Integrating the XXX-Heisenberg spin chain while building conserved charges

by Sylvain Labopin

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Quantum system proposed by Heisenberg in 1928

The $XXX_{\frac{1}{2}}$ Heisenberg spin chain :

$$\mathcal{H} = \bigotimes_{i=1}^{L} \mathbb{C}^2$$

(Hilbert space)

$$H = L - 2\sum_{i=1}^{L} \mathcal{P}_{i,i+1}$$

(hamiltonian)

where :

$$\mathcal{P}_{i,i+1}\left(v_1 \otimes \ldots \otimes v_L\right) = v_1 \otimes \ldots \otimes \underbrace{v_{i+1}}_{i^{\text{th}}\text{ factor}} \otimes \underbrace{v_i}_{(i+1)^{\text{th}}\text{ factor}} \otimes \ldots \otimes v_L$$

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Building operators with auxiliary space and partial trace

$\mathcal{A}$: finite dimensional $\mathbb{C}$-vector space with a basis $(e_k)_{k\in[\![1;d]\!]}$

$\mathcal{A}$ is called the auxiliary space.

$$\mathcal{O} \in \mathcal{L}\left(\left(\bigotimes_{i=1}^{L} \mathbb{C}^k\right) \bigotimes \mathcal{A}\right) \xrightarrow{?} Tr_{\mathcal{A}}\mathcal{O} \in \mathcal{L}\left(\bigotimes_{i=1}^{L} \mathbb{C}^k\right)$$

Definition : (partial trace of $\mathcal{O}$ with respect to $\mathcal{A}$) :

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where $(e_{i_1} \otimes \ldots \otimes e_{i_L})_{(i_u)_u\in[\![1:k]\!]^{[\![1:L]\!]}}$ canonical basis of $\bigotimes_{i=1}^{L} \mathbb{C}^k$

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Choose for auxiliary space : $\mathcal{H}_{a_1} \bigotimes \mathcal{H}_{a_2}$ with $\forall i,\ \mathcal{H}_{a_i} \simeq \mathbb{C}^k$

$$\forall u \in \mathbb{C},\ \forall (i,j) \in \left([\![1;L]\!] \cup \{a_1;a_2\} \right)^2,\ \boxed{\mathbf{R_{i,j}(u) := u + \mathcal{P}_{i,j}}}$$

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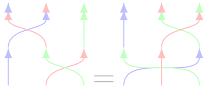
From $\mathcal{P}_{i,j}\mathcal{P}_{j,k} = \mathcal{P}_{j,k}\mathcal{P}_{k,i} \iff$  , we get :

$$\boxed{R_{i,j}(u-v)R_{i,k}(u)R_{j,k}(v) = R_{j,k}(v)R_{i,k}(u)R_{i,j}(u-v)}$$

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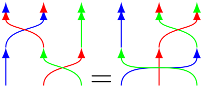
From $\mathcal{P}_{i,j}\mathcal{P}_{j,k} = \mathcal{P}_{j,k}\mathcal{P}_{k,i} \iff$  , we get :

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where $L^{(k)}_{p,q}(w) = R_{p,a_k}(w) \dots R_{q,a_k}(w)$ for $p \geq q$

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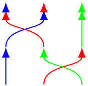
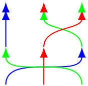
From $\mathcal{P}_{i,j}\mathcal{P}_{j,k} = \mathcal{P}_{j,k}\mathcal{P}_{k,i} \iff$  $=$  , we get :

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The first T-operators (commuting with each others)

Assume $R(u-v)$ reversible. With last relation with $i=0$:

$$L^{(1)}(u)L^{(2)}(v) = (R_{a_1,a_2}(u-v))^{-1} L^{(2)}(v)L^{(1)}(u)R_{a_1,a_2}(u-v)$$

With a property on the partial trace :

$$Tr_{\mathcal{H}_{a_1}\otimes\mathcal{H}_{a_2}}\left(L^{(1)}(u)L^{(2)}(v)\right) = Tr_{\mathcal{H}_{a_1}\otimes\mathcal{H}_{a_2}}\left(L^{(2)}(v)L^{(1)}(u)\right)$$

With identifications : $L^{(l)}(w) \in \mathcal{L}\left(\left(\bigotimes_{i=1}^{L}\mathbb{C}^k\right)\bigotimes\mathcal{H}_{a_l}\right)$. Then :

$$T(u)T(v) = T(v)T(u)$$

Where $T(w) := Tr_{\mathcal{H}_{a_l}}\left(L^{(l)}(w)\right)$ (called T-operator)

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Definition : (right-action of $\mathfrak{S}_L$ on $\bigotimes_{u=1}^{L} \mathbb{C}^k$)

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$$T(0) = \mathcal{P}_{(1\ 2\ \ldots\ L)} \qquad \partial_u T(u)|_{u=0} = \sum_i \mathcal{P}_{(1\ 2\ \ldots\ L)\circ(i\ i+1)}$$

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Generalization with a twist g, i.e. $g \in GL(\mathbb{C}^k)$:

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Auxiliary space non isomorphic to $\mathbb{C}^k$

With $\mathcal{A} \not\simeq \mathbb{C}^k$, we need to generalize the action of $\mathfrak{S}_{[\![1,L]\!]\cup\{a\}}$:

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$\overline{\pi}_\lambda$ preserve the Lie bracket $\implies$ $\boxed{[\mathcal{P}_{ij} + \mathcal{P}_{ia_2}; \mathcal{P}_{ja_2}] = 0}$

Yang-Baxter identity is then preserved :

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Differentiation with respect to the twist

Definition : (Coderivative)

$$\forall\ V\ \mathbb{C}\text{-vector space},\ \forall f \in \mathcal{C}^\infty\left(GL(\mathbb{C}^k), \mathcal{L}(V)\right),$$

$$\boxed{\widehat{D}\ \otimes f(g) := \sum_{\alpha,\beta} e_{\alpha\beta} \otimes \partial_t \left[f\left(e^{\phi + t e_{\beta\alpha}} g\right)\right]\Big|_{\substack{t=0\\ \phi=0}} \in \mathcal{L}\left(\mathbb{C}^k \otimes V\right)}$$

Remark : It allows to build operator of the spin chain.

$$\forall f \in \mathcal{C}^\infty\left(GL(\mathbb{C}^k), \mathcal{L}(\mathbb{C})\right),\ \left(\bigotimes_{i=1}^{L} \widehat{D}\right) \otimes f(g) \in \mathcal{L}\left(\bigotimes_{i=1}^{L} \mathbb{C}^k\right)$$

T-operators in terms of co-derivative

Considering $(u,g) \in \mathbb{C} \times GL\left(\mathbb{C}^k\right)$ fixed,

$T_g^{\pi}(u)$ can be built only from the character χ_π of π :

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Irreducible representations of $GL\left(\mathbb{C}^k\right)$

Definition : (Tensor representations of $GL(\mathbb{C}^k)$)

$$\begin{array}{rcl} \pi_M : GL\left(\mathbb{C}^k\right) & \rightarrow & GL\left(\bigotimes_{i=1}^{M} \mathbb{C}^k\right) \\ A & \mapsto & \left(\bigotimes_{i=1}^{M} v_i \mapsto \bigotimes_{i=1}^{M} (A.v_i)\right) \end{array}$$

Definition : A Young diagram is a 0-stationary non-increasing sequence of integers $\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{|\lambda|} \neq 0, \lambda_{|\lambda|+1} = 0, 0, 0, ...)$

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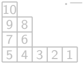
Young symmetrisors : (example)



$$s := \left(\sum_{\sigma \in \mathfrak{S}_{\{5,7,9,10\}}} \epsilon(\sigma)\mathcal{P}_\sigma\right) \left(\sum_{\sigma \in \mathfrak{S}_{\{4,6,8\}}} \epsilon(\sigma)\mathcal{P}_\sigma\right) \left(\mathbf{1} + \mathcal{P}_{(8,9)}\right) \left(\mathbf{1} + \mathcal{P}_{(6,7)}\right) \left(\sum_{\sigma \in \mathfrak{S}_{[\![1;5]\!]}} \mathcal{P}_\sigma\right)$$

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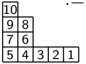
s 

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Theorem : Each irreducible representation of $GL\left(\mathbb{C}^k\right)$ is isomorphic to an unique restriction of a tensor representation to an image of a Young symmetrisor.

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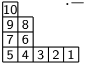

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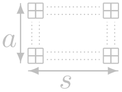
$(a, s) \equiv a, s \equiv as \equiv$ 

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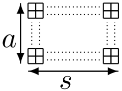
$(a, s) \equiv a, s \equiv as \equiv$ 

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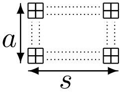
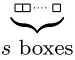
$(a,s) \equiv a, s \equiv as \equiv$ 

About the symmetric representations : (i.e: $(1,s) =$ )

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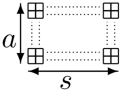
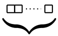


About the symmetric representations : (i.e: $(1, s) =$ 

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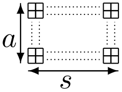
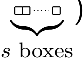


About the symmetric representations : (i.e: $(1, s) =$ 

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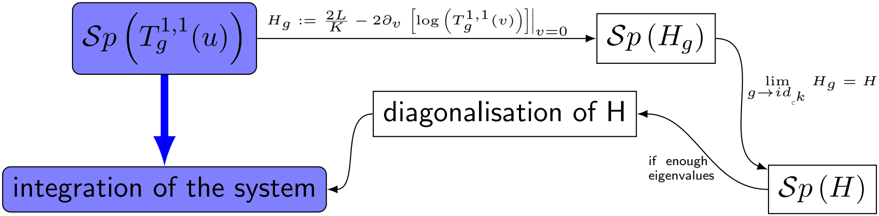

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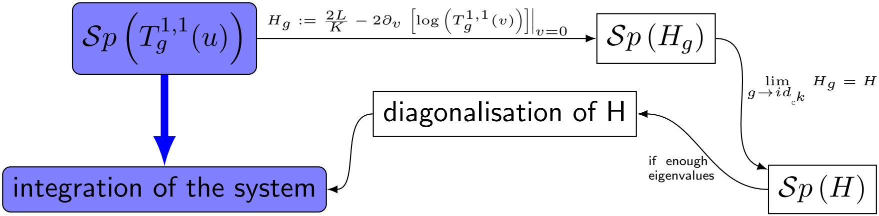

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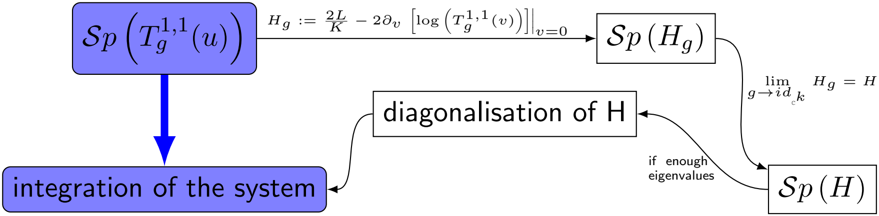

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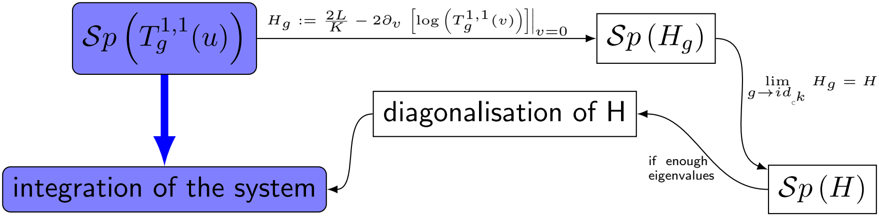

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For $L = 0$: $\boxed{T_g^{as}(u) = \chi_{as}(g)}$

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Generalization to spin chains with non zero length

For $L > 0$, we generalize the Bäcklund flow system by :

$$\left\{ \begin{array}{l} T^{a+1,s}_{I\cup\{j\}}(u)T^{a,s}_{I}(u) = T^{a,s}_{I\cup\{j\}}(u)T^{a+1,s}_{I}(u) + x_j T^{a+1,s-1}_{I\cup\{j\}}(u+1)T^{a,s+1}_{I}(u-1) \\ T^{a,s+1}_{I\cup\{j\}}(u)T^{a,s}_{I}(u) = T^{a,s}_{I\cup\{j\}}(u)T^{a,s+1}_{I}(u) + x_j T^{a+1,s}_{I\cup\{j\}}(u+1)T^{a-1,s+1}_{I}(u-1) \end{array} \right.$$

$$\forall I \in \mathcal{P}([\![1,k]\!]),\ \forall j \in [\![1,k]\!] \setminus I,\ \forall (a,s) \in \mathbb{N}^2, \forall u \in \mathbb{C}$$

with $T^{as} = T^{as}_{[\![1;k]\!]}$

Theorem/Definition/Notation :

- There exists a polynomial solution $\left(T^{a,s}_{I}\right)_{a,s}$ to that system, called the Bäcklund flow , such that :

$$\boxed{\forall I \in \mathcal{P}\left([\![1,k]\!]\right), \forall s > 0,\ \forall a > \mathrm{card}\left(I\right),\ T^{a,s}_{I} = 0}$$

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Generalization to spin chains with non zero length

For $L > 0$, we generalize the Bäcklund flow system by :

$$\begin{cases} T^{a+1,s}_{I\cup\{j\}}(u)T^{a,s}_{I}(u) = T^{a,s}_{I\cup\{j\}}(u)T^{a+1,s}_{I}(u) + x_j T^{a+1,s-1}_{I\cup\{j\}}(u+1)T^{a,s+1}_{I}(u-1) \\ T^{a,s+1}_{I\cup\{j\}}(u)T^{a,s}_{I}(u) = T^{a,s}_{I\cup\{j\}}(u)T^{a,s+1}_{I}(u) + x_j T^{a+1,s}_{I\cup\{j\}}(u+1)T^{a-1,s+1}_{I}(u-1) \end{cases}$$

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Bethe equations as conditions satisfied by the initial datas

Definition/Notation : We define the family of the Q-functions :

$$(Q_I)_{I\in\mathcal{P}([\![1,k]\!])} \equiv \left(T_I^{00}\right)_{I\in\mathcal{P}([\![1,k]\!])} \equiv \left(\alpha_I \prod_{n=1}^{d_I}\left(u-u_I^{(n)}\right)\right)_{I\in\mathcal{P}([\![1,k]\!])}$$

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For $\sigma \in \mathfrak{S}_k$ "nesting path", denoting $I_m = \{\sigma(1), \dots, \sigma(m)\}\ \forall m \in [\![0,k]\!]$:

$$\frac{Q_{I_m}\left(u_{I_m}^{(n)}+1\right) Q_{I_{m+1}}\left(u_{I_m}^{(n)}\right) Q_{I_{m-1}}\left(u_{I_m}^{(n)}-1\right)}{Q_{I_m}\left(u_{I_m}^{(n)}-1\right) Q_{I_{m+1}}\left(u_{I_m}^{(n)}+1\right) Q_{I_{m-1}}\left(u_{I_m}^{(n)}\right)} = -\frac{x_{\sigma(m+1)}}{x_{\sigma(m)}}$$

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Recover the Bethe equations for $XXX_{\frac{1}{2}}$ Heisenberg spin chain

For $k = 2,\ g = diag(x_1, x_2)$:

$$Q_{\{1;2\}} = u^L, \qquad Q_{\{1\}} = \alpha_1 \prod_n \left(u - u^{(n)}\right), \qquad Q_\emptyset = 1$$

Bethe equations $\iff \frac{Q_{\{1\}}\left(u^{(n)}+1\right)}{Q_{\{1\}}\left(u^{(n)}-1\right)} \left(\frac{u^{(n)}}{u^{(n)}+1}\right)^L = -\frac{x_2}{x_1}$

$$\iff \left(\frac{u^{(n)}}{u^{(n)}+1}\right)^L = \frac{x_2}{x1} \prod_{m \neq n} \frac{u^{(n)} - u^{(m)} - 1}{u^{(n)} - u^{(m)} + 1}$$

$u^{(n)} \equiv \frac{e^{\mathbf{i}p_n}}{1-e^{\mathbf{i}p_n}},\ g \to id_{\mathbb{C}^2} \implies$ Bethe equation Hans Bethe discovered in 1931

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Integration of the Heisenberg $XXX_{\frac{1}{2}}$ spin chain in 1931

In 1931, Hans Bethe found the eigenstates in the form :

$$\left|p_1, p_2, ..., p_M; (\mathcal{A}_\sigma)_{\sigma \in \mathfrak{S}_M}\right\rangle \equiv \sum_{1 \leq j_1 < ... < j_M \leq M} \Psi\left(j_1, ..., j_M\right) \left|\{j_1, ..., j_M\}\right\rangle$$

Where

$$\Psi\left(j_1, ..., j_M\right) \equiv \sum_{\sigma \in \mathfrak{S}_M} \mathcal{A}_\sigma \exp\left(\mathbb{i} \sum_{k=1}^{M} p_{\sigma(k)} j_k\right)$$

$$\mathcal{A}_\sigma \equiv \epsilon(\sigma) \prod_{j<k} \left(1 + e^{\mathbb{i}\left(p_{\sigma(j)} + p_{\sigma(k)}\right)} - 2e^{\mathbb{i} p_{\sigma(k)}}\right)$$

and denoting $(\uparrow; \downarrow)$ the canonical basis of $\mathbb{C}^2$,

$$\left|\{j_1, ..., j_M\}\right\rangle \equiv \downarrow \otimes ... \otimes \downarrow \otimes \underbrace{\uparrow}_{j_1^{\text{th}} \text{ factor}} \otimes \downarrow \otimes ... \otimes \downarrow \otimes \underbrace{\uparrow}_{j_2^{\text{th}} \text{ factor}} \otimes \downarrow \otimes ...$$

The momenta p_j must satisfy the Bethe equations of 1931 :

$$\forall j \in [\![1; M]\!], \ e^{\mathbb{i} L p_j} = \prod_{k \neq j} \frac{1 + e^{\mathbb{i}(p_j + p_k)} - 2e^{\mathbb{i} p_j}}{1 + e^{\mathbb{i}(p_j + p_k)} - 2e^{\mathbb{i} p_k}}$$