Nonlocal Aspects in PDEs and Applications

Besancon, May 20-21, 2010

Nonlocal Dissipation:

Modelling Issues and Numerical Analysis

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Plan of the Talk

1) Radiation Hydrodynamics

2) Radiation Hydrodynamics: A Model Problem

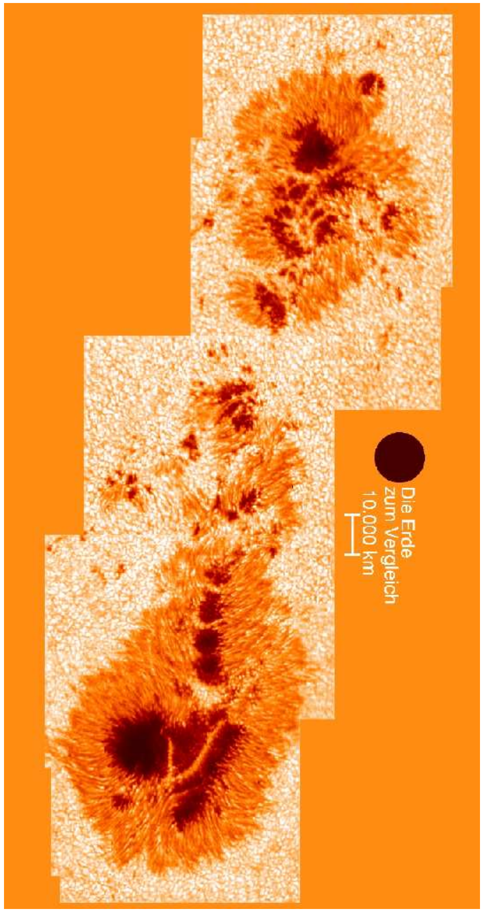
3) Compressible Liquid-Vapour Flow

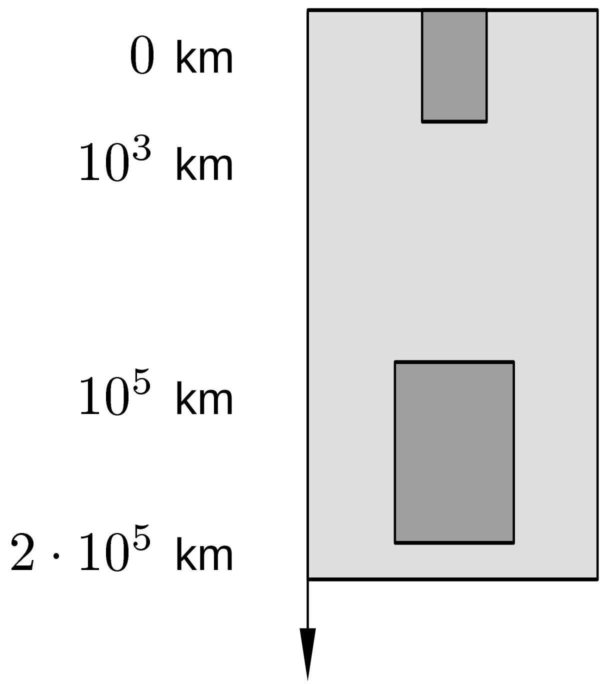
4) Compressible Liquid-Vapour Flow: A Model Problem

1) Radiation Hydrodynamics

(joint work with W.-A. Yong, A. Dedner)

Sun spots and flux tubes:





Photosphere:

Plasma	Ma ≈ 1
Magnetic fields	Rm $\approx 10^5$
Radiation	

Convection zone:

Plasma	Ma $\approx 10^{-3}$
Magnetic fields	Rm $\approx 10^{10}$

(Material: Kiepenheuer-Institut für Sonnenphysik)

A Mathematical Model for Compressible RHD: (simplified, nondimensional)

$$\left.\begin{aligned} \rho_t &+ \nabla\cdot(\rho\mathbf{v}) &&= 0 \\ (\rho\mathbf{v})_t &+ \nabla\cdot(\rho\mathbf{v}\mathbf{v}^t + p\mathcal{I}) &&= 0 \\ (\rho e)_t &+ \nabla\cdot(\rho e\mathbf{v} + p\mathbf{v}) &&= \mathcal{P}\kappa\int_{\mathcal{S}^{d-1}}(I - B(\vartheta))\,d\mu \end{aligned}\right\} \text{ in } \mathbb{R}^2\times(0,T)$$

$$\frac{1}{c}I_t + \mu\cdot\nabla I = \kappa\,(B(\vartheta) - I) \quad \text{in } \mathbb{R}^2\times(0,T)\times\mathcal{S}^1$$

Unknown functions

- $\rho = \rho(\mathbf{x},t) > 0$: Density
- $\mathbf{v} = \mathbf{v}(\mathbf{x},t) \in \mathbb{R}^d$: Velocity
- $e = e(\mathbf{x},t) > 0$: Energy
- $I = I(\mathbf{x},t,\mu) \geq 0$: Radiation intensity

Coefficients

- $p = p(\rho,\vartheta)$: Pressure
- $\kappa = \kappa(\rho,\vartheta)$: Absorption
- $B = B(\vartheta)$: Planck function
- $c > 0$: Speed of light

$$\mathcal{P} = a\frac{T_{ref}^4}{\rho_{ref}v_{ref}^2}$$

Formal Non-Relativistic Limit for the RHD-System: $\left(\mathcal{P} = \mathcal{O}(1)\right)$

$$
\begin{aligned}
\rho_t &+ \nabla \cdot (\rho \mathbf{v}) &= 0 \\
(\rho \mathbf{v})_t &+ \nabla \cdot (\rho \mathbf{v}\mathbf{v}^t + p\mathcal{I}) &= 0 \\
(\rho e)_t &+ \nabla \cdot (\rho e \mathbf{v} + p\mathbf{v}) &= \mathcal{P}\kappa \int_{\mathcal{S}^{d-1}} (I - B(\vartheta))\, d\mu \\
\frac{1}{c} I_t &+ \mu \cdot \nabla_{\mathbf{x}} I &= \kappa \left(B(\vartheta) - I\right)
\end{aligned}
$$

Formal Non-Relativistic Limit for the RHD-System: $(\mathcal{P} = \mathcal{O}(1))$

$$\begin{aligned} \rho_t &+& \nabla \cdot (\rho \mathbf{v}) &=& 0 \\ (\rho \mathbf{v})_t &+& \nabla \cdot (\rho \mathbf{v}\mathbf{v}^t + p\mathcal{I}) &=& 0 \\ (\rho e)_t &+& \nabla \cdot (\rho e \mathbf{v} + p\mathbf{v}) &=& \mathcal{P}\kappa \int_{\mathcal{S}^{d-1}} (I - B(\vartheta))\, d\mu \\ && \mu \cdot \nabla_{\mathbf{x}} I &=& \kappa\, (B(\vartheta) - I) \end{aligned}$$

Formal Non-Relativistic Limit for the RHD-System: $\big(\mathcal{P} = \mathcal{O}(1)\big)$

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(\rho e)_t &+ \nabla \cdot (\rho e \mathbf{v} + p\mathbf{v}) &= \mathcal{P}\kappa \int_{\mathcal{S}^{d-1}} (I - B(\vartheta))\, d\mu \\
& \mu \cdot \nabla_{\mathbf{x}} I &= \kappa \left(B(\vartheta) - I\right)
\end{aligned}
$$

If $\vartheta \in L^\infty$ is given we obtain by the characteristic method:

$$
I(\mathbf{x}, t, \mu) = \int_{-\infty}^{0} e^{\kappa s} \kappa B(\vartheta(t, \mathbf{x} + s\mu))\, ds
$$

Formal Non-Relativistic Limit for the RHD-System: $(\mathcal{P} = \mathcal{O}(1))$

$$\begin{aligned} \rho_t &+ \nabla\cdot(\rho\mathbf{v}) &= 0 \\ (\rho\mathbf{v})_t &+ \nabla\cdot(\rho\mathbf{v}\mathbf{v}^t + p\mathcal{I}) &= 0 \\ (\rho e)_t &+ \nabla\cdot(\rho e\mathbf{v} + p\mathbf{v}) &= \mathcal{P}\kappa\int_{\mathcal{S}^{d-1}} (I - B(\vartheta))\,d\mu \\ & \mu\cdot\nabla_{\mathbf{x}} I &= \kappa\,(B(\vartheta) - I) \end{aligned}$$

Integrated intensity:

$$\begin{aligned} \int_{\mathcal{S}^{d-1}} I(\mathbf{x},t,\mu)\,d\mu &= \int_{\mathbb{R}^d} \frac{\kappa}{\mathcal{B}(d)} \frac{e^{-\kappa|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^{d-1}} B(\vartheta(t,\mathbf{y}))\,d\mathbf{y} \\ &=: \int_{\mathbb{R}^d} K_\kappa(\mathbf{x}-\mathbf{y}) B(\vartheta(t,\mathbf{y}))\,d\mathbf{y} \end{aligned}$$

Weakly singular kernel:

$$K(\mathbf{x}) = \frac{1}{\mathcal{B}(d)} \frac{e^{-|\mathbf{x}|}}{|\mathbf{x}|}, \qquad K_\kappa(\mathbf{x}) = \kappa^d K(\kappa\mathbf{x})$$

A Basic Nonlocal Model in Radiation Hydrodynamics: ($\mathcal{P} = \mathcal{O}(1)$)

$$\begin{aligned} \rho_t &+ \nabla\cdot(\rho\mathbf{v}) &= 0 \\ (\rho\mathbf{v})_t &+ \nabla\cdot(\rho\mathbf{v}\mathbf{v}^t + p\mathcal{I}) &= 0 \\ (\rho e)_t &+ \nabla\cdot(\rho e\mathbf{v} + p\mathbf{v}) &= \mathcal{P}\mathbf{L}[\vartheta(.,t)] \end{aligned}$$

Nonlocal convolution operator:

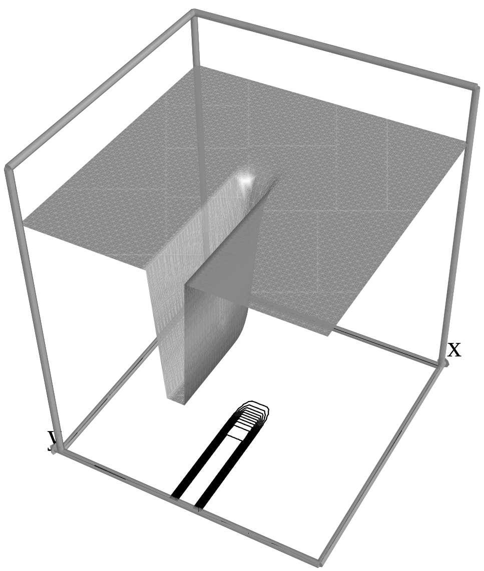
$$\mathbf{L}[\vartheta(.,t)](\mathbf{x}) = \kappa[K_\kappa * B(\vartheta(.,t))](\mathbf{x}) - \kappa B(\vartheta(\mathbf{x},t))$$

A numerical experiment with the RHD-system: (A. Dedner)

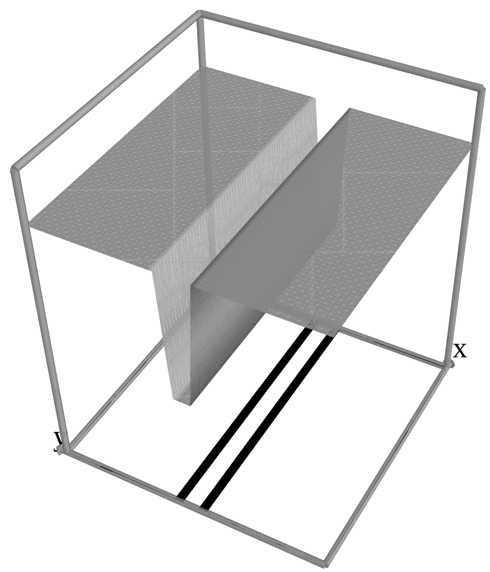
The RHD-system in 2D:

$$\begin{array}{rcccl} \rho_t & + & \nabla \cdot (\rho \mathbf{v}) & = & 0 \\ (\rho \mathbf{v})_t & + & \nabla \cdot (\rho \mathbf{v}\mathbf{v}^t + p\mathcal{I}) & = & 0 \\ (\rho e)_t & + & \nabla \cdot (\rho e \mathbf{v} + p\mathbf{v}) & = & \mathcal{P}\mathbf{L}[\vartheta(.,t)] \end{array}$$

Computation with no radiation, $\kappa = 0$:



Density at t=0.3



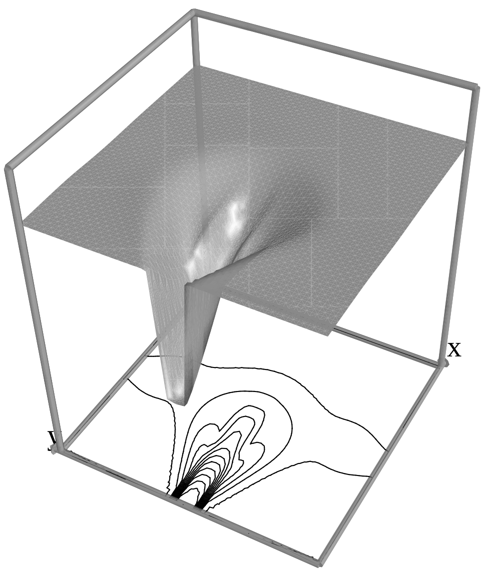
Density at t=1.0

A numerical experiment with the RHD-system: (A. Dedner)

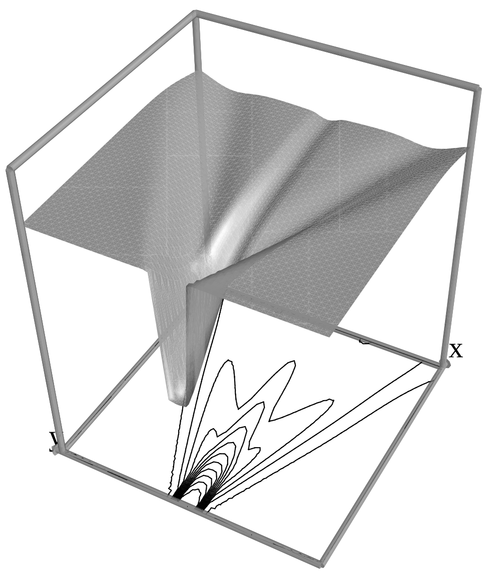
The RHD-system in 2D:

$$
\begin{aligned}
\rho_t &+ \nabla \cdot (\rho \mathbf{v}) &= 0 \\
(\rho \mathbf{v})_t &+ \nabla \cdot (\rho \mathbf{v}\mathbf{v}^t + p\mathcal{I}) &= 0 \\
(\rho e)_t &+ \nabla \cdot (\rho e \mathbf{v} + p\mathbf{v}) &= \mathcal{P}\mathbf{L}[\vartheta(.,t)]
\end{aligned}
$$

Computation with radiation, $\kappa = 0.5$:



Density at t=0.3



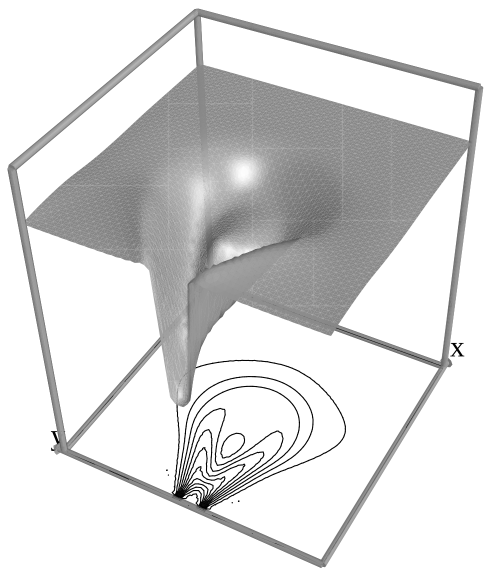
Density at t=1.0

A numerical experiment with the RHD-system: (A. Dedner)

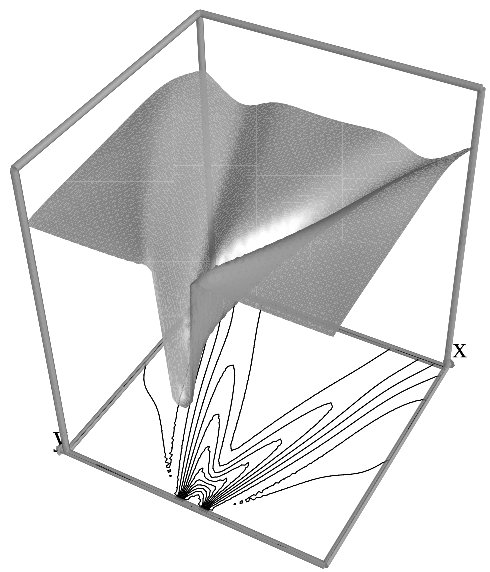
The RHD-system in 2D:

$$
\begin{aligned}
\rho_t &+ \nabla\cdot(\rho\mathbf{v}) &= 0\\
(\rho\mathbf{v})_t &+ \nabla\cdot(\rho\mathbf{v}\mathbf{v}^t + p\mathcal{I}) &= 0\\
(\rho e)_t &+ \nabla\cdot(\rho e\mathbf{v} + p\mathbf{v}) &= \mathcal{P}\mathbf{L}[\vartheta(.,t)]
\end{aligned}
$$

Computation with radiation, $\kappa = 2.0$:



Density at t=0.3



Density at t=1.0

A Very Old Numerical Result on Threshold Behaviour:

(from Heaslet&Baldwin,

Phys. Fluids 6, 1963)

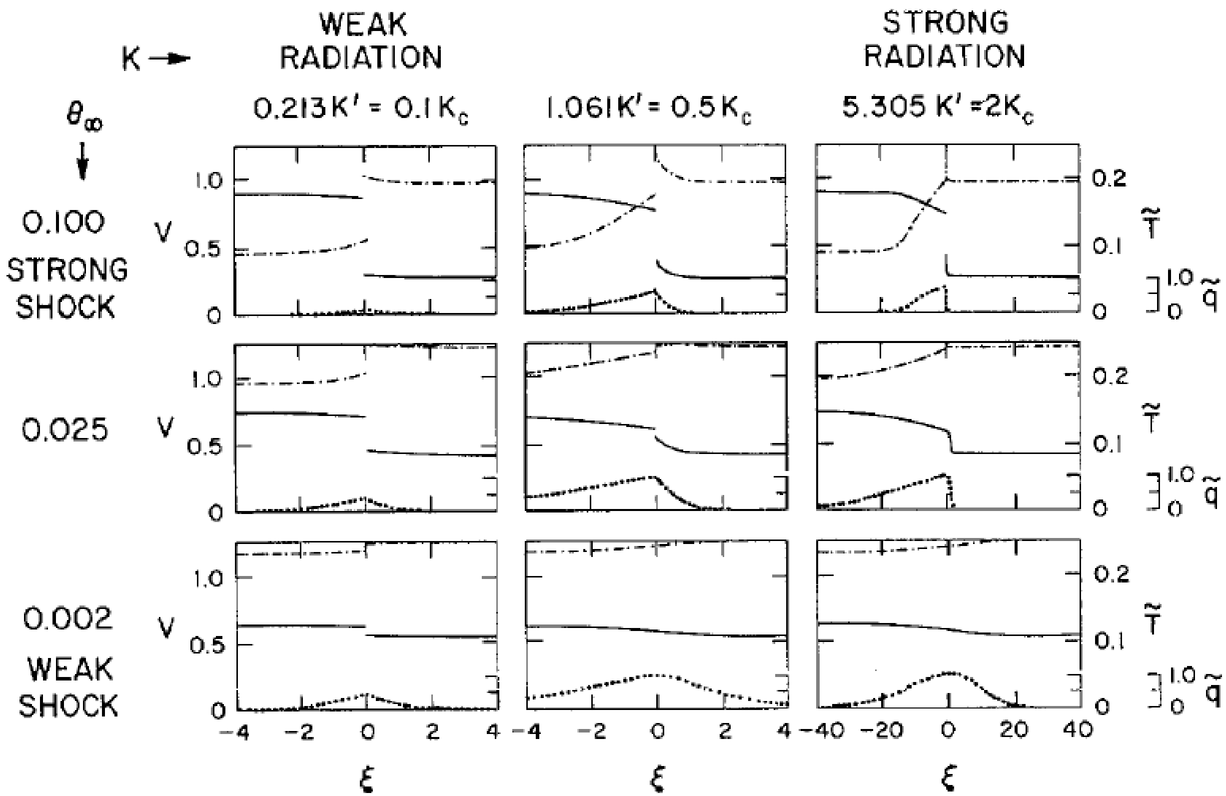


Fig. 104.6 Dimensionless velocity (solid curves), temperature (dash-dot curves), and heat flux (dotted curves) as a function of optical depth in shocks of different strengths and different amounts of radiation. From (**H3**) by permission.

Related Models I: Equilibrium Diffusion($\mathcal{P} = \mathcal{O}(\kappa)$)

$$\begin{aligned} \rho_t &+ \nabla \cdot (\rho \mathbf{v}) &= 0 \\ (\rho \mathbf{v})_t &+ \nabla \cdot (\rho \mathbf{v}\mathbf{v}^t + p\mathcal{I}) &= 0 \\ (\rho e)_t &+ \nabla \cdot (\rho e \mathbf{v} + p\mathbf{v}) &= \mathcal{P}\kappa[K_\kappa * B(\vartheta) - \kappa B(\vartheta)] \end{aligned}$$

We consider the situation $d = 1$ (for simplicity):

$$\begin{aligned} K_\kappa * B(\vartheta) - B(\vartheta) &= \int_{\mathbb{R}} \frac{\kappa}{2} e^{-\kappa|x-y|}(B(\theta(y)) - B(\theta(x)))dy \\ &\approx \int_{\mathbb{R}} \frac{1}{2} e^{-|z|}((B(\theta(x))_x \frac{z}{\kappa} + B(\theta(x))_{xx} \frac{z^2}{2\kappa^2})dy \\ &= \frac{1}{4\kappa^2} \left(\int_{\mathbb{R}} e^{-|z|} z^2 \, dz \right) (B(\theta(x)))_{xx} \end{aligned}$$

For $\mathcal{P} = \mathcal{O}(\kappa)$ (and $d \geq 1$) the energy balance becomes

$$(\rho e)_t + \nabla \cdot (\rho e \mathbf{v} + p\mathbf{v}) = C \Delta B(\vartheta)$$

Related Models II: Hyperbolic-Elliptic Model:

$$
\begin{aligned}
\rho_t &+ \nabla \cdot (\rho \mathbf{v})_x &= 0 \\
(\rho \mathbf{v})_t &+ \nabla \cdot (\rho \mathbf{v}\mathbf{v}^t + p\mathcal{I}) &= 0 \\
(\rho e)_t &+ \nabla \cdot (\rho e \mathbf{v} + p\mathbf{v}) &= \kappa \int_{\mathcal{S}^{d-1}} (I - B(\vartheta))\, d\mu \\
& \mu \cdot \nabla_{\mathbf{x}} I &= \kappa\, (B(\vartheta) - I)
\end{aligned}
$$

Related Models II: Hyperbolic-Elliptic Model: ($d = 1$!)

$$
\begin{aligned}
\rho_t &+ \rho v_x &= 0 \\
(\rho \mathbf{v})_t &+ (\rho v^2 + p)_x &= 0 \\
(\rho e)_t &+ (\rho e + pv)_x &= \kappa(I^+ - B(\vartheta) + I^- - B(\vartheta)) \\
& \pm I^\pm_x &= \kappa\left(B(\vartheta) - I^\pm\right)
\end{aligned}
$$

Related Models II: Hyperbolic-Elliptic Model: ($d = 1$!)

$$
\begin{aligned}
\rho_t &+ \rho v_x &= 0 \\
(\rho \mathbf{v})_t &+ (\rho v^2 + p)_x &= 0 \\
(\rho e)_t &+ (\rho e + p v)_x &= -(I_x^+ - I_x^-) \\
& \pm I_x^{\pm} &= \kappa\,(B(\vartheta) - I^{\pm})
\end{aligned}
$$

Related Models II: Hyperbolic-Elliptic Model: ($d = 1$!)

$$\begin{aligned}
\rho_t &+ \rho v_x &= 0 \\
(\rho \mathbf{v})_t &+ (\rho v^2 + p)_x &= 0 \\
(\rho e)_t &+ (\rho e + pv)_x &= -q_x \\
& \pm I^{\pm}_x &= \kappa \left(B(\vartheta) - I^{\pm}\right)
\end{aligned}$$

We define $q := I^+ - I^-$ and get for the radiation equation

$$q_x = 2B(\vartheta) - (I^+ + I^-) \Rightarrow -q_{xx} - \underbrace{\left(I^+ + I^-\right)_x}_{=q} = 2B(\vartheta)_x .$$

Related Models II: Hyperbolic-Elliptic Model: ($d = 1$!)

$$
\begin{aligned}
\rho_t &+ \rho v_x &= 0 \\
(\rho \mathbf{v})_t &+ (\rho v^2 + p)_x &= 0 \\
(\rho e)_t &+ (\rho e + pv)_x &= -q_x \\
& -q_{xx} + q &= 2B(\vartheta)_x
\end{aligned}
$$

We define $q := I^+ - I^-$ and get for the radiation equation

$$
q_x = 2B(\vartheta) - (I^+ + I^-) \Rightarrow -q_{xx} - \underbrace{(I^+ + I^-)_x}_{=q} = 2B(\vartheta)_x .
$$

Note: Hamer ’71, Kawashima et al. ’85, Serre et al. ’03,...

2) Radiation Hydrodynamics
A Model Problem

Cauchy Problem for *Relativistic* Model System:

Let $\mathbf{f} \in C^2(\mathbb{R}, \mathbb{R}^d)$ and $B \in C^1(\mathbb{R})$ with $B, B' \geq 0$.
For $c > 0$ find $u^c = u^c(\mathbf{x}, t)$ and $I^c = I^c(\mathbf{x}, t, \mu)$ with

$$\begin{aligned} u_t^c + \nabla \cdot \mathbf{f}(u^c) &= \int_{\mathcal{S}^{d-1}} I^c(., \mu) - B(u^c)\, d\mu \quad \text{in } \mathbb{R}^d \times (0,\infty), \\ \tfrac{1}{c} I_t^c + \mu \cdot \nabla I^c &= B(u^c) - I^c \quad \text{in } \mathbb{R}^d \times (0,\infty) \times \mathcal{S}^{d-1}, \\ u^c(., 0) = u_0, &\quad I^c(., 0, \mu) = I_0(., \mu). \end{aligned} \qquad (P_c)$$

Theorem:[R&Yong08]
Let $a, b \in \mathbb{R}$ such that $(u_0(\mathbf{x}), I_0(\mathbf{x}, \mu)) \in [a, b] \times [B(a), B(b)]$.
For each $c > 0$ there is an entropy solution (u^c, I^c) of (P_c) with

(i) $(u^c(\mathbf{x}, t), I^c(\mathbf{x}, t, \mu)) \in [a, b] \times [B(a), B(b)]$ a.e.,

(ii) $|u^c(., t)|_{BV} \leq |u_0|_{BV} + \frac{|\mathcal{S}^1|}{c} \text{ess sup}_{\mu \in \mathcal{S}^{d-1}} |I_0(., \mu)|_{BV}$.

Proof: (Almost) standard via finite-volume scheme on Cartesian mesh.

Classical Solutions for Small Amplitude Data:

Theorem: [R.&Tiemann&Yong06]
Let $\bar{u} > 0$ and $d = 2$.
For all initial data $(u_0, I_0(\cdot, \mu))$ sufficiently close to $(\bar{u}, B(\bar{u}))$ with

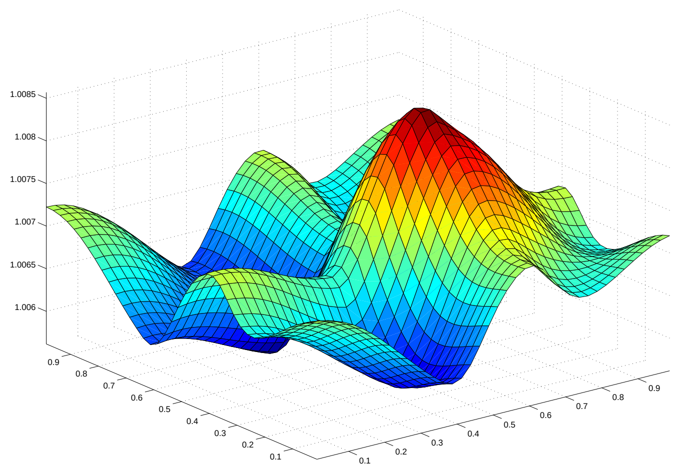
$$u_0 - \bar{u}, I_0(., \mu) - B(\bar{u}) \in H^3(\mathbb{R}^2),\ \mu \in \mathcal{S}^1,$$

there is a unique **classical** solution $(u^c, I^c(., \mu)) \in C(0, \infty; H^3(\mathbb{R}^2))$ of (P_c).

Proof: Use theory of W.-A. Yong for relaxation systems.
Numerical Experiment: $(d = 2)$

$$f_1(u) = f_2(u) = u^2/2$$

$$\begin{aligned} u_0(\mathbf{x}) &= \begin{cases} 1.5 : |\mathbf{x} - (0,5,0.5)^T| \\ 1 : \text{elsewhere} \end{cases} \\ I_0(\mathbf{x}, \mu) &= 1.0 \end{aligned}$$



The Nonrelativistic Limit:

Theorem: [R.&Yong 06]
Let $\{(u^c, I^c)\}_{c>0}$ be a family of entropy solutions for (P_c).
Then there exists a function $u = u(\mathbf{x}, t)$ such that

(i) $\lim_{c\to\infty} \|u^c - u\|_{L^1_{loc}(\mathbb{R}^d\times(0,\infty))} = 0$,

(ii) u is the (unique) entropy solution of

$$\begin{aligned} u_t + \nabla\cdot\mathbf{f}(u) &= K * B(u) - B(u) \quad \text{in } \mathbb{R}^d \times (0,\infty), \\ u(x,0) &= u_0 \quad \text{in } \mathbb{R}^d, \end{aligned}$$

i.e. for all entropies $\eta \in C^2(\mathbb{R})$ with associated entropy-flux $\mathbf{q} \in C^2(\mathbb{R},\mathbb{R}^d)$ holds

$$\eta(u)_t + \nabla\cdot\mathbf{q}(u) \le \eta'(u)(K * B(u) - B(u)) \quad \text{in } \mathcal{D}'(\mathbb{R}^d \times [0,\infty)).$$

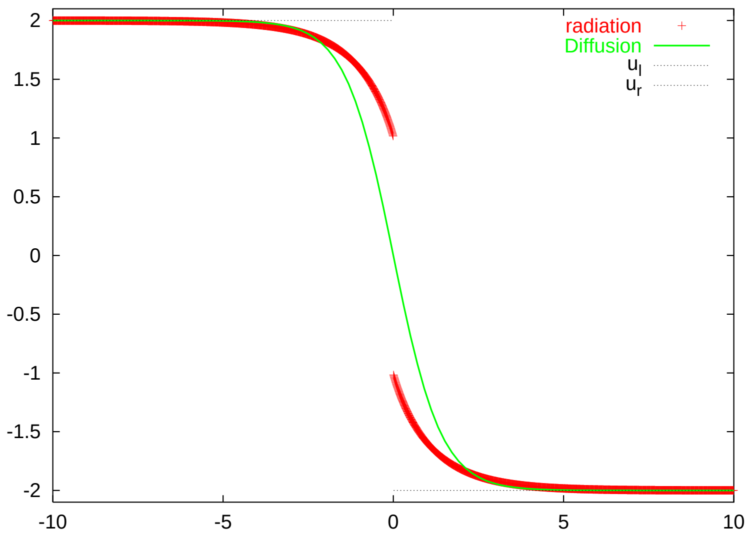
Note 1: No strong but weak compactness for $\{I^c\}_{c>0}$ is needed.
Note 2: The rate of convergence with respect to c is not known.

Limit Problem and Smoothing Effect for $d = 1$ (Dedner):

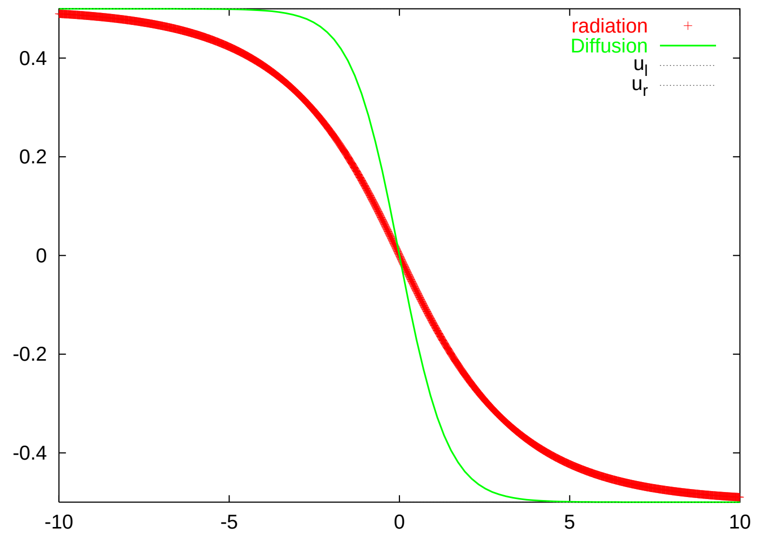
$$u_0(x) = \begin{cases} a & : \quad x < 0 \\ -a & : \quad x > 0 \end{cases}, \qquad\qquad f(u) = u^2/2.$$

Limit Problem and Smoothing Effect for $d = 1$ (Dedner):

$$u_0(x) = \begin{cases} a & : \quad x < 0 \\ -a & : \quad x > 0 \end{cases}, \qquad f(u) = u^2/2.$$

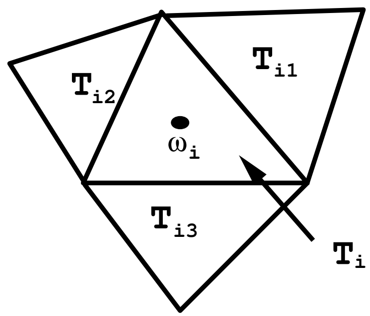


$a = 2$



$a = 0.5$

Finite Volume Scheme for the Homogeneous Case:



$$\begin{aligned} u_t + \nabla \cdot \mathbf{f}(u) &= 0 && \text{in } \mathbb{R}^d \times (0,\infty), \\ u(x,0) &= u_0 && \text{in } \mathbb{R}^d, \end{aligned}$$

Define $u_h : \mathbb{R}^2 \times [0,T] \to \mathbb{R}$ iteratively by

$$\begin{aligned} u_i^0 &= \frac{1}{|T_i|} \int_{T_i} u_0(\mathbf{x})\, d\mathbf{x}, \\ u_i^{n+1} &= u_i^n - \frac{\Delta t}{|T_i|} \sum_{l \in \mathcal{N}(i)} F_{il}(u_i^n, u_l^n) \end{aligned}$$

Discretization of the Convolution Operator

$$\mathbf{L}[w](\mathbf{x}) := \int_{\mathbb{R}^2} K(\mathbf{x}-\mathbf{y})(B(w(\mathbf{y})) - B(w(\mathbf{x})))\, d\mathbf{y}$$

Definition: [Discrete Convolution]
For a **compact** subset Ω of $\mathbb{R}^2$ we define $\mathbf{L}_{\Omega,h}$ for $\mathbf{x} \in T_j$ by

$$\mathbf{L}_{\Omega,h}[w](\mathbf{x}) = \chi_\Omega(\mathbf{x}) \sum_{i \neq j} |T_i| K_h(\omega_j - \omega_i)(B(w(\omega_i)) - B(w(\omega_j)))$$

Thereby we used $K_h(\mathbf{x}-\mathbf{y}) = \exp(-|\mathbf{x}-\mathbf{y}|)/(|\mathbf{x}-\mathbf{y}|+h)$.
This implies for $w \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$

(i) $\quad \exists\, C = C(\|w\|_{L^\infty}) > 0 : \int_{T_i} |(\mathbf{L} - \mathbf{L}_{\Omega,h})[w]|\, d\mathbf{x} \leq C|T_i| h \ln|h|,$

(ii) $\quad \text{supp}\big(\mathbf{L}_{\Omega,h}[w]\big) \subset \Omega.$

The Fully-Discrete Finite Volume Scheme:

Define $u_{\Omega,h} : \mathbb{R}^2 \times [0,T] \to \mathbb{R}$ iteratively by

$$
\begin{aligned}
u_i^0 &= \frac{1}{|T_i|} \int_{T_i} u_0(\mathbf{x})\, d\mathbf{x}, \\
u_i^{n+1} &= u_i^n - \frac{\Delta t}{|T_i|} \sum_{l \in \mathcal{N}(i)} F_{il}(u_i^n, u_l^n) + \boldsymbol{\Delta} t \mathbf{L}_{\Omega,h}[u_{\Omega,h}(.,t^n)](\omega_i).
\end{aligned}
$$

Theorem: [Dedner&Rohde '04]
Let $u_0 \in L^1 \cap L^\infty \cap BV$ with compact support.
For an appropriate CFL-condition we have for $t \in [0,T]$ the estimate

$$\operatorname*{ess\,inf}_{\mathbf{x}\in\mathbb{R}^2}\{u_0(\mathbf{x})\} \leq u_{\Omega,h}(\mathbf{x},t) \leq \operatorname*{ess\,sup}_{\mathbf{x}\in\mathbb{R}^2}\{u_0(\mathbf{x})\}.$$

Furthermore we have for $C > 0$

$$\|u - u_{\Omega,h}\|_1 \leq C\Big(|\Omega| h^{\frac{1}{4}} + |\Omega| h \ln|h| + \operatorname*{ess\,sup}_{t\in[0,T]} \int_{\mathbb{R}^2\setminus\Omega} |\mathbf{L}[u(.,t)]|\Big).$$

The choice $|\Omega| = \mathcal{O}(\ln|h|)$ leads to the estimate

$$\|u - u_{\Omega,h}\|_1 \leq C h^{1/4} |\ln(h)|.$$

(provided the exact solutions decays exponentially for $|\mathbf{x}| \to \infty$)

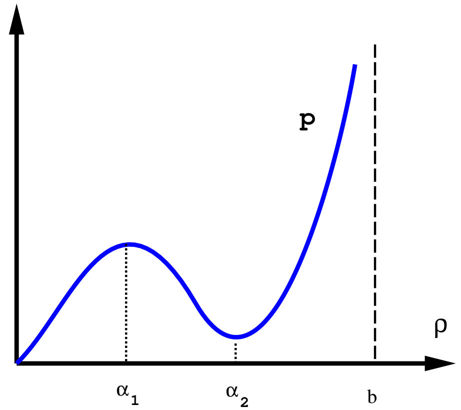
3) Compressible Liquid-Vapour Flow

Dynamic Local Diffuse Interface Model (Navier-Stokes-Korteweg)

$$\begin{aligned} \rho_t \;+\; & \operatorname{div}(\rho\mathbf{v}) &= 0 \\ (\rho\mathbf{v})_t \;+\; & \operatorname{div}(\rho\mathbf{v}\mathbf{v}^T + p(\rho)\mathcal{I}) &= \varepsilon\operatorname{div}(\tau) + \rho\nabla D^\varepsilon[\rho] \end{aligned}$$

$\rho = \rho(\mathbf{x}, t) > 0$: Density

$\mathbf{v} = \mathbf{v}(\mathbf{x}, t) \in \mathbb{R}^d$: Velocity



Van-der-Waals pressure

$$p(\rho) = \rho W'(\rho) - W(\rho).$$

$$D^\varepsilon_{\text{local}}[\rho] = \varepsilon^2\Delta\rho$$

Energy inequality

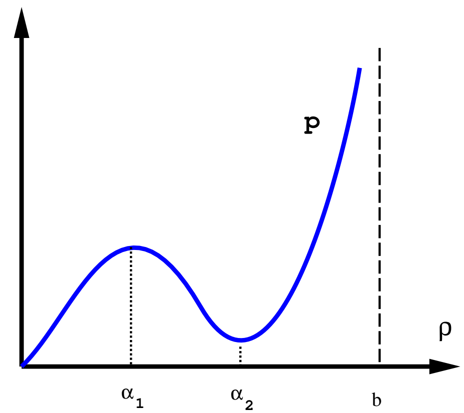
$$\frac{d}{dt}\Big(\int_{\mathbb{R}^d} W(\rho) + \frac{1}{2}\rho|\mathbf{v}|^2 + \frac{\varepsilon^2}{2}|\nabla\rho|^2\,d\mathbf{x}\Big) \le 0$$

Dynamic Nonlocal Diffuse Interface Model I

$$\begin{aligned} \rho_t \; + \; & \text{div}(\rho \mathbf{v}) &= \; & 0 \\ (\rho\mathbf{v})_t \; + \; & \text{div}(\rho\mathbf{v}\mathbf{v}^T + p(\rho)\mathcal{I}) &= \; & \varepsilon\text{div}(\tau) + \rho\nabla D^\varepsilon[\rho] \end{aligned}$$

$\rho = \rho(\mathbf{x}, t) > 0$: Density

$\mathbf{v} = \mathbf{v}(\mathbf{x}, t) \in \mathbb{R}^d$: Velocity



Van-der-Waals pressure

$$D^\varepsilon_{\text{global}}[\rho] = K_\varepsilon * \rho - \rho, \quad K_\varepsilon(\mathbf{x}) = \varepsilon^{-d} K(\mathbf{x}/\varepsilon)$$

Energy inequality

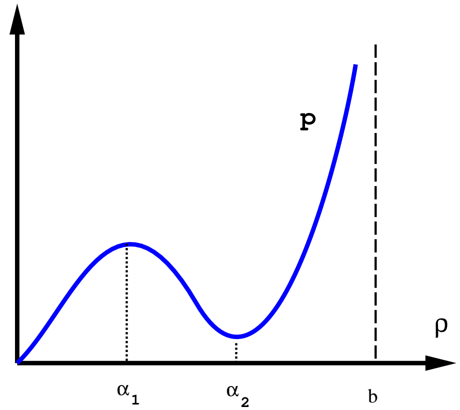
$$\frac{d}{dt}\int_{\mathbb{R}^d} \Big(W(\rho) + \frac{1}{2}\rho|\mathbf{v}|^2 + \frac{1}{4}\int_{\mathbb{R}^d} K_\varepsilon(.\mathbf{x}, \mathbf{y})(\rho(\mathbf{y}, t) - \rho(\mathbf{x}, t))^2 \, d\mathbf{y} \, d\mathbf{x} \Big) \leq 0$$

Examples for K: Gauss- or Newton kernels, fractional Laplacian.

Liquid Vapour Flow

Dynamic Nonlocal Diffuse Interface Model II

$$\begin{aligned} \rho_t + \operatorname{div}(\rho \mathbf{v}) &= 0 \\ (\rho \mathbf{v})_t + \operatorname{div}(\rho \mathbf{v} \mathbf{v}^T + p(\rho)\mathcal{I}) &= \varepsilon \operatorname{div}(\tau) + \rho \nabla D^{\varepsilon}[\rho] \\ 0 &= \frac{\varepsilon^2}{\tau^2}\Delta c - c + \rho \end{aligned}$$



Van-der-Waals pressure

$\rho = \rho(\mathbf{x}, t) > 0$: Density

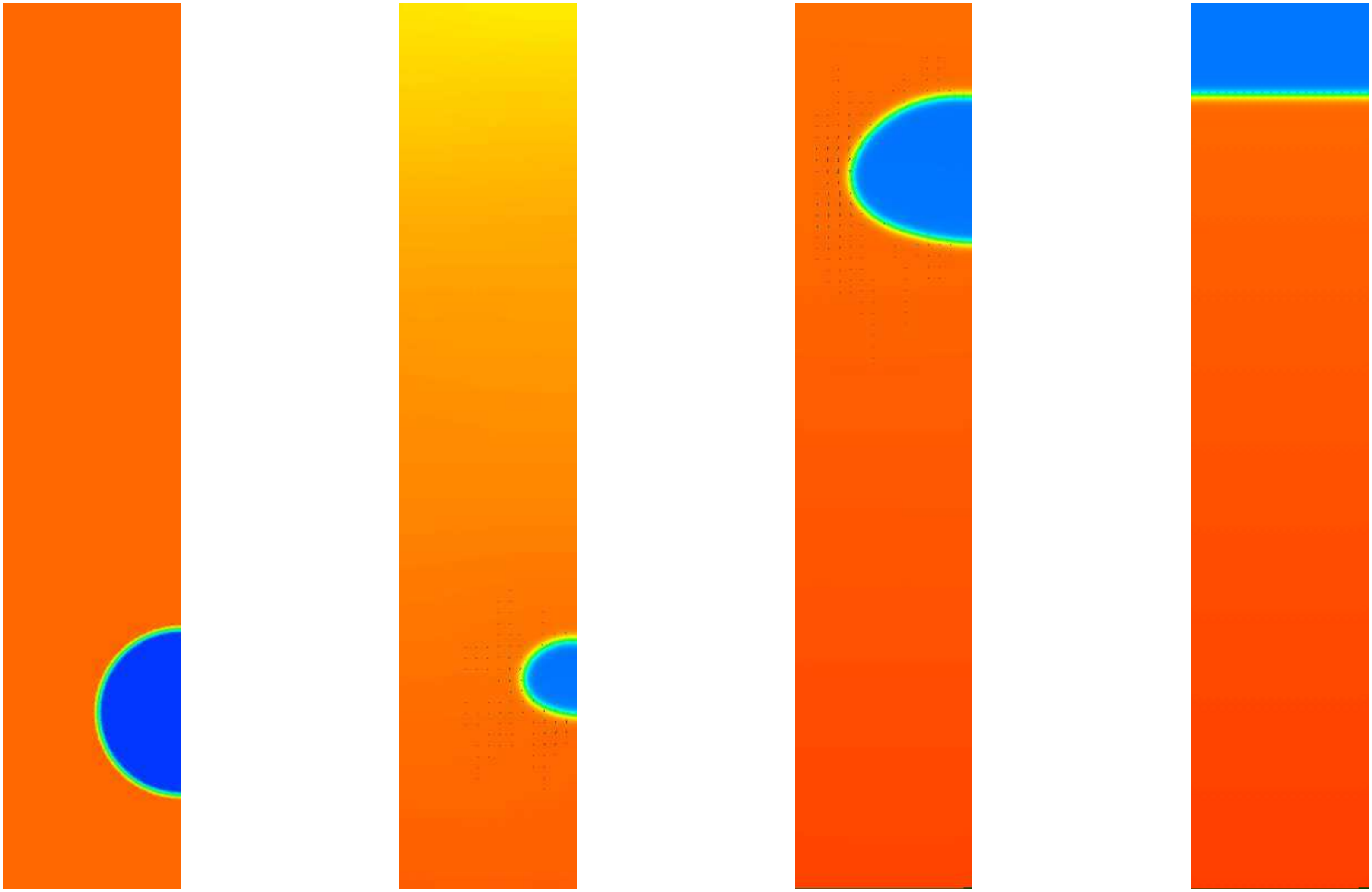
$\mathbf{v} = \mathbf{v}(\mathbf{x}, t) \in \mathbb{R}^d$: Velocity

$$D^{\varepsilon}_{\text{order}}[\rho] = \tau^2(\rho - c)$$

Energy inequality

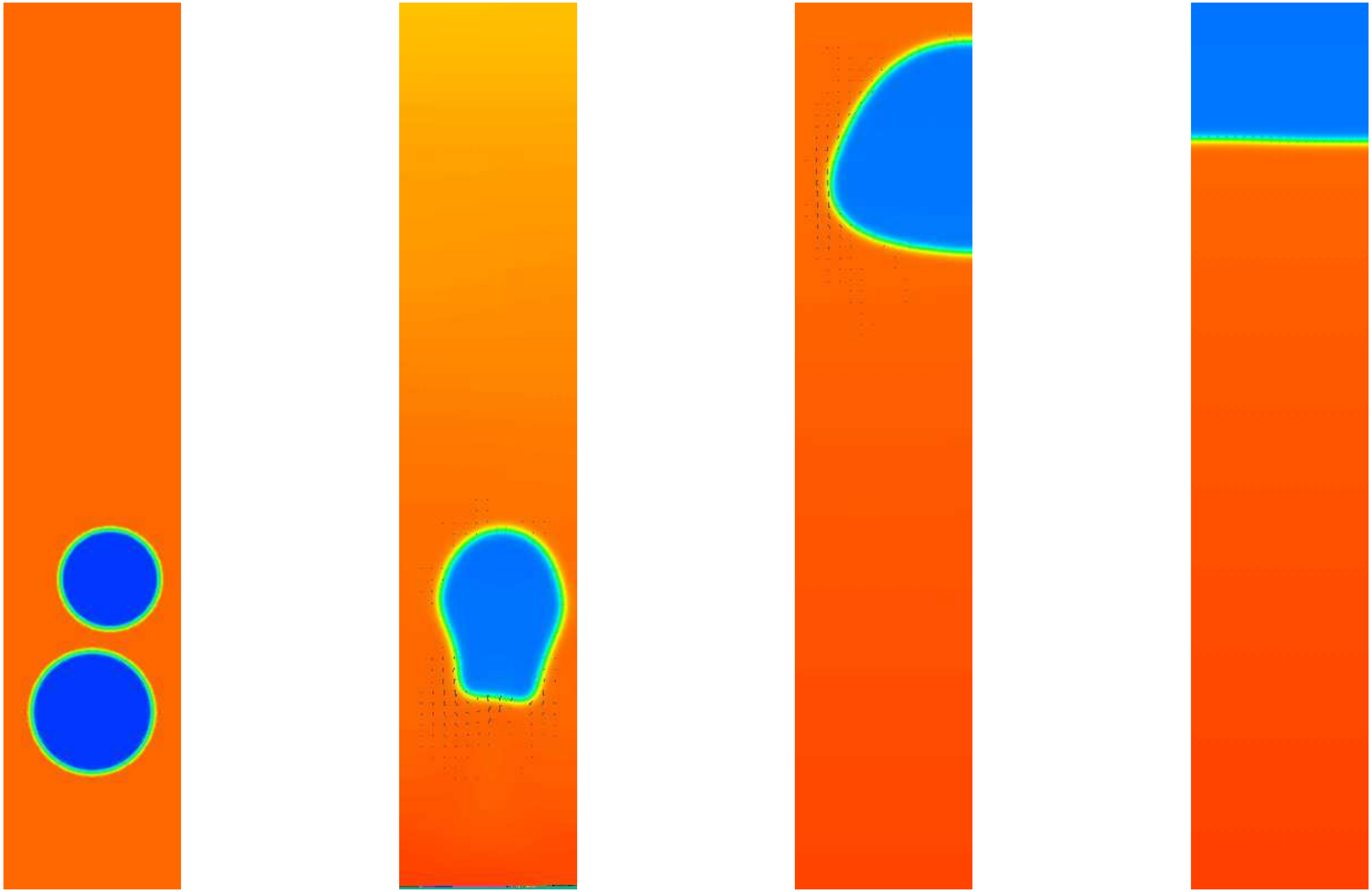
$$\frac{d}{dt}\Big(\int_{\mathbb{R}^d} W(\rho) + \frac{1}{2}\rho|\mathbf{v}|^2 + \frac{\tau^2}{2}(\rho - c)^2 + \frac{\varepsilon^2}{2}|\nabla c|^2 \, d\mathbf{x} \Big) \leq 0$$

Numerical Experiment: (Rising bubble at boundary, Haink ’09)



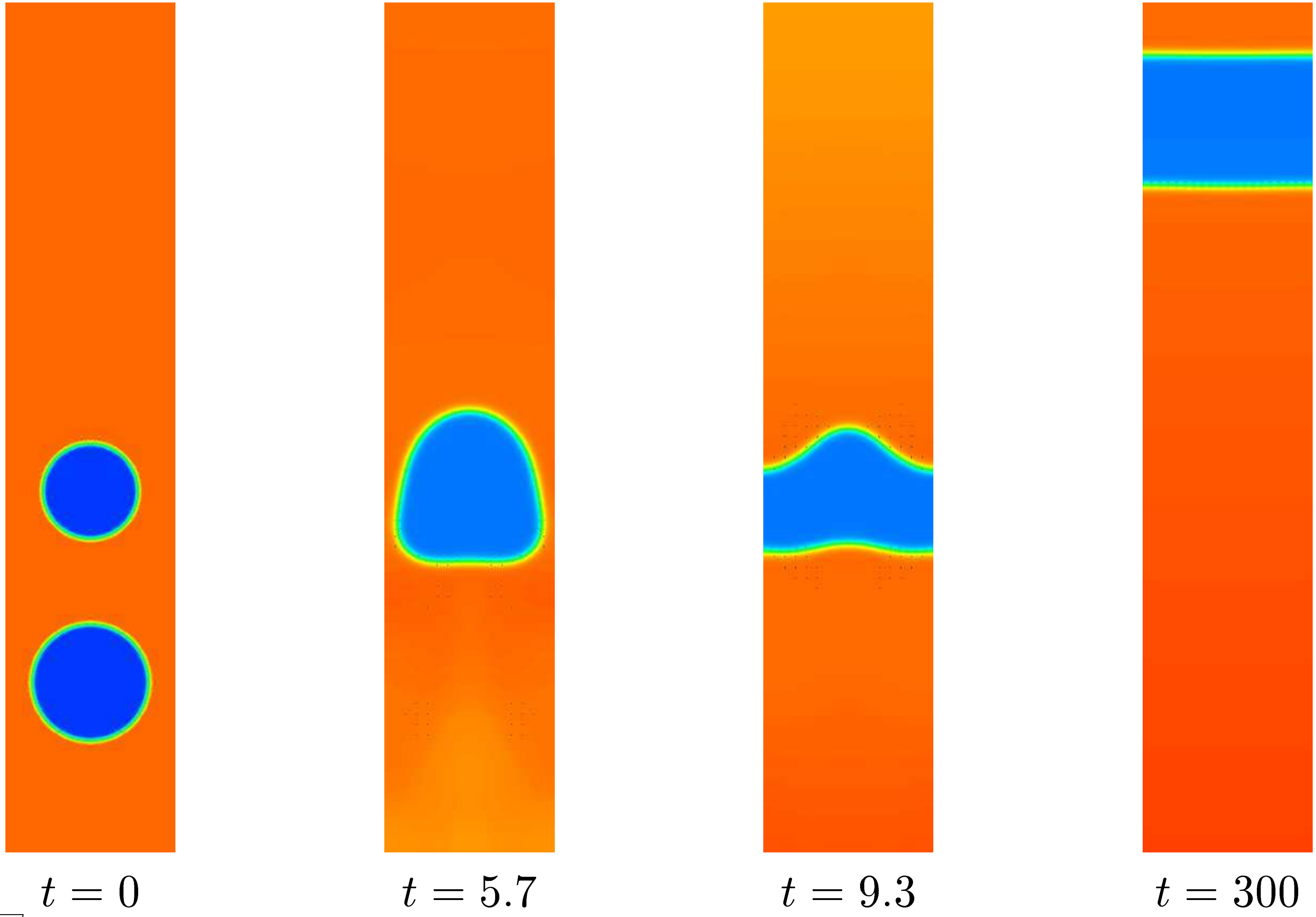
$t = 0$ $t = 4$ $t = 38$ $t = 200$

Numerical Experiment: (Two bubbles at one boundary, Haink '09)



$t = 0$ $t = 4.6$ $t = 50$ $t = 100$

Numerical Experiment: (Rising centered bubbles, Haink ’09)



4) Compressible Liquid-Vapour Flow
A Model Problem

The Nonlocal Model Problem

For $\varepsilon > 0$ find u^ε with

$$\begin{aligned} u_t^\varepsilon + (f(u^\varepsilon))_x &= \varepsilon u_{xx}^\varepsilon + \gamma(K_\varepsilon * u^\varepsilon - u^\varepsilon)_x && \text{in } \mathbb{R} \times (0,\infty), \\ u(.,0) &= u_0 && \text{in } \mathbb{R}. \end{aligned} \qquad (P_\varepsilon)$$

Theorem: [R'06]

Let $K \in C^\infty(\mathbb{R})$ and $u_0 \in L^\infty(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ with $r_\infty = \|u_0\|_{L^\infty}$. Then there exists a number $r_2 = r_2(r_\infty) > 0$ such that for $\|u_0\|_{L^2} < r_2$ there exists a unique classical solution u^ε of (P_ε) in $\mathbb{R}^d \times [0,T]$ which satisfies for $t \in [0,T]$ the estimate

$$\frac{1}{2}\|u^\varepsilon(.,t)\|_{L^2} + \varepsilon \|\nabla u^\varepsilon\|_{L^2(\mathbb{R}^d \times [0,t])} = \|u_0\|_{L^2}.$$

Note: See Schonbek, Shearer et al., LeFloch et al.,... for local version.

The Vanishing Diffusion-Dispersion Limit

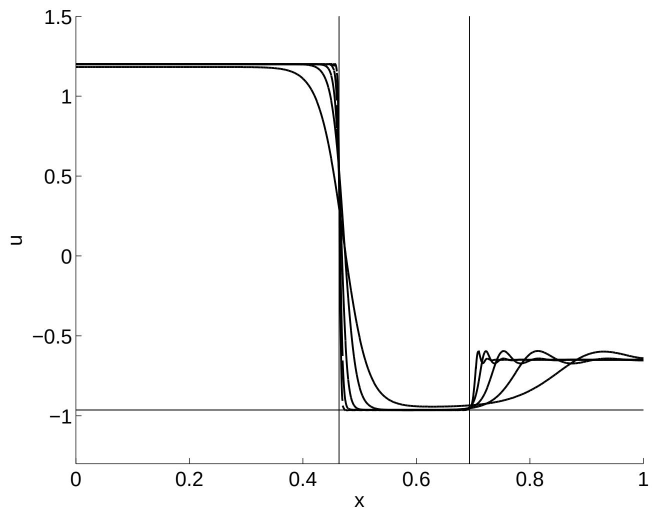
Theorem: [R '06]

Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a sequence of classical solutions for (P_ε) for $f(u) = u^3$.
Then there exists a function $u \in L^4(\mathbb{R})$ and a subsequence $\{u^\varepsilon\}_{\varepsilon>0}$ such that

$$u^\varepsilon \to u \text{ in } L^r_{loc}(\mathbb{R}^d \times [0,T]) \quad (r \in [1,4)).$$

Moreover u is a weak solution of

$$u_t + f(u)_x = 0.$$



Proof: (with compensated compactness)
A-priori estimates:

$$\|u^\varepsilon(.,t)\|_{L^2} + \|u^\varepsilon(.,t)\|_{L^4} + \sqrt{\varepsilon}\|u_x^\varepsilon\|_{L^2} < C \quad (t > 0).$$

Estimates of entropy dissipation:

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \subset \text{ compact set in } W^{-1,2}(K)$$

Proof: (with compensated compactness)
A-priori estimates:

$$\|u^\varepsilon(.,t)\|_{L^2} + \|u^\varepsilon(.,t)\|_{L^4} + \sqrt{\varepsilon}\|u^\varepsilon_x\|_{L^2} < C \quad (t > 0).$$

Estimates of entropy dissipation:

$$\begin{aligned}\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \quad &= \quad \varepsilon\eta(u^\varepsilon)_{xx} - \varepsilon\eta''(u^\varepsilon)(u^\varepsilon_x)^2 \\ &\quad + \alpha[\eta'(u^\varepsilon)(\phi_\varepsilon * u^\varepsilon - u^\varepsilon)]_x - \alpha\eta''(u^\varepsilon)u^\varepsilon_x(\phi_\varepsilon * u^\varepsilon - u^\varepsilon).\end{aligned}$$

Proof: (with compensated compactness)
A-priori estimates:

$$\|u^\varepsilon(.,t)\|_{L^2} + \|u^\varepsilon(.,t)\|_{L^4} + \sqrt{\varepsilon}\|u^\varepsilon_x\|_{L^2} < C \quad (t > 0).$$

Estimates of entropy dissipation:

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$$\begin{aligned} \left|\alpha \int_K \left[\eta'(u^\varepsilon)[\phi_\varepsilon * u^\varepsilon](x) - u^\varepsilon\right]_x \theta\, dxdt\right| &\leq C\alpha\|\phi_\varepsilon * u^\varepsilon - u^\varepsilon\|_2\|\theta_x\|_2 \\ &\leq C\alpha\varepsilon\|u^\varepsilon_x\|_2\|\theta\|_{W^{1,2}(K)} \\ &\to 0. \end{aligned}$$

Generalities on Local-Discontinuous-Galerkin(LDG) Schemes (Cockburn&Shu98)

Let $\{I_j\}$ be a partition of the interval I.

We seek an approximation $u_h(.,t) : I \to \mathbb{R}$ in the space

$$\mathcal{V}_h^{\mathbf{p}} := \{\phi_h \,|\, \phi_h|_{I_j} \text{ is a polynomial of degree} \leq \mathbf{p} \text{ for all } j \in \mathbb{Z}\,\}.$$

Legendre-polynomial ansatz:

$$u_h(.,t)\Big|_{I_j} = \sum_{k=0}^{\mathbf{p}} \alpha_k^j(t)\phi_k^j(.),$$

Idea: Derive an equivalent first-order enlarged system for (P_ε) and discretize the weak formulation.

Source-Like LDG-Scheme for the Model Problem

$$u_t + (f(u) - \varepsilon q)_x - \lambda\gamma(\Phi_\varepsilon * q - q) = 0, \quad q - u_x = 0$$

Source-Like LDG-Scheme for the Model Problem

$$\int_I u_t \varphi \, dx - \int_I (f(u) - \varepsilon q)\varphi_x \, dx + [\![f(u) - \varepsilon q]\!] - \int_I \gamma(\Phi_\varepsilon * q - q)\, dx = 0$$

Find $u_h(.,t) : I \to \mathbb{R}$ with

$$\int_{I_j} \big(u_{h,t} - \gamma([K_\varepsilon * q_h] - q_h)\big)\phi_h \, dx - \int_{I_j} \big(f(u_h) - \varepsilon q_h\big)\phi_{h,x} \, dx$$
$$= -\tilde{f}_{j+1/2}\phi_h(x^-_{j+1/2}) + \tilde{f}_{j-1/2}\phi_h(x^+_{j-1/2})$$
$$+\varepsilon\tilde{q}_{j+1/2}\phi_h(x^-_{j+1/2}) - \varepsilon\tilde{q}_{j-1/2}\phi_h(x^+_{j-1/2}),$$
$$\int_{I_j} q_h\phi_h \, dx + \int_{I_j} u_h\phi_{h,x} \, dx = \tilde{u}_{j+1/2}\phi_h(x^-_{j+1/2}) - \tilde{u}_{j-1/2}\phi_h(x^+_{j-1/2})$$

for all $\phi_h \in \mathcal{V}_h^{\mathbf{p}}$, $j \in \mathbb{Z}$.

Flux-Like LDG-Scheme for the Model Problem

$$u_t + (f(u) - \varepsilon q - \lambda\gamma(\Phi_\varepsilon * u - u))_x = 0, \quad q - u_x = 0.$$

Find $u_h(.,t), q_h(.,t) \in \mathcal{V}_h^{\mathbf{p}}$ such that

$$\int_{I_j} u_{h,t}\phi_h \, dx - \int_{I_j} \big(f(u_h) - \varepsilon q_h - \lambda\gamma([K_\varepsilon * u_h] - u_h)\big)\phi_{h,x} \, dx$$
$$= -\tilde{f}_{j+1/2}\phi_h(x^-_{j+1/2}) + \tilde{f}_{j-1/2}\phi_h(x^+_{j-1/2})$$
$$+\varepsilon\tilde{q}_{j+1/2}\phi_h(x^-_{j+1/2}) - \varepsilon\tilde{q}_{j-1/2}\phi_h(x^+_{j-1/2})$$
$$+\gamma[K_\varepsilon * u_h](x_{j+1/2})\phi_h(x^-_{j+1/2}) - \gamma[K_\varepsilon * u_h](x_{j-1/2})\phi_h(x^+_{j-1/2})$$
$$-\gamma\tilde{u}_{j+1/2}\phi_h(x^-_{j+1/2}) + \gamma\tilde{u}_{j-1/2}\phi_h(x^+_{j-1/2}),$$

$$\int_{I_j} q_h\phi_h \, dx + \int_{I_j} u_h\phi_{h,x} \, dx = \tilde{u}_{j+1/2}\phi_h(x^-_{j+1/2}) - \tilde{u}_{j-1/2}\phi_h(x^+_{j-1/2})$$

for all $\phi_h \in \mathcal{V}_h^{\mathbf{p}}, j \in \mathbb{Z}$.

Theorem for the Flux-Like Version [Haink&R.09]
Let $u_h \in \mathcal{V}_h^{\mathbf{p}}$ be the solution of the flux-like LDG-scheme with central flux $\tilde{q}$ and a monotone flux $\tilde{f}$. Then there are functions

$$g_{j+1/2} = g(u_h(x_{j+1/2}^-, t), u_h(x_{j+1/2}^+, t), q_h(x_{j+1/2}^-, t), q_h(x_{j+1/2}^+, t))$$

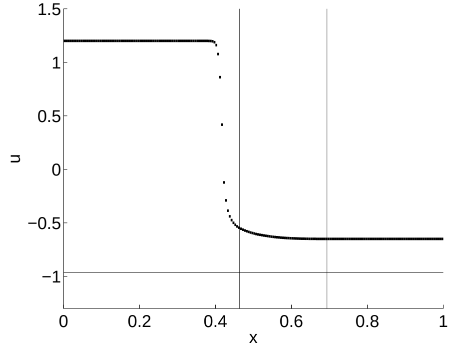
with $g(w, w, 0, 0) = f(w)w - F(w)$, $F' = f$, such that u_h satisfies

$$\frac{d}{dt}\int_{I_j} \frac{u_h^2}{2}\, dx + g_{j+1/2} - g_{j-1/2} \leq -\varepsilon \int_{I_j} q_h^2\, dx$$
$$- \gamma \left(\int_{I_j} [K_\varepsilon * u_h] u_{h,x}\, dx + [K_\varepsilon * u_h](x_{j-1/2}) \left(u_h(x_{j-1/2}^+) - u_h(x_{j-1/2}^-) \right) \right)$$

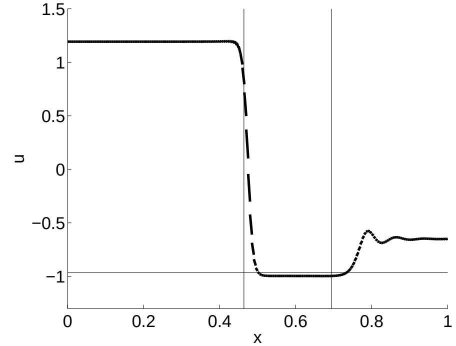
for all $j \in \mathbb{Z}$. Adding up yields

$$\frac{d}{dt}\int_{\mathbb{R}} \frac{u_h^2}{2}\, dx \leq 0.$$

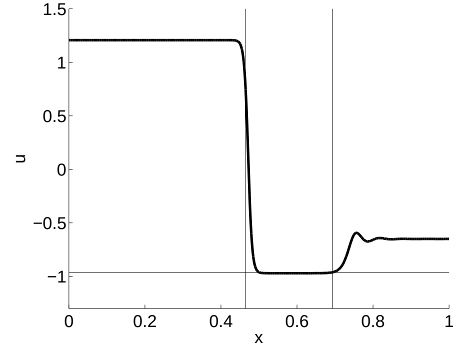
Numerical Experiment with the Two Variants: (Shock-Shock)



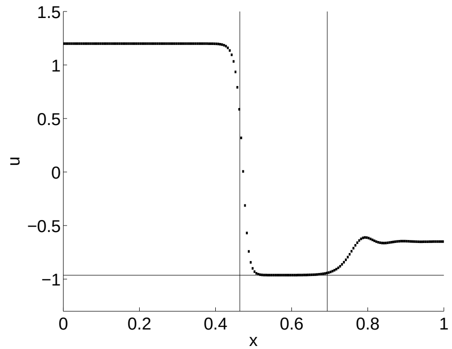
() $u_{h,f} \in \mathcal{V}_h^0$.



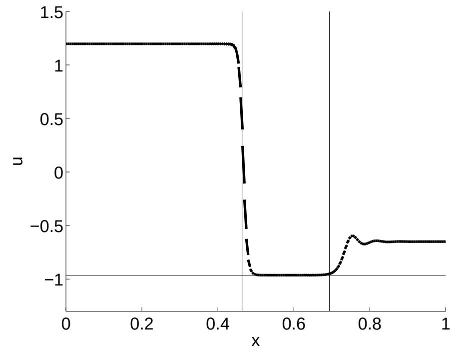
() $u_{h,f} \in \mathcal{V}_h^1$.



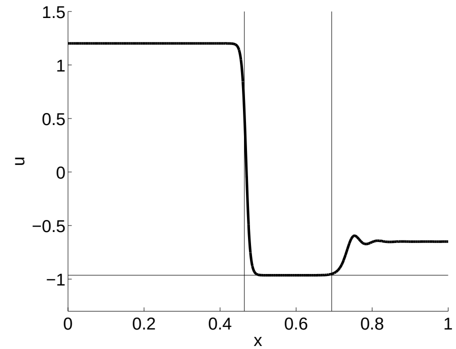
() $u_{h,f} \in \mathcal{V}_h^2$.



() $u_{h,s} \in \mathcal{V}_h^0$.

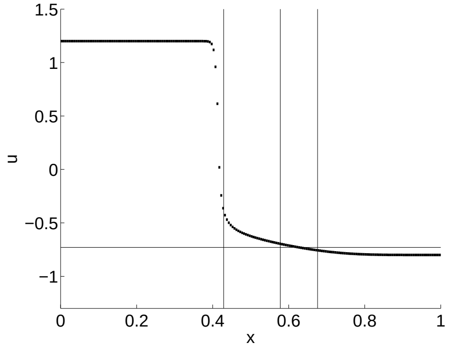


() $u_{h,s} \in \mathcal{V}_h^1$.

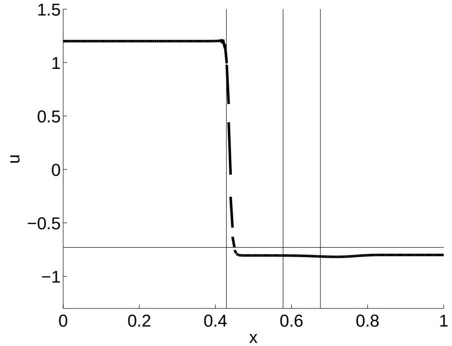


() $u_{h,s} \in \mathcal{V}_h^2$.

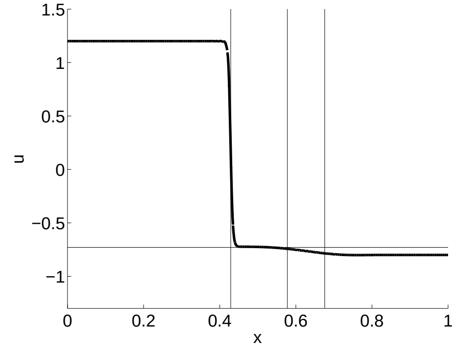
Numerical Experiment with the Two Variants: (Shock-Rarefaction)



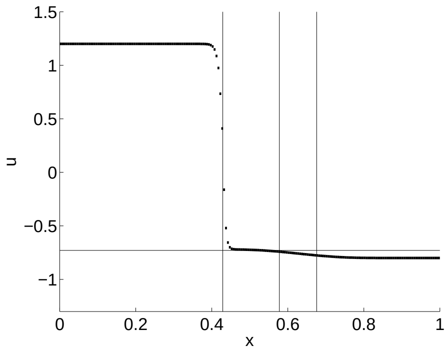
() $u_{h,f} \in \mathcal{V}_h^0$.



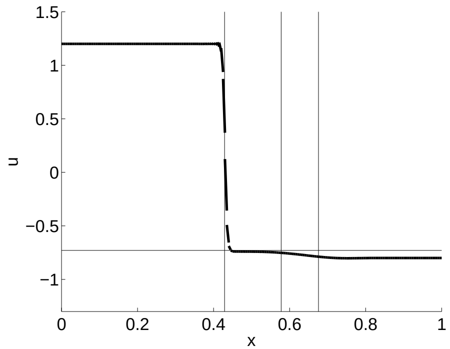
() $u_{h,f} \in \mathcal{V}_h^1$.



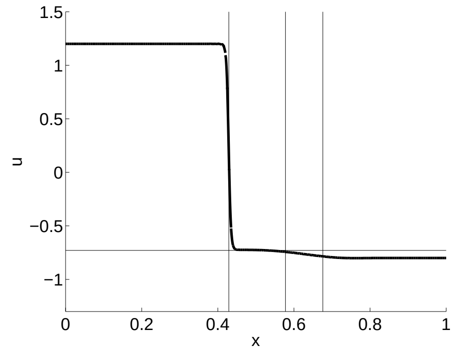
() $u_{h,f} \in \mathcal{V}_h^2$.



() $u_{h,s} \in \mathcal{V}_h^0$.



() $u_{h,s} \in \mathcal{V}_h^1$.



() $u_{h,s} \in \mathcal{V}_h^2$.

Some Conclusions:

(i) At least some theory on the level of model problems available...

(ii) Nonlocal Modelling leads to less severe time-step restrictions (but of course has a sampling problem)

(iii) Not much known for strongly singular kernels

(iv) Nonlocal approach leads to more general models (nondefininite, anisotropic kernels)