^2(y)dxdy \\ &\leq \frac{\delta\lambda}{2(2-\lambda)}\|\widetilde{u}_1\|_2^2 \cdot \left\|v_0^\lambda\right\|_2^2 \\ &\leq \frac{\lambda}{2}\|\widetilde{u}_1\|_2^2 \cdot \left\|v_0^\lambda\right\|_2^2 \end{aligned}$$

and then (5.12) holds. Now from (5.11), (5.12) and using (5.9) we see that

$$A(w_\lambda) \to A(\widetilde{u}_1),\ B(w_\lambda) \to B(\widetilde{u}_1) \text{ and } C(w_\lambda) \to C(\widetilde{u}_1),\ \text{ as } \lambda \to 0. \tag{5.13}$$

Thus from Lemma 5.2 we have that, fixing $\lambda > 0$ small enough,

$$\max_{t>0} F(w_\lambda^t) \leq \max_{t>0} F(\widetilde{u}_1^t) + \frac{\varepsilon}{4}. \tag{5.14}$$

Now, using Lemma 5.1, (5.14), (5.8), (5.5) and (5.4) we have that

$$\begin{aligned}\gamma(c_2) \leq \max_{t>0} F(w_\lambda^t) &\leq \max_{t>0} F(\widetilde{u}_1^t) + \frac{\varepsilon}{4} \\ &\leq \max_{t>0} F(u_1^t) + \frac{\varepsilon}{2} \\ &= F(u_1) + \frac{\varepsilon}{2} \leq \gamma(c_1) + \varepsilon\end{aligned}$$

and this ends the proof. □

Lemma 5.4. *When $p \in (\frac{10}{3}, 6)$, $c \mapsto \gamma(c)$ is continuous at each $c > 0$.*

Proof. Since, by Lemma 5.3, $c \to \gamma(c)$ is non increasing proving that it is continuous at $c > 0$ is equivalent to show that for any sequence $c_n \to c^+$

$$\gamma(c) \leq \lim_{c_n \to c^+} \gamma(c_n). \tag{5.15}$$

Let $\varepsilon > 0$ be arbitrary but fixed. By Lemma 2.3 we know that there exists $u_n \in V(c_n)$ such that

$$F(u_n) \leq \gamma(c_n) + \frac{\varepsilon}{2}. \tag{5.16}$$

We define $\widetilde{u}_n = \frac{c}{c_n} \cdot u_n := \rho_n \cdot u_n$. Then $\widetilde{u}_n \in S(c)$ and $\rho_n \to 1^-$. In addition

$$\begin{aligned}\gamma(c) &\leq \max_{t>0} F(\widetilde{u}_n^t) \\ &= \max_{t>0} \{\frac{t^2}{2}\rho_n^2 A(u_n) + \frac{t}{4}\rho_n^4 B(u_n) + \frac{t^{\frac{3p-6}{2}}}{p}\rho_n^p C(u_n)\}.\end{aligned} \tag{5.17}$$

Since $u_n \in V(c_n)$ and $c_n \to c^+$, using the identity

$$F(u_n) - \frac{2}{3(p-2)}Q(u_n) = \frac{3p-10}{6(p-2)}A(u_n) + \frac{3p-8}{12(p-2)}B(u_n), \tag{5.18}$$

it is not difficult to check that $A(u_n), B(u_n)$ and $C(u_n)$ are bounded both from above and from zero. Thus without restriction we can get that

$$A(u_n) \to A > 0,\ B(u_n) \to B \geq 0 \quad \text{and} \quad C(u_n) \to C < 0.$$

Indeed, $A \geq 0, B \geq 0, C \leq 0$ are trivial and it is also easy to verify by contradiction that $A \neq 0, C \neq 0$ from (4.6), (5.18) and the fact

$$Q(u_n) = A(u_n) + \frac{1}{4}B(u_n) + \frac{3p-6}{2p}C(u_n) = 0.$$

Now recording that $\rho_n \to 1^-$, using Lemma 5.2 twice, we get from (5.17), for any $n \in \mathbb{N}$ sufficiently large

$$\begin{aligned}\max_{t>0} F(\widetilde{u}_n^t) &\leq \max_{t>0}\{(\frac{A}{2})t^2 + (\frac{B}{4})t - (-\frac{C}{p})t^{\frac{3}{2}(p-2)}\} + \frac{\varepsilon}{4} \\ &\leq \max_{t>0}\{(\frac{A(u_n)}{2})t^2 + (\frac{B(u_n)}{4})t - (-\frac{C(u_n)}{p})t^{\frac{3}{2}(p-2)}\} + \frac{\varepsilon}{2} \\ &= \max_{t>0} F(u_n^t) + \frac{\varepsilon}{2} = F(u_n) + \frac{\varepsilon}{2}. \qquad (5.19)\end{aligned}$$

Now from (5.16) and (5.19) it follows that $\gamma(c) \leq \gamma(c_n)+\varepsilon$ for $n \in \mathbb{N}$ large enough and since $\varepsilon > 0$ is arbitrary (5.15) holds. □

Lemma 5.5. *Let $p \in (\frac{10}{3}, 6)$ and $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ solves*

$$-\Delta v - \lambda v + (|x|^{-1} * |v|^2)v - |v|^{p-2}v = 0 \ \textit{in}\ \mathbb{R}^3,$$

with $F(u_c) = \inf_{u \in V(c)} F(u) = \gamma(c)$. Then $\lambda_c \leq 0$ and moreover if $\lambda_c < 0$ the function $c \to \gamma(c)$ is strictly decreasing in a neighborhood of c.

Proof. To prove the lemma it suffices to show that if $\lambda_c < 0$ ($\lambda_c > 0$) the function $c \to \gamma(c)$ is strictly decreasing (increasing) in a neighborhood of c. Indeed, in view of Lemma 5.3 the case $\lambda_c > 0$ is then impossible.

The strict monotonicity of the function $c \to \gamma(c)$ when $\lambda_c \neq 0$ is obtained as a consequence of the Implicit Function Theorem.
Let us consider the following rescaled functions $u_{t,\theta}(x) = \theta^{\frac{3}{2}} t^{\frac{1}{2}} u_c(\theta x) \in S(tc)$ with $\theta \in (0,\infty)$ and $t \in (0,\infty)$. We define the following quantities

$$\alpha(t,\theta) = F(u_{t,\theta}), \qquad (5.20)$$

$$\beta(t,\theta) = Q(u_{t,\theta}). \qquad (5.21)$$

Simple calculus shows that

$$\frac{\partial \alpha(t,\theta)}{\partial t}_{|(1,1)} = \frac{1}{2}\left(A(u_c) + B(u_c) + C(u_c)\right) = \frac{1}{2}\lambda_c c \qquad (5.22)$$

$$\frac{\partial \alpha(t,\theta)}{\partial \theta}_{|(1,1)} = 0, \quad \frac{\partial^2 \alpha(t,\theta)}{\partial^2 \theta}_{|(1,1)} < 0. \qquad (5.23)$$

Following the classical Lagrange Theorem we get, for any $\delta_t \in \mathbb{R}$, $\delta_\theta \in \mathbb{R}$,

$$\alpha(1+\delta_t, 1+\delta_\theta) = \alpha(1,1) + \delta_t \frac{\partial \alpha(t,\theta)}{\partial t}_{|(\bar{t},\bar{\theta})} + \delta_\theta \frac{\partial \alpha(t,\theta)}{\partial \theta}_{|(\bar{t},\bar{\theta})} \qquad (5.24)$$

where $|1-\bar{t}| \leq |\delta_t|$ and $|1-\bar{\theta}| \leq |\delta_\theta|$, and by continuity, for sufficiently small $\delta_t > 0$ and sufficiently small $|\delta_\theta|$,

$$\alpha(1+\delta_t, 1+\delta_\theta) < \alpha(1,1) \quad \text{if} \quad \lambda_c < 0 \qquad (5.25)$$

$$\alpha(1-\delta_t, 1+\delta_\theta) < \alpha(1,1) \quad \text{if} \quad \lambda_c > 0. \tag{5.26}$$

To conclude the proof it is enough to show that $\beta(t,u) = 0$ in a neighborhood of $(1,1)$ is the graph of a function $g : [1-\varepsilon, 1+\varepsilon] \to \mathbb{R}$ with $\varepsilon > 0$, such that $\beta(t, g(t)) = 0$ for $t \in [1-\varepsilon, 1+\varepsilon]$. Indeed in this case we have when $\lambda_c < 0$ by (5.25)

$$\gamma((1+\varepsilon)c) = \inf_{u \in V((1+\varepsilon)c)} F(u) \leq F(u_{1+\varepsilon, g(1+\varepsilon)}) < F(u_c) = \gamma(c)$$

and when $\lambda_c > 0$ we have by (5.26)

$$\gamma((1-\varepsilon)c) = \inf_{u \in V((1-\varepsilon)c)} F(u) \leq F(u_{1-\varepsilon, g(1-\varepsilon)}) < F(u_c) = \gamma(c).$$

To show the graph property by the Implicit Function Theorem it is sufficient to show that

$$\frac{\partial \beta(t,\theta)}{\partial \theta}_{|(1,1)} \neq 0. \tag{5.27}$$

By simple calculus we get

$$\frac{\partial \beta(t,\theta)}{\partial \theta}_{|(1,1)} = 2A(u_c) + \frac{B(u_c)}{4} + \frac{1}{p}(\frac{3}{2}(p-2))^2 C(u_c).$$

Using the fact that $Q(u_c) = 0$ we then obtain

$$\frac{\partial \beta(t,\theta)}{\partial \theta}_{|(1,1)} = (5 - \frac{3}{2}p)A(u_c) + (1 - \frac{3}{8}p)B(u_c).$$

Then, since $p > \frac{10}{3}$ we see that to have

$$\frac{\partial \beta(t,\theta)}{\partial \theta}_{|(1,1)} = 0$$

necessarily $A(u_c) = 0$ and $B(u_c) = 0$. Thus the derivative is never zero. □

Lemma 5.6. *We have $\gamma(c) \to \infty$ as $c \to 0$.*

Proof. By Theorem 1.1 we know that for any $c > 0$ sufficiently small there exists a couple $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}^-$ solution of (1.2) with $||u_c||_2^2 = c$ and $F(u_c) = \gamma(c)$. In addition by Lemma 4.2, $Q(u_c) = 0$. Thus $u_c \in H^1(\mathbb{R}^3)$ fulfills

$$0 = Q(u_c) = A(u_c) + \frac{1}{4}B(u_c) + \frac{3(p-2)}{2p}C(u_c) \tag{5.28}$$

$$\gamma(c) = F(u_c) = \frac{1}{2}A(u_c) + \frac{1}{4}B(u_c) + \frac{1}{p}C(u_c). \tag{5.29}$$

We deduce from (5.28) that $A(u_c) \leq -\frac{3(p-2)}{2p}C(u_c)$ and thus it follows from Gagliardo-Nirenberg inequality that

$$\|\nabla u_c\|_2^2 \leq \frac{3(p-2)}{2p}\|u_c\|_p^p \leq \widetilde{C}(p)\cdot\|\nabla u_c\|_2^{\frac{3(p-2)}{2}}\cdot\|u_c\|_2^{\frac{6-p}{2}},$$

i.e

$$1 \leq \widetilde{C}(p)\cdot\|\nabla u_c\|_2^{\frac{3p-10}{2}}\cdot c^{\frac{6-p}{4}}. \tag{5.30}$$

Since $p \in (\frac{10}{3}, 6)$, we obtain that

$$\|\nabla u_c\|_2^2 \to \infty, \quad \text{as } \ c \to 0. \tag{5.31}$$

Now from (5.28) and (5.29) we deduce that

$$\gamma(c) = F(u_c) = \frac{3p-10}{6(p-2)}A(u_c) + \frac{3p-8}{12(p-2)}B(u_c). \tag{5.32}$$

and thus from (5.31) we get immediately that $\gamma(c) \to \infty$ as $c \to 0$. □

6. Proof of Theorem 1.4 and Lemma 1.1

In this section we prove Theorem 1.4. Let us first show

Lemma 6.1. *Let $p \in (\frac{10}{3}, 6)$, for each $u_c \in \mathcal{M}_c$ there exists a $\lambda_c \in \mathbb{R}$ such that $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ solves (1.2).*

Proof. From Lagrange multiplier theory, to prove the lemma, it suffices to show that any $u_c \in \mathcal{M}_c$ is a critical point of $F(u)$ constrained on $S(c)$.

Let $u_c \in \mathcal{M}_c$ and assume, by contradiction, that $\|F'|_{S(c)}(u_c)\|_{H^{-1}(\mathbb{R}^3)} \neq 0$. Then, by the continuity of F', there exist $\delta > 0, \mu > 0$ such that

$$v \in B_{u_c}(3\delta) \implies \|F'|_{S(c)}(v)\|_{H^{-1}(\mathbb{R}^3)} \geq \mu,$$

where $B_{u_c}(\delta) := \{v \in S(c) \ : \ \|v - u_c\| \leq \delta\}$.

Let $\varepsilon := \min\{\gamma(c)/4, \mu\delta/8\}$. We claim that it is possible to construct a deformation on $S(c)$ such that

(i) $\eta(1, v) = v$ if $v \notin F^{-1}([\gamma(c) - 2\varepsilon, \gamma(c) + 2\varepsilon])$,
(ii) $\eta(1, F^{\gamma(c)+\varepsilon} \bigcap B_{u_c}(\delta)) \subset F^{\gamma(c)-\varepsilon}$,
(iii) $F(\eta(1, v)) \leq F(v)$, $\forall\ v \in S(c)$.

Here, $F^d := \{u \in S(c) : F(u) \leq d\}$. For this we use the pseudo gradient flow on $S(c)$ defined in (3.5) but where now $g : S(c) \to [0, \delta]$ satisfies

$$g(v) = \begin{cases} \delta & \text{if } v \in B_{u_c}(2\delta) \bigcap F^{-1}([\gamma(c) - \varepsilon, \gamma(c) + \varepsilon]) \\ 0 & \text{if } v \notin F^{-1}([\gamma(c) - 2\varepsilon, \gamma(c) + 2\varepsilon]). \end{cases}$$

With this definition clearly (i) and (iii) hold. To prove (ii) first observe that if $v \in F^{\gamma(c)+\varepsilon} \bigcap B_{u_c}(\delta)$, then $\eta(t,v) \in B_{u_c}(2\delta)$ for all $t \in [0,1]$. Indeed

$$\begin{aligned}\|\eta(t,v)-v\| &= \left\| \int_0^t -g(\eta(s,v)) \frac{Y(\eta(s,v))}{\|Y(\eta(s,v))\|} ds \right\| \\ &\leq \int_0^t \|g(\eta(s,v))\| \, ds \leq t\delta \leq \delta.\end{aligned}$$

In particular for $s \in [0,1], g(\eta(s,v)) = \delta$ as long as $F(\eta(s,v)) \geq \gamma(c) - \varepsilon$. Thus if we assume that there exists a $v \in F^{\gamma(c)+\varepsilon} \bigcap B_{u_c}(\delta)$ such that $F(\eta(1,v)) > \gamma(c) - \varepsilon$ we have

$$\begin{aligned}F(\eta(1,v)) &= F(v) + \int_0^1 \frac{d}{dt} F(\eta(t,v)) dt \\ &= F(v) + \int_0^1 \langle dF(\eta(t,v)), -g(\eta(t,v)) \frac{Y(\eta(t,v))}{\|Y(\eta(t,v))\|} \rangle dt \\ &\leq F(v) - \frac{\mu\delta}{4} \leq \gamma(c) + \varepsilon - \frac{\mu\delta}{4} \leq \gamma(c) - \varepsilon,\end{aligned}$$

i.e. $\eta(1,v) \in F^{\gamma(c)-\varepsilon}$. This contradiction proves that (ii) also hold.

Now let $g \in \Gamma_c$ be the path constructed in the proof of Lemma 2.3 by choosing $v = u_c \in V(c)$. We claim that

$$\max_{t \in [0,1]} F(\eta(1, g(t))) < \gamma(c). \tag{6.1}$$

By (i) and Remark 2.1 we have $\eta(1, g(t)) \in \Gamma_c$. Thus if (6.1) holds, it contradicts the definition of $\gamma(c)$. To prove (6.1), we distinguish three cases:
a) If $g(t) \in S(c) \setminus B_{u_c}(\delta)$, then using (iii) and Lemma 2.2 (6),

$$F(\eta(1, g(t))) \leq F(g(t)) < F(u_c) = \gamma(c).$$

b) If $g(t) \in F^{\gamma(c)-\varepsilon}$, then by (iii)

$$F(\eta(1, g(t))) \leq F(g(t)) \leq \gamma(c) - \varepsilon.$$

c) If $g(t) \in F^{-1}([\gamma(c) - \varepsilon, \gamma(c) + \varepsilon]) \bigcap B_{u_c}(\delta)$, then by (ii)

$$F(\eta(1, g(t))) \leq \gamma(c) - \varepsilon.$$

Note that since $F(g(t)) \leq \gamma(c)$, for all $t \in [0,1]$ one of the three cases above must occurs. This proves that (6.1) hold and the proof of the lemma is completed. □

Proof of Theorem 1.4. We know from Lemma 6.1 that to each $u_c \in \mathcal{M}_c$ is associated a $\lambda_c \in \mathbb{R}$ such that $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ is solution of (1.2). Now using Lemmas 5.5 we deduce that necessarily $\lambda_c \leq 0$. □

Proof of Lemma 1.1. Let $u_c \in H^1(\mathbb{R}^3, \mathbb{C})$ with $u_c \in V(c)$. Since $\|\nabla|u_c|\|_2 \leq \|\nabla u_c\|_2$ we have that $F(|u_c|) \leq F(u_c)$ and $Q(|u_c|) \leq Q(u_c) = 0$. In addition, by Lemma 2.2, there exists $t_0 \in (0,1]$ such that $Q(|u_c|^{t_0}) = 0$. We claim that

$$F(|u_c|^{t_0}) \leq t_0 \cdot F(u_c). \tag{6.2}$$

Indeed, due (2.2) and since $Q(|u_c|^{t_0}) = Q(u_c) = 0$, we have

$$\begin{aligned} F(|u_c|^{t_0}) &= t_0^2 \cdot \frac{3p-10}{6(p-2)}\|\nabla|u_c|\|_2^2 + t_0 \cdot \frac{3p-8}{12(p-2)}T(|u_c|) \\ &= t_0 \cdot \left(t_0 \cdot \frac{3p-10}{6(p-2)}\|\nabla|u_c|\|_2^2 + \frac{3p-8}{12(p-2)}T(u_c) \right) \\ &\leq t_0 \cdot \left(\frac{3p-10}{6(p-2)}\|\nabla u_c\|_2^2 + \frac{3p-8}{12(p-2)}T(u_c) \right) \\ &= t_0 \cdot F(u_c). \end{aligned}$$

Thus if $u_c \in H^1(\mathbb{R}^3, \mathbb{C})$ is a minimizer of $F(u)$ on $V(c)$ we have

$$F(u_c) = \inf_{u \in V(c)} F(u) \leq F(|u_c|^{t_0}) \leq t_0 \cdot F(u_c),$$

which implies $t_0 = 1$ since $t_0 \in (0,1]$. Then $Q(|u_c|) = 0$ and we conclude that

$$\|\nabla|u_c|\|_2 = \|\nabla u_c\|_2 \quad \text{and} \quad F(|u_c|) = F(u_c). \tag{6.3}$$

Thus point (i) follows. Now since $|u_c|$ is a minimizer of $F(u)$ on $V(c)$ we know by Theorem 1.4 that it satisfies (1.2) for some $\lambda_c \leq 0$. By elliptic regularity theory and the maximum principle it follows that $|u_c| \in C^1(\mathbb{R}^3, \mathbb{R})$ and $|u_c| > 0$. At this point, using that $\|\nabla|u_c|\|_2 = \|\nabla u_c\|_2$ the rest of the proof of point (ii) is exactly the same as in the proof of Theorem 4.1 of [18]. ☐

7. Proof of Theorems 1.2 and 1.3

In [19] the authors consider the functional $F(u)$ as a free functional defined in the real space

$$E := \{u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}dxdy < \infty\}$$

equipped with the norm

$$||u||_E := \Big(\int_{\mathbb{R}^3} |\nabla u(x)|^2 dx + \Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}dxdy \Big)^{\frac{1}{2}} \Big)^{\frac{1}{2}}.$$

Clearly $H^1(\mathbb{R}^3, \mathbb{R}) \subset E$. They show, see Theorem 1.1 and Proposition 3.4 in [19], that $F(u)$ has in E a least energy solution whose energy is given by the mountain pass level

$$m := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} F(\gamma(t)) > 0 \tag{7.1}$$

where

$$\Gamma := \{\gamma \in C([0,1], E), \gamma(0) = 0, F(\gamma(1)) < 0\}.$$

Lemma 7.1. *For any $c > 0$ we have $\gamma(c) \geq m$ where $m > 0$ is given in (7.1).*

Proof. We fix an arbitrary $c > 0$. From Lemma 1.1 we know that the infimum of $F(u)$ on $V(c)$ is reached by real functions. As a consequence in the definition of $\gamma(c)$, see in particular (5.1), we can restrict ourself to paths in $H^1(\mathbb{R}^N, \mathbb{R})$ instead of $H^1(\mathbb{R}^N, \mathbb{C})$. To prove the lemma it suffices to show that for any $g \in \Gamma_c$ there exists a $\gamma \in \Gamma$ such that

$$\max_{t\in[0,1]} F(g(t)) \geq \max_{t\in[0,1]} F(\gamma(t)). \tag{7.2}$$

Let $v \in S(c)$ be arbitrary but fixed. Letting $v^\theta(x) = \theta^{\frac{3}{2}} v(\theta x)$ we have $v^\theta \in S(c)$ for any $\theta > 0$. Also taking $\theta > 0$ sufficiently small, $v^\theta \in A_{K_c}$. Now for $g \in \Gamma_c$ arbitrary but fixed, let $\gamma_\theta(t) \in C([\frac{1}{4}, \frac{1}{2}], A_{K_c})$ satisfies $\gamma_\theta(\frac{1}{4}) = v^\theta, \gamma_\theta(\frac{1}{2}) = g(0)$, and consider $\gamma(t)$ given by

$$\gamma(t) = \begin{cases} 4tv^\theta, & 0 \leq t \leq \frac{1}{4}, \\ \gamma_\theta(t), & \frac{1}{4} \leq t \leq \frac{1}{2}, \\ g(2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Since $S(c) \subset H^1(\mathbb{R}^3) \subset E$ by construction $\gamma \in \Gamma$. Now direct calculations show that, taking $\theta > 0$ small enough, $F(4tv^\theta) \leq F(v^\theta)$ for any $t \in [0, \frac{1}{4}]$. Thus

$$\max_{t\in[0,1]} F(\gamma(t)) = \max_{t\in[\frac{1}{4},1]} F(\gamma(t)).$$

Recalling that $\gamma_\theta(t) \in A_{K_c}$ for any $t \in [\frac{1}{4}, \frac{1}{2}]$, we conclude from Theorem 2.1 that

$$\max_{t\in[0,1]} F(\gamma(t)) = \max_{t\in[\frac{1}{2},1]} F(\gamma(t)) = \max_{t\in[0,1]} F(g(t))$$

and (7.2) holds. This proves the lemma.

□

Lemma 7.2. *There exists $\gamma(\infty) > 0$ such that $\gamma(c) \to \gamma(\infty)$ as $c \to \infty$.*

Proof. The existence of a limit follows directly from the fact that $c \to \gamma(c)$ is non-increasing. Now because of Lemma 7.1 the limit is strictly positive. □

Proof of Theorem 1.3. As we already mentioned this proof is largely due to L. Dupaigne. It also uses arguments from [12] and [16]. We divide the proof into two steps.

Step 1: Regularity and vanishing: let (u, λ) with $u \in E$ and $\lambda \leq 0$ solves (1.2), then $u \in L^\infty(\mathbb{R}^3) \bigcap C^1(\mathbb{R}^3)$ and $u(x) \to 0$, as $|x| \to \infty$.

We set $\phi_u(x) = \frac{1}{4\pi|x|} * u^2$. Clearly since $u \in E$ then $\phi_u \in D^{1,2}(\mathbb{R}^3)$. We denote $H = -\Delta + (1-\lambda)$. Since $\lambda \leq 0$, H^{-1} exists in $L^\eta(\mathbb{R}^3)$ for all $\eta \in (1,\infty)$. The operators H and $-\Delta$ being closed in $L^\eta(\mathbb{R}^3)$ with domain $D(H) \subset D(-\Delta)$, it follows from the Closed Graph Theorem that there exists a constant $\widetilde{C} > 0$ such that

$$\|\Delta u\|_\eta \leq \widetilde{C}\|Hu\|_\eta, \tag{7.3}$$

for any $u \in D(H)$. Now we write (1.2) as

$$u = H^{-1}u - H^{-1}(\phi_u u) + H^{-1}(|u|^{p-2}u) \tag{7.4}$$

and we claim that

$$H^{-1}u \in L^3 \cap L^\infty(\mathbb{R}^3) \text{ and } H^{-1}(\phi_u u) \in L^2 \cap L^\infty(\mathbb{R}^3). \tag{7.5}$$

Indeed, $u \in L^q(\mathbb{R}^3)$ for all $q \in [3,6]$, see [32], and from (7.3) and Sobolev's embedding theorem, we obtain

$$H^{-1}u \in W^{2,q}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3),\ \forall\ q \in [3,6]. \tag{7.6}$$

Now since $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, by Hölder inequality, $\phi_u u \in L^t(\mathbb{R}^3)$ holds for any $t \in [2,3]$ and we have

$$H^{-1}(\phi_u u) \in W^{2,t}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3),\ \forall\ t \in [2,3]. \tag{7.7}$$

At this point the claim is proved. Next we denote

$$v := u + H^{-1}(\phi_u u) - H^{-1}u. \tag{7.8}$$

By interpolation, and using (7.5), we see that $v \in L^q(\mathbb{R}^3)$ for all $q \in [3,6]$. Now since $u \in L^q(\mathbb{R}^3)$, for all $q \in [3,6]$, (7.4) implies that

$$Hv = |u|^{p-2}u \in L^{\frac{q}{p-1}}(\mathbb{R}^3). \tag{7.9}$$

By (7.3) and Sobolev's embedding theorem, we conclude from (7.9) that

$$v \in L^r(\mathbb{R}^3), \quad \text{for all } r \geq \frac{q}{p-1} \text{ such that } \frac{1}{r} \geq \frac{p-1}{q} - \frac{2}{3}. \tag{7.10}$$

Next we follow the arguments of Cazenave [12] to increase the index r.

For $j \geq 0$, we define r_j as:

$$\frac{1}{r_j} = -\delta(p-1)^j + \frac{2}{3(p-2)}, \quad \text{with } \delta = \frac{2}{3(p-2)} - \frac{1}{p}.$$

Since $p \in [3,6)$, then $\delta > 0$ and $\frac{1}{r_j}$ is decreasing with $\frac{1}{r_j} \to -\infty$ as $j \to \infty$. Thus there exists some $k > 0$ such that

$$\frac{1}{r_i} > 0 \text{ for } 0 \leq i \leq k; \ \frac{1}{r_{k+1}} \leq 0.$$

Now we claim that $v \in L^{r_k}(\mathbb{R}^3)$. Indeed, $r_0 = p$ as $j = 0$ and it is trivial that $v \in L^{r_0}(\mathbb{R}^3)$. If we assume that $v \in L^{r_i}(\mathbb{R}^3)$ for $0 \le i < k$, then by (7.8) and (7.5), we have $u \in L^{r_i}(\mathbb{R}^3)$. Thus following (7.10) we obtain

$$v \in L^r(\mathbb{R}^3), \quad \text{for all } r \ge \frac{r_i}{p-1} \text{ such that } \frac{1}{r} \ge \frac{p-1}{r_i} - \frac{2}{3} = \frac{1}{r_{i+1}}.$$

In particular, $v \in L^{r_{i+1}}(\mathbb{R}^3)$ and we conclude this claim by induction. Now since $v \in L^{r_k}(\mathbb{R}^3)$ it follows from (7.8) and (7.5) that $u \in L^{r_k}(\mathbb{R}^3)$ and we get that

$$v \in L^r(\mathbb{R}^3), \quad \text{for all } r \ge \frac{r_k}{p-1} \text{ such that } \frac{1}{r} \ge \frac{p-1}{r_k} - \frac{2}{3} = \frac{1}{r_{k+1}}.$$

Since $1/r_{k+1} < 0$ we obtain that $v \in \bigcap_{3 \le \alpha \le \infty} L^\alpha(\mathbb{R}^3)$ and thus also $u \in \bigcap_{3 \le \alpha \le +\infty} L^\alpha(\mathbb{R}^3)$.

At this point we have shown that

$$Hu = u - \phi_u u + |u|^{p-2}u$$

with for all $\alpha \in [3, \infty]$,

$$u \in L^\alpha \cap L^\infty(\mathbb{R}^3), \quad \phi_u u \in L^{\frac{6\alpha}{6+\alpha}}(\mathbb{R}^3) \quad \text{and} \quad |u|^{p-2}u \in L^{\frac{\alpha}{p-1}} \cap L^\infty(\mathbb{R}^3).$$

Since $\frac{6\alpha}{6+\alpha} \in [3, 6]$ for $\alpha \in [6, \infty]$, by interpolation and (7.3) we obtain that

$$u \in W^{2, \frac{6\alpha}{6+\alpha}}(\mathbb{R}^3) \text{ for any } \alpha \in [6, +\infty]. \tag{7.11}$$

Thus by Sobolev's embedding, $u \in L^\infty(\mathbb{R}^3) \bigcap C^1(\mathbb{R}^3)$. Also there exists a sequence $\{u_n\} \subset C_c^1(\mathbb{R}^3)$ such that $u_n \to u$ in $W^{2, \frac{6\alpha}{6+\alpha}}(\mathbb{R}^3)$. When $\alpha > 6$, $W^{2, \frac{6\alpha}{6+\alpha}}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$. Thus $u_n \to u$ uniformly in $\mathbb{R}^3$ and we conclude that $u(x) \to 0$ as $|x| \to \infty$.

Step 2: Exponential decay estimate.

First we show that $\phi_u \in C^{0,\gamma}(\mathbb{R}^3)$, $\forall \gamma \in (0, 1)$ and that there exists a constant $C_0 > 0$ such that

$$\phi_u \ge \frac{C_0}{|x|}, \quad \text{for all } |x| \ge 1. \tag{7.12}$$

Since $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ solves the equation $-\Delta \Phi = 4\pi |u|^2$ and $u \in L^6(\mathbb{R}^3)$ by elliptic regularity $\phi_u \in W^{2,3}_{loc}(\mathbb{R}^3)$, Thus by Sobolev's embedding, $\phi_u \in C^{0,\gamma}(\mathbb{R}^3)$, $\forall \gamma \in (0, 1)$. In particular

$$C_0 = \min_{\partial B_1} \phi_u(x) > 0$$

where $B_R = \{x \in \mathbb{R}^3 : |x| \le R\}$. Indeed, if $\phi_u(x_0) = 0$ at some point $x_0 \in \mathbb{R}^3$ with $|x_0| = 1$, then $u(x) = 0$ a.e. in $\mathbb{R}^3$.

Now for an arbitrary $R_0 > 0$, let $w_1 = \phi_u - \frac{C_0}{|x|}$. Then

$$\begin{cases} -\Delta w_1 = 4\pi u^2 \geq 0, & \text{in} \quad B_{R_0} \setminus B_1; \\ w_1 \geq 0, & \text{on} \quad \partial B_1; \\ w_1 \geq -\frac{C_0}{R}, & \text{on} \quad \partial B_{R_0} \end{cases}$$

and the maximum principle yields that

$$w_1 \geq -\frac{C_0}{R_0}, \quad \text{in } B_{R_0} \setminus B_1.$$

Letting $R_0 \to \infty$, it follows that $w_1 \geq 0$ in $\mathbb{R}^3 \setminus B_1$ and thus (7.12) holds.

Now we denote by $u^+(u^-)$ the positive (negative) part of u, namely $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \max\{-u(x), 0\}$.

By Kato's inequality, we know that $\Delta u^+ \geq \chi[u \geq 0]\Delta u$, see [4]. Thus

$$-\Delta u^+ - \lambda u^+ + \phi_u \cdot u^+ \leq (u^+)^{p-1} \quad \text{in} \quad \mathbb{R}^3. \tag{7.13}$$

Let us show that there exist constants $\widetilde{C} > 0$ and $R_1 > 0$ such that

$$u^+(x) \leq \widetilde{C}\phi_u(x) \quad \text{for } |x| > R_1. \tag{7.14}$$

To prove this, we consider $w_2 = u^+ - \phi_u - \frac{d}{|x|}$, for a constant $d > 0$. Then (7.13) and $\lambda \leq 0$ imply that

$$-\Delta w_2 \leq (u^+)^{p-1} - 4\pi u^2, \quad \text{in } |x| \geq 1.$$

Since $\lim_{|x|\to\infty} u(x) \to 0$ and $p > 3$, then $(u^+)^{p-1} - 4\pi u^2 \leq 0$ holds in $|x| \geq R_1$ for some $R_1 > 0$ large enough. Thus for any $R \geq R_1$ and taking $d > 0$ large enough we have

$$\begin{cases} -\Delta w_2 \leq 0, & \text{in} \quad B_R \setminus B_{R_1}; \\ w_2 \leq 0, & \text{on} \quad \partial B_{R_1}; \\ w_2 \leq \max_{\partial B_R} u^+ - \frac{d}{R}, & \text{on} \quad \partial B_R. \end{cases}$$

Then by the maximum principle, we have $w_2 \leq \max_{\partial B_R} u^+ - \frac{d}{R}$ in $B_R \setminus B_{R_1}$. Letting $R \to \infty$ we conclude that $w_2 \leq 0$ in $\mathbb{R}^3 \setminus B_{R_1}$. This, together with (7.12), implies (7.14).

¿From (7.13) we have for any $\sigma > 0$ and since $\lambda \leq 0$,

$$\begin{aligned} -\Delta u^+ + \frac{\sigma}{|x|}u^+ &\leq \frac{\sigma}{|x|}u^+ - \phi_u u^+ + \lambda u^+ + (u^+)^{p-1} \\ &\leq \left(\frac{\sigma}{|x|} - \phi_u + (u^+)^{p-2}\right) u^+. \end{aligned} \tag{7.15}$$

Using (7.12) and (7.14), for $|x| \geq R_1 > 1$, by choosing $0 < \sigma < C_0$, we have

$$\begin{aligned} \frac{\sigma}{|x|} - \phi_u + (u^+)^{p-2} &\leq \frac{\sigma}{C_0} \cdot \phi_u - \phi_u + (u^+)^{p-2} \\ &\leq -(1 - \frac{\sigma}{C_0})\widetilde{C}^{-1} u^+ + (u^+)^{p-2} \\ &= \left(-(1 - \frac{\sigma}{C_0})\widetilde{C}^{-1} + (u^+)^{p-3}\right) \cdot u^+, \end{aligned}$$

where $(1 - \frac{\sigma}{C_0})\widetilde{C}^{-1} > 0$. Since $p \geq 3$ and $u(x) \to 0$ as $|x| \to \infty$, for $R_1 > 1$ sufficiently large, we obtain that $-(1 - \frac{\sigma}{C_0})\widetilde{C}^{-1} + (u^+)^{p-3} \leq 0$ in $|x| \geq R_1$. Thus it follows from (7.15) that

$$-\Delta u^+ + \frac{\sigma}{|x|} u^+ \leq 0, \quad \text{in } \mathbb{R}^3 \setminus B_{R_1}. \tag{7.16}$$

If we denote $\bar{C}_1 = \max_{\partial B_{R_1}} u^+$, applying the maximum principle, we thus obtain

$$u^+ \leq \bar{C}_1 \cdot \bar{w}, \quad \text{in } \mathbb{R}^3 \setminus B_{R_1} \tag{7.17}$$

where $\bar{w}$ is the radial solution of

$$\begin{cases} -\Delta \bar{w} + \frac{\sigma}{|x|} = 0, & \text{if} \quad |x| > R_1; \\ \bar{w}(x) = 1, & \text{if} \quad |x| = R_1; \\ \bar{w}(x) \to 0, & \text{if} \quad |x| \to \infty. \end{cases}$$

Now $\bar{w}$ satisfies (cfr. [2] Section 4),

$$\bar{w}(x) \leq \frac{C}{|x|^{3/4}} e^{-2C'\sqrt{|x|}}, \quad \forall\ |x| > R' \tag{7.18}$$

for some $C > 0, C' > 0$ and $R' > 0$.

Finally we observe that if u is a solution of (1.2), then $-u$ is also a solution. Thus since $u^- = (-u)^+$, following the same arguments, we obtain that there exists a constant $\bar{C}_2 > 0$ such that

$$u^- \leq \bar{C}_2 \cdot \bar{w}, \quad \text{in } \mathbb{R}^3 \setminus B_{R_1}. \tag{7.19}$$

Hence $|u| = u^+ + u^- \leq (\bar{C}_1 + \bar{C}_2)\bar{w}$, in $\mathbb{R}^3 \setminus B_{R_1}$ for $R_1 > 0$ sufficiently large. At this point we see from (7.18) that $u \in E$ satisfies the exponential decay (1.6). In particular $u \in L^2(\mathbb{R}^3)$ and then also $u \in H^1(\mathbb{R}^3)$. □

Lemma 7.3. *There exists $c_\infty > 0$ such that for all $c \geq c_\infty$ the function $c \to \gamma(c)$ is constant. Also if for a $c \geq c_\infty$ there exists a couple $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ solution of (1.2) with $||u_c||_2^2 = c$ and $F(u_c) = \gamma(c)$ then necessarily $\lambda_c = 0$.*

Proof. ¿From Lemma 1.1 and [19] we know that there exists grounds states of the free functional $F(u)$ which are real. From Theorem 1.3 we know that any ground state belongs to $H^1(\mathbb{R}^3)$. Let $u_0 \in H^1(\mathbb{R}^3)$ be one of these ground states and set $c_0 = ||u_0||_2^2$. Then, by Lemma 4.2, $u_0 \in V(c_0)$ and using Lemma 7.1 we get

$$F(u_0) \geq \gamma(c_0) \geq m = F(u_0).$$

Thus necessarily $\gamma(c_0) = m$. Now since $c \to \gamma(c)$ is non increasing, still by Lemma 7.1, we deduce that $\gamma(c) = \gamma(c_0)$ for all $c \geq c_0$. Now let $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ be a solution of (1.2) with $||u_c||_2^2 = c$ and $F(u_c) = \gamma(c)$. Thus by Lemma 5.5, $\lambda_c \leq 0$. But we note that $\lambda_c < 0$ will not happen since by Lemma 5.5 it would imply that $c \to \gamma(c)$ is strictly decreasing around $c > 0$ in contradiction with the fact that $\gamma(c)$ is constant. Then necessarily $\lambda_c = 0$. □

Remark 7.1. We see, from Theorem 1.4 and Lemma 7.3, that if $\gamma(c)$ is reached, say by a $u_c \in H^1(\mathbb{R}^3)$ with $c > 0$ large enough, then u_c is a ground state of $F(u)$ defined on E. It is unlikely that ground states exist for an infinity of value of $c > 0$. So we conjecture that there exists a $c_{lim} > 0$ such that for $c \geq c_{lim}$ there are no critical points for $F(u)$ constrained to $S(c)$ at the ground state level $\gamma(c)$.

Proof of Theorem 1.2. Obviously, points (i), (ii), (iv), (v) of Theorem 1.2 follow directly from Lemmas 5.3, 5.4, 5.6, 7.2, 7.3 and Lemmas 4.2, 5.5 conclude point (iii). □

8. Global existence and strong instability

We introduce the following result about the locally well-posedness of the Cauchy problem to the equation (1.1) (see Cazenave [12], *Theorem 4.4.6* and *Propostion 6.5.1* or Kikuchi's Doctoral thesis [23], *Chapter 3*).

Proposition 8.1. *Let $p \in (2,6)$, for any $u_0 \in H^1(\mathbb{R}^3,\mathbb{C})$, there exists $T = T(\|u_0\|_{H^1}) > 0$ and a unique solution $u(t) \in C([0,T), H^1(\mathbb{R}^3,\mathbb{C}))$ of the equation* (1.1) *with initial datum $u(0) = u_0$ satisfying*

$$F(u(t)) = F(u_0),\ \|u(t)\|_2 = \|u_0\|_2 \text{ for any } t \in [0,T).$$

In addition, if $u_0 \in H^1(\mathbb{R}^3,\mathbb{C})$ satisfies $|x|u_0 \in L^2(\mathbb{R}^3,\mathbb{C})$, then the virial identity

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \|xu(t)\|_2^2 = 8Q(u),$$

holds for any $t \in [0,T)$.

Proof of Theorem 1.5. Let $u(x,t)$ be the solution of (1.1) with $u(x,0) = u_0$ and $T_{max} \in (0,\infty]$ its maximal time of existence. Then classically we have either

$$T_{max} = +\infty$$

or

$$T_{max} < +\infty \quad \text{and} \quad \lim_{t \to T_{max}} ||\nabla u(x,t)||_2^2 = \infty. \tag{8.1}$$

Since

$$F(u(x,t)) - \frac{2}{3(p-2)}Q(u(x,t)) = \frac{3p-10}{6(p-2)}A(u(x,t)) + \frac{3p-8}{12(p-2)}B(u(x,t))$$

and $F(u(x,t)) = F(u_0)$ for all $t < T_{max}$, if (8.1) happens then, we get

$$\lim_{t \to T_{max}} Q(u(x,t)) = -\infty.$$

By continuity it exists $t_0 \in (0, T_{max})$ such that $Q(u(x,t_0)) = 0$ with $F(u(x,t_0)) = F(u_0) < \gamma(c)$. This contradicts the definition $\gamma(c) = \inf_{u \in V(c)} F(u)$. □

Remark 8.1. For $p \in (\frac{10}{3}, 6)$ and any $c > 0$ the set $\mathcal{O}$ is not empty. Indeed for an arbitrary but fixed $u \in S(c)$ $u^t(x) = t^{\frac{3}{2}}u(tx)$. Then $u^t \in S(c)$ for all $t > 0$ and

$$Q(u^t) = t^2 A(u) + \frac{t}{4}B(u) - \frac{3(p-2)}{2p}t^{\frac{3(p-2)}{2}}C(u),$$

$$F(u^t) = \frac{t^2}{2}A(u) + \frac{t}{4}B(u) - \frac{t^{\frac{3(p-2)}{2}}}{p}C(u).$$

We observe that $F(u^t) \to 0$ as $t \to 0$. Also, since $\frac{3(p-2)}{2} > 1$, we have $Q(u^t) > 0$ when $t > 0$ is sufficiently small. This proves that $\mathcal{O}$ is not empty.

Proof of the Theorem 1.6. For any $c > 0$, let $u_c \in \mathcal{M}_c$ and define the set

$$\Theta = \left\{ v \in H^1(\mathbb{R}^3) \setminus \{0\} : \ F(v) < F(u_c), \ \|v\|_2^2 = \|u_c\|_2^2, \ Q(v) < 0 \right\}.$$

The set Θ contains elements arbitrary close to u_c in $H^1(\mathbb{R}^3)$. Indeed, letting $v_0(x) = u_c^\lambda = \lambda^{\frac{3}{2}}u_c(\lambda x)$, with $\lambda < 1$, we see from Lemma 2.2 that $v_0 \in \Theta$ and that $v_0 \to u_c$ in $H^1(\mathbb{R}^3)$ as $\lambda \to 1$.

Let $v(t)$ be the maximal solution of (1.1) with initial datum $v(0) = v_0$ and $T \in (0, \infty]$ the maximal time of existence. Let us show that $v(t) \in \Theta$ for all $t \in [0, T)$. From the conservation laws

$$\|v(t)\|_2^2 = \|v_0\|_2^2 = \|u_c\|_2^2,$$

and

$$F(v(t)) = F(v_0) < F(u_c).$$

Thus it is enough to verify $Q(v(t)) < 0$. But $Q(v(t)) \neq 0$ for any $t \in (0, T)$. Otherwise, by the definition of $\gamma(c)$, we would get for a $t_0 \in (0, T)$ that $F(v(t_0)) \geq F(u_c)$ in contradiction with $F(v(t)) < F(u_c)$. Now by continuity of Q we get that $Q(v(t)) < 0$ and thus that $v(t) \in \Theta$ for all $t \in [0, T)$. Now we claim that there exists $\delta > 0$, such that

$$Q(v(t)) \leq -\delta, \ \forall t \in [0, T). \tag{8.2}$$

Let $t \in [0, T)$ be arbitrary but fixed and set $v = v(t)$. Since $Q(v) < 0$ we know by Lemma 2.2 that $\lambda^\star(v) < 1$ and that $\lambda \longmapsto F(v^\lambda)$ is concave on $[\lambda^\star, 1)$. Hence

$$\begin{aligned} F(v^{\lambda^\star}) - F(v) &\leq (\lambda^\star - 1)\frac{\partial}{\partial\lambda}F(v^\lambda) \mid_{\lambda=1} \\ &= (\lambda^\star - 1)Q(v). \end{aligned}$$

Thus, since $Q(v(t)) < 0$, we have

$$F(v) - F(v^{\lambda^\star}) \geq (1 - \lambda^\star)Q(v) \geq Q(v).$$

It follows from $F(v) = F(v_0)$ and $v^{\lambda^\star} \in V(c)$ that

$$Q(v) \leq F(v) - F(v^{\lambda^\star}) \leq F(v_0) - F(u_c).$$

Then letting $\delta = F(u_0) - F(v_0) > 0$ the claim is established. To conclude the proof of the theorem we use Proposition 8.1. Since $v_0(x) = u_c^\lambda$ we have that

$$\int_{\mathbb{R}^3} |x|^2|v_0|^2dx = \int_{\mathbb{R}^3} |x|^2|u_c^\lambda|^2dx = \lambda^2 \int_{\mathbb{R}^3} |y|^2|u_c(y)|^2dy.$$

Thus, from Lemma 6.1 and Theorem 1.3, we obtain that

$$\int_{\mathbb{R}^3} |x|^2|v_0|^2dx < \infty. \tag{8.3}$$

Applying Proposition 8.1 it follows that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left\|xv(t)\right\|_2^2 = 8Q(v).$$

Now by (8.2) we deduce that $v(t)$ must blow-up in finite time, namely that (8.1) hold. Recording that v_0 has been taken arbitrarily close to u_c, this ends the proof of the theorem. □

Proof of Theorem 1.7. For $p \in (\frac{10}{3}, 6)$, let u_0 be a ground state of equation (1.4). From Theorem 1.3 we know that $u_0 \in H^1(\mathbb{R}^3)$, thus we can set

$$c_0 := \|u_0\|_2^2.$$

¿From Lemma 4.2, we have $Q(u_0) = 0$. Thus $u_0 \in V(c_0)$ and it follows from (1.3) and Lemma 7.1 that

$$F(u_0) \geq \gamma(c_0) \geq m = F(u_0).$$

Hence $F(u_0) = \inf_{u \in V(c_0)} F(u)$, which means that u_0 minimizes $F(u)$ on $V(c_0)$. Thus applying Theorem 1.6, we end the proof. □

9. Comparison with the nonlinear Schrödinger case

In [24] the existence of critical points of

$$\tilde{F}(u) = \frac{1}{2}\,\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p, \qquad u \in H^1(\mathbb{R}^N). \tag{9.1}$$

constrained to $S(c)$ was considered under the condition:

$$(C): \frac{2N+4}{N} < p < \frac{2N}{N-2}, \text{ if } N \geq 3 \text{ and } \frac{2N+4}{N} < p \text{ if } N = 1,2.$$

In our notation it is proved in [24] that $\tilde{F}(u)$ has a mountain pass geometry on $S(c)$ in the sense that

$$\tilde{\gamma}(c) = \inf_{g \in \Gamma_c} \max_{t \in [0,1]} \tilde{F}(g(t)) > \max\{\tilde{F}(g(0)), \tilde{F}(g(1))\}$$

where

$$\tilde{\Gamma}_c = \{g \in C([0,1], S(c)),\ g(0) \in A_{K_c}, \tilde{F}(g(1)) < 0\},$$

and $A_{K_c} = \{u \in S(c) :\ \|\nabla u\|_2^2 \leq K_c\}$. Also we have

Lemma 9.1. *([24] Theorem 2) For $N \geq 1$ and any $c > 0$, under the condition (C), the functional $\tilde{F}(u)$ admits a critical point u_c at the level $\tilde{\gamma}(c)$ with $\|u_c\|_2^2 = c$ and there exists $\lambda_c < 0$ such that (λ_c, u_c) solves weakly the following Euler-Lagrange equation associated to the functional $\tilde{F}(u)$:*

$$-\Delta u - \lambda u = |u|^{p-2}u. \tag{9.2}$$

Lemma 9.2. *([24] Corollary 3.1 and Theorem 3.2) For $N \geq 1$, as $c \to 0$*

$$\begin{cases} \|\nabla u_c\|_2^2 \to \infty, \\ \lambda_c \to -\infty. \end{cases}$$

and as $c \to +\infty$,

$$\begin{cases} \|\nabla u_c\|_2^2 \to 0, \\ \lambda_c \to 0. \end{cases}$$

Using the above two results we now prove

Lemma 9.3. *For $N \geq 1$, under the condition (C), the function $c \longmapsto \tilde{\gamma}(c)$ is strictly decreasing. In addition, we have*

$$\begin{cases} \tilde{\gamma}(c) \to +\infty, & as \quad c \to 0, \\ \tilde{\gamma}(c) \to 0, & as \quad c \to \infty. \end{cases} \tag{9.3}$$

Proof. Arguing as in the proof of Lemma 5.1 we can deduce that

$$\tilde{\gamma}(c) = \inf_{u \in S(c)} \max_{t>0} \tilde{F}(u^t) = \inf_{u \in \tilde{V}(c)} \tilde{F}(u). \tag{9.4}$$

Here $\tilde{V}(c) = \{u \in H^1(\mathbb{R}^N) : \tilde{Q}(u) = 0\}$ with

$$\tilde{Q}(u) = \|\nabla u\|_2^2 - \frac{N(p-2)}{2p}\|u\|_p^p$$

and $u^t(x) = t^{\frac{N}{2}} u(tx)$ for $t > 0$. To show that $c \to \tilde{\gamma}(c)$ is strictly decreasing we just need to prove that: for any $c_1 < c_2$, there holds $\tilde{\gamma}(c_2) < \tilde{\gamma}(c_1)$. By (9.4) we have

$$\tilde{\gamma}(c_1) = \inf_{u \in S(c_1)} \max_{t>0} \tilde{F}(u^t) \quad \text{and} \quad \tilde{\gamma}(c_2) = \inf_{u \in S(c_2)} \max_{t>0} \tilde{F}(u^t)$$

where

$$\tilde{F}(u^t) = \frac{t^2}{2}\|\nabla u\|_2^2 - \frac{t^{\frac{N}{2}(p-2)}}{p}\|u\|_p^p.$$

After a simple calculation, we get

$$\max_{t>0} \tilde{F}(u^t) = \widetilde{c}(p) \cdot \left(\frac{1}{2}\|\nabla u\|_2^2\right)^{\frac{N(p-2)}{N(p-2)-4}} \cdot \left(\frac{1}{p}\|\nabla u\|_p^p\right)^{-\frac{4}{N(p-2)-4}} \tag{9.5}$$

with

$$\widetilde{c}(p) = \left(\frac{4}{N(p-2)}\right)^{\frac{4}{N(p-2)-4}} \cdot \frac{N(p-2)-4}{N(p-2)} > 0.$$

By Lemma 9.1, we know that $\gamma(c_1)$ is attained, namely that there exists $u_1 \in S(c_1)$, such that $\tilde{\gamma}(c_1) = \tilde{F}(u_1) = \max_{t>0} \tilde{F}(u_1^t)$. Then using the scaling $u_\theta(x) = \theta^{1-\frac{N}{2}} u_1(\frac{x}{\theta})$, we have

$$\|u_\theta\|_2^2 = \theta^2\|u_1\|_2^2, \quad \|\nabla u_\theta\|_2^2 = \|\nabla u_1\|_2^2 \quad \text{and} \quad \|u_\theta\|_p^p = \theta^{(1-\frac{N}{2})p+N}\|u_1\|_p^p.$$

Thus we can choose $\theta > 1$ such that $u_\theta \in S(c_2)$. Under the condition (C), we have $(1-\frac{N}{2})p + N > 0$ for $N \geq 1$ and thus $\|u_\theta\|_p^p > \|u_1\|_p^p$. Now we have

$$\begin{aligned} \max_{t>0} \tilde{F}(u_\theta^t) &= \widetilde{c}(p) \cdot \left(\frac{1}{2}\|\nabla u_\theta\|_2^2\right)^{\frac{N(p-2)}{N(p-2)-4}} \cdot \left(\frac{1}{p}\|u_\theta\|_p^p\right)^{-\frac{4}{N(p-2)-4}} \\ &< \widetilde{c}(p) \cdot \left(\frac{1}{2}\|\nabla u_1\|_2^2\right)^{\frac{N(p-2)}{N(p-2)-4}} \cdot \left(\frac{1}{p}\|u_1\|_p^p\right)^{-\frac{4}{N(p-2)-4}} \\ &= \max_{t>0} \tilde{F}(u_1^t), \end{aligned}$$

which implies that

$$\tilde{\gamma}(c_1) = \max_{t>0} \tilde{F}(u_1^t) > \max_{t>0} \tilde{F}(u_\theta^t) \geq \tilde{\gamma}(c_2). \tag{9.6}$$

Finally, from Lemma 2.7 of [24] we know that, for any $c > 0$, $\tilde{Q}(u_c) = 0$. Thus we can write

$$\tilde{\gamma}(c) = \frac{N(p-2)-4}{2N(p-2)}\|\nabla u_c\|_2^2$$

and (9.3) directly follows from Lemma 9.2. □

Finally in analogy with Theorems 1.4 and 1.6 we have

Remark 9.1. Let

$$\tilde{\mathcal{M}}_c := \{u_c \in \tilde{V}(c) \ : \ \tilde{F}(u_c) = \inf_{u \in \tilde{V}(c)} \tilde{F}(u)\}. \tag{9.7}$$

Then for any $u_c \in \tilde{\mathcal{M}}_c$ there exists a $\lambda_c < 0$ such that $(u_c, \lambda_c) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ solves (9.2) and the standing wave solution $e^{-i\lambda_c t}u_c$ of (1.8) is strongly unstable.

The proof of these statements is actually simpler than the ones for (1.1) and thus we just indicate the main lines. We proceed as in Lemma 6.1 to show that for any $u_c \in \tilde{\mathcal{M}}_c$ there exists a $\lambda_c \in \mathbb{R}$ such that $(u_c, \lambda_c) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ solves (9.2). Indeed a version of Lemma 2.2 (and thus of Lemma 2.3) holds when $\tilde{F}(u)$ replaces $F(u)$ and this is precisely Lemma 8.2.5 in [12]. Now if for a $\lambda \in \mathbb{R}$, $u \in S(c)$ solves

$$-\Delta u - |u|^{p-2}u = \lambda u, \tag{9.8}$$

on one hand, multiplying (9.8) by $u \in S(c)$ and integrating we obtain

$$||\nabla u||_2^2 - ||u||_p^p = \lambda c. \tag{9.9}$$

On the other hand, since solutions of (9.8) satisfy $\tilde{Q}(u) = 0$, we have

$$||\nabla u||_2^2 - \frac{N(p-2)}{2p}||u||_p^p = 0. \tag{9.10}$$

Thus, since under (C) $N(p-2)/2p < 1$, we deduce that necessarily $\lambda < 0$. To conclude the proof we just have to show that the standing wave $e^{-i\lambda_c t}u_c$ is strongly unstable. This can be done following the same lines as in the proof of Theorem 1.6. Here the fact that $\lambda_c < 0$ insures the exponential decay at infinity of $u_c \in S(c)$ which permits to use the virial identity in the blow-up argument (see also [7]).

References

[1] A. Azzollini, A. Pomponio, P. d'Avenia, On the Schrödinger-Maxwell equations under the effect of a general nonlinear term, Ann. Inst. H. Poincaré Anal. Non Linéaire 27 (2010), no. 2, 779-791.

[2] A. Ambrosetti, A. Malchiodi, D. Ruiz, Bound states of nonlinear Schrödinger equations with potential vanishing at infinity, J. Anal. Math. 98 (2006), no. 1, 317-348.

[3] C. Bardos, F. Golse, A. D. Gottlieb, N. Mauser, Mean field dynamics of fermions and the time-dependent Hartree-Fock equation, J. Math. Pures Appl. (9) 82 (2003), no. 6, 665-683.

[4] H. Brezis, A. C. Ponce, Kato's inequality when Δu is a measure, (English, French summary) C. R. Math. Acad. Sci. Paris. 338 (2004), no. 8, 599-604.

[5] J. Bellazzini, G. Siciliano, Stable standing waves for a class of nonlinear Schrödinger-Poisson equations, Z. Angew. Math. Phys., 62 (2011), no. 2, 267-280.

[6] J. Bellazzini, G. Siciliano, Scaling properties of functionals and existence of constrained minimizers, J. Funct. Analysis, 261 (2011), no. 9, 2486-2507.
[7] H. Berestycki, T. Cazenave, Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaire, C. R. Acad. Sci. Paris Sér. I Math. 293 (1981), no. 9, 489-492.
[8] H. Berestycki, P.L. Lions, Nonlinear scalar field equations I, Arch. Ration. Mech. Anal., 82, (1983), no. 4, 313-346.
[9] H. Berestycki, P. L. Lions, Nonlinear Scalar Field Equations II, Existence of Infinitely Many Solutions, Arch. Rat. Mech. Anal., 82 (1983), no. 4, 347-375.
[10] H. Brezis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals. Proc. Amer. Math. Soc. 88 (1983), no. 3, 486-490.
[11] I. Catto, P. L. Lions, Binding of atoms and stability of molecules in Hartree and Thomas-Fermi type theories. I. A necessary and sufficient condition for the stability of general molecular systems, Comm. Partial Differential Equations, 17 (1992), no. 7-8, 1051-1110.
[12] T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003. xiv+323 pp. ISBN: 0-8218-3399-5.
[13] T. Cazenave, P. L. Lions, Orbital Stability of Standing Waves for Some Non linear Schrödinger Equations, Commun. Math. Phys. 85 (1982), no. 4, 549-561.
[14] T. D'Aprile, D. Mugnai, Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrodinger-Maxwell equations, Proc. Roy. Soc. Edinburgh Sect. A 134 (2004), no. 5, 893-906.
[15] L. Dupaigne, Private communication.
[16] R. Fukuizumi, T. Ozawa, Exponential decay of the solutions to nonlinear elliptic equations with potential. Z. Angew. Math. Phys. 56 (2005), no. 6, 1000-1011.
[17] V. Georgiev, F. Prinari, N. Visciglia, On the radiality of constrained minimizers to the Schrödinger-Poisson-Slater energy, preprint.
[18] H. Hajaiej and C.A. Stuart, On the variational approach to the stability of standing waves for the nonlinear Schrödinger equation, Adv. Nonlinear Stud. 4 (2004), no. 4, 469-501.
[19] I. Ianni, D. Ruiz, Ground and bound states for a static Schrödinger-Poisson-Slater problem, to appear in Comm. Comp. Math, DOI No: 10.1142/S0219199712500034.
[20] M. Kaminaga, M. Ohta, Stability of standing waves for nonlinear Schrödinger equation with attractive delta potential and repulsive nonlinearity, Saitama Math. J. 26 (2009), 39-48.
[21] T. Kato, Growth properties of the solutions of the reduced wave equation with a variable coefficient, Comm. Pure Applied Math. 12 (1959), 403-425.
[22] H. Kikuchi, Existence and stability of standing waves for Schrödinger-Poisson-Slater equation, Adv. Nonlinear Stud. 7 (2007), no. 3, 403-437.
[23] H. Kikuchi, Existence and orbital stability of the standing waves for nonlinear Schrödinger equations via the variational method, Doctoral Thesis (2008).
[24] L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, Nonlinear Analysis, Theory T. M.& A. 28 (1997), no. 10, 1633-1659.
[25] L. Jeanjean, M. Squassina, An approach to minimization under a constraint: the added mass technique, Calc. Var. Partial Differential Equations 41 (2011), no. 3-4, 511-534.
[26] E. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, Studies in Appl. Math. 57 (1977), no. 2, 93-105.
[27] E. H. Lieb, B. Simon, The Thomas - Fermi theory of atoms, molecules, and solids, Advances in Math. 23 (1977), no. 1, 22-116.

[28] P. L. Lions, The concentration-compactness principle in the Calculus of Variation. The locally compact case, part I and II, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984), no. 2, 109-145 and no. 4, 223-283.
[29] P. L. Lions, Solutions of Hartree-Fock Equations for Coulomb Systems, Comm. Math. Phys. 109 (1987), no. 1, 33-97.
[30] N. J. Mauser, The Schrödinger-Poisson-Xα equation, Appl. Math. Lett. 14 (2001), no. 6, 759-763.
[31] D. Ruiz, The Schrödinger-Poisson equation under the effect of a nonlinear local term, J. Funct. Anal. 237 (2006), no. 2, 655-674.
[32] D. Ruiz, On the Schrödinger-Poisson-Slater System: Behavior of Minimizers, Radial and Nonradial Cases, Arch. Rational Mech. Anal. 198 (2010), no. 1, 349-368.
[33] O. Sanchez, J. Soler, Long-time dynamics of the Schrödinger-Poisson-Slater system, J. Statist. Phys., 114 (2004), no. 1-2, 179-204.
[34] W. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys. 55 (1977), no. 2, 149-162.
[35] L. Zhao, F. Zhao, On the existence of solutions for the Schrödinger-Poisson equations, J. Math. Anal. Appl. 346 (2008), no. 1, 155-169.

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