Analysis of Discontinuous Galerkin Methods for Fractional Conservation Laws and Related Equations

Kenneth H. Karlsen

University of Oslo
Centre of Mathematics for Applications

Joint work with Simone Cifani (Trondheim) & Espen Jakobsen (Trondheim)

Fractional degen convection-diffusion equations

$$\begin{cases} u_t + f(u)_x = (a(u)u_x)_x + b\mathcal{L}[u], \\ u(x,0) = u_0(x) \end{cases}$$

$f, a : \mathbb{R} \to \mathbb{R},\ a \geq 0$ bounded, Lipschitz continuous

$b \geq 0$ is a constant, and $\mathcal{L}$ is a nonlocal operator

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$\mathcal{L}$ is fractional Laplacian:

$$\widehat{\mathcal{L}[u(\cdot,t)]}(\xi) = -|\xi|^\lambda \hat{u}(\xi,t).$$

$$\lambda \in (0,1)$$

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Another way to represent $\mathcal{L}$ (Landkof [72])

$$\mathcal{L}[u(x,t)] = c_\lambda \int_{|z|>0} \frac{u(x+z,t) - u(x,t)}{|z|^{1+\lambda}}\, dz.$$

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Pseudodifferential operator $\mathcal{P}$ with a symbol $a(\omega) \geq 0$:

$$\widehat{\mathcal{P}v}(\omega) = a(\omega)\widehat{v}(\omega)$$

Lévy-Khintchine formula

$$a(\omega) = ib\cdot\omega + q(\omega) + \int_{\mathbb{R}^d\setminus\{0\}} \left(1 - e^{-iz\cdot\omega} - iz\cdot\omega\,\mathbf{1}_{|z|<1}(z)\right)\,\pi(dz),$$

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drift diffusion jump (Lévy) part Lévy measure

Special cases

Fractional diffusion equation

$$u_t = -(-\Delta)^\gamma, \qquad \gamma \in (0,1). \qquad\qquad u|_{t=0} = u_0.$$

Solution given by Greens' function! Solution is smooth ...

Conservation laws $u_t + f(u)_x = 0$

Discontinuous solutions, shock waves

Weak solutions, entropy conditions

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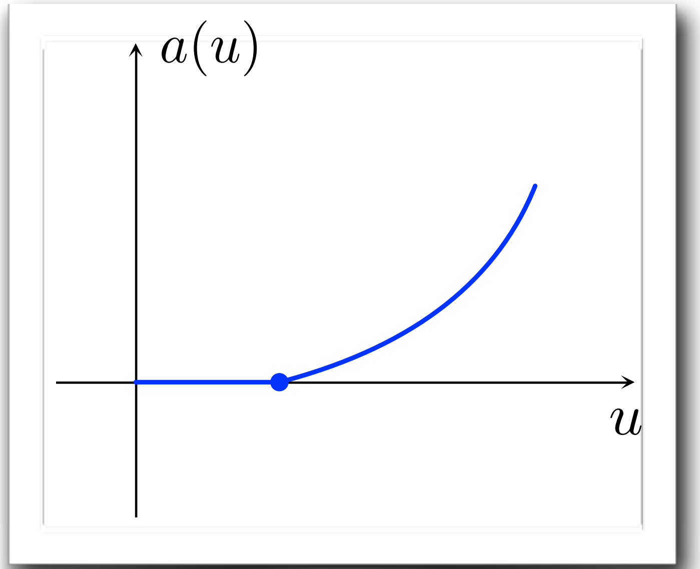
Discontinuous solutions, shock waves

Weak solutions, entropy conditions

For all convex η with $q' = \eta' f'$

$$\partial_t \eta(u) + \partial_x q(u) \leq 0 \quad \text{weakly}$$

Mixed hyperbolic-parabolic



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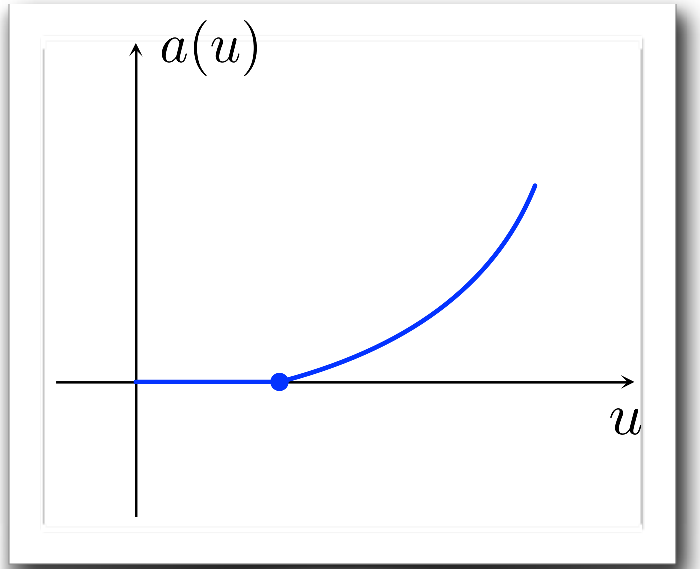


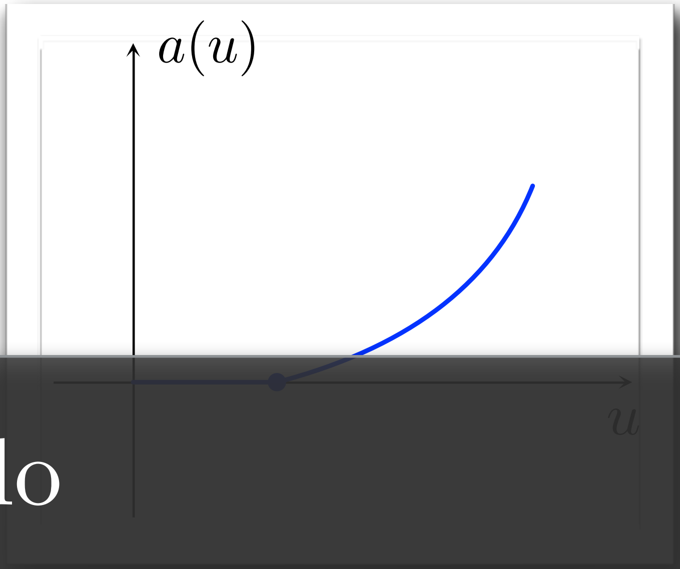
For all convex η with $q' = \eta' f'$, $r' = \eta' a$

$$\partial_t \eta(u) + \partial_x q(u) \leq \partial_x^2 r(u) - \eta''(u) a(u) (\partial_x u)^2 \quad \text{weakly}$$

Well-posedness theory in $L^1 \cap L^\infty$ of existence, uniqueness, L^1 contraction of entropy solutions ... (Carillo ...)

Mixed hyperbolic-parabolic

$$u_t + f(u)_x = (a(u)u_x)_x.$$



Entropy solution ala Carrillo

- $u \in L_t^\infty(L_x^1) \cap L^\infty \cap C_t(L_x^1)$
- $\nabla A(u) \in L^2 \qquad A = \int a$
- $\partial_t |u-k| + \partial_x \left(\operatorname{sgn}(u-k)(f(u)-f(k))\right.$
  $\left. - \operatorname{sgn}(u-k)\partial_x A(u)\right) \leq 0 \quad \text{weakly}$

Regularity issues for fractional Burgers equation

$$\partial_t u + \partial_x (u^2/2) = -(-\Delta)^\lambda u,$$

Subcritical ($\lambda > 1/2$), critical ($\lambda = 1/2$) cases:

solutions smooth in $t > 0$

(Droniou-Gallouet-Vovelle, Kiselev-Nazarov-Shterenberg
Chan-Czubak, Dong-Du-Li, ...)

[many parallel results quasi-geostrophic equation,
Kiselev et al., Caffarelli-Vasseur, ...]

Regularity issues for fractional Burgers equation

$$\partial_t u + \partial_x (u^2/2) = -(-\Delta)^\lambda u,$$

Supercritical case ($\lambda < 1/2$):

singularities indeed occur

(Alibaud-Droniou-Vovelle, Kiselev-Nazarov-Shterenberg, Dong-Du-Li)

$\longrightarrow$ Weak (distributional) solutions

Regularity issues for fractional Burgers equation

$$\partial_t u + \partial_x (u^2/2) = -(-\Delta)^\lambda u,$$

Weak solution: $u \in L^\infty$

$$\iint u \partial_t \phi + \frac{u^2}{2} \partial_x \phi - u(-\Delta)^\lambda [\phi] \, dx dt = 0, \qquad \forall \phi \in C_c^\infty$$

+ initial condition $u_0 \in L^\infty$

$\longrightarrow$ Weak (distributional) solutions

Entropy solutions for fractional PDE

Conservation laws

$$u_t + f(u)_x = \mathcal{L}[u]$$

$$\mathcal{L}[u](t,x) = \int_{\mathbb{R}\setminus\{0\}} \left[u(t,x+z) - u(t,x) - z\partial_x u\,\mathbf{1}_{|z|<1}\right]\,\pi(dz)$$

Viscosity regularized version

$$\partial_t u_\rho + \partial_x f(u_\rho) = \mathcal{L}[u_\rho(t,\cdot)] + \rho\Delta u_\rho, \qquad \varrho > 0.$$

Fix convex entropy η, entropy-flux q by $q' = \eta' f'$. Then

$$\partial_t \eta(u_\rho) + \partial_x q(u_\rho) = \mathcal{L}[\eta(u_\rho)] + \rho \Delta \eta(u_\rho) - \nu_\rho,$$

where $\nu_\rho = \nu_\rho^1 + \nu_\rho^2 + \nu_\rho^3$ consists of three parts:

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Entropy dissipation term

$$\nu_\rho^1 := \rho \Delta \eta(u_\rho) - \rho \eta'(u_\rho) \Delta u_\rho = \rho \eta''(u_\rho) \left| \nabla u_\rho \right|^2;$$

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where $\nu_\rho = \nu_\rho^1 + \nu_\rho^2 + \nu_\rho^3$ consists of three parts:

Fractional parabolic dissipation term

$$\nu_\rho^3 = \int_{\mathbb{R}^d \setminus \{0\}} \overline{\eta''}(u_\rho; z) \left(u_\rho(t, x+z) - u_\rho(t,x)\right)^2 \, \pi(dz),$$

$$\overline{\eta''}(u_\rho; z) = \int_0^1 (1-\tau)\eta''((1-\tau)u_\rho(t,x) + \tau u_\rho(t, x+z)) \, d\tau$$

The commutator $\mathcal{L}[\eta(u_\rho)] - \eta'(u_\rho)\mathcal{L}[u_\rho]$ equals ν_ρ^3, since

$$\eta(b) - \eta(a) = \eta'(a)\,(b-a) + \left(\int_0^1 (1-\tau)\eta''((1-\tau)a + \tau b)\,d\tau\right)(b-a)^2 .$$

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Entropy solution

$u \in L^\infty$ is an entropy solution if

$\forall$ convex entropies η, entropy-fluxes q, $q' = \eta' f$

$$\partial_t \eta(u) + \partial_x q(u) \leq \mathcal{L}[\eta(u)] - m^{u,\eta} \quad \text{weakly}$$

$$m^{u,\eta} = \int_{\mathbb{R}\setminus\{0\}} \overline{\eta''}(u;z)\,\big(u(t,x+z) - u(t,x)\big)^2\ \pi(dz),$$

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$\forall$ convex entropies η, entropy-fluxes q, $q' = \eta' f$

$\phi = \phi(t,x) \in C_c^\infty$

Can replace the term $\displaystyle\iint_{Q_T} \Big(\eta(u)\mathcal{L}[\varphi] - m^{u,\eta}\Big)\, dx\, dt$ by

$$\iint_{Q_T} \int_{|z|<r} \eta(u)[\varphi(t,x+z) - \varphi(t,x) - \nabla\varphi \cdot z]\, \pi(dz)\, dx\, dt,$$
$$+ \iint_{Q_T} \int_{|z|\geq r} \eta'(u)[u(t,x+z) - u(t,x)]\, \pi(dz)\, dx\, dt, \quad \forall r \in (0,1),$$

Entropy formulation due to Alibaud (existemce, uniqueness, etc.)

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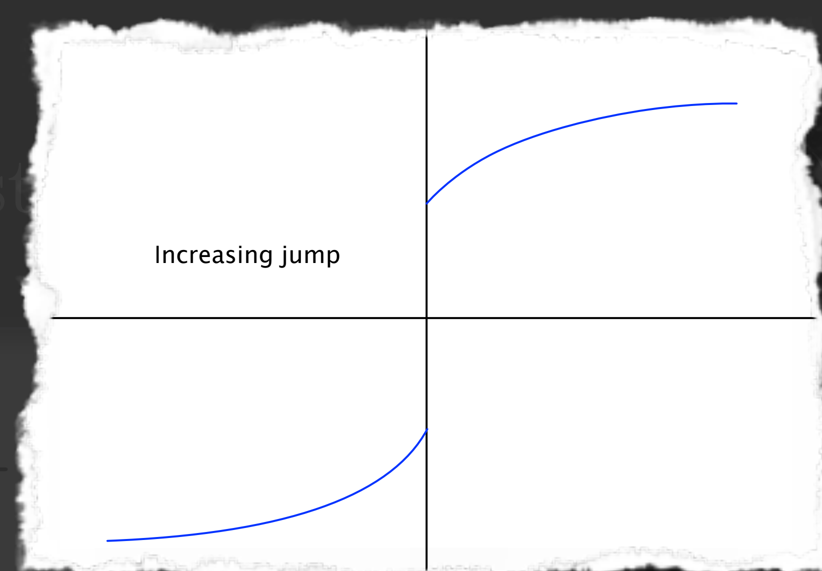
Not all weak solutions are entropy solutions !

Alibaud-Andreianov construct
a stationary weak solution
violating Oleinik's E condition



$$m^{u,\eta} = \int_{\mathbb{R}\setminus\{0\}} \overline{\eta''}(u;z)\,(u(t,x+z) - u(t,x))^2\ \pi(dz),$$

Fractional convection-diffusion equations

$$u_t + f(u)_x = (a(u)u_x)_x + \mathcal{L}[u], \qquad a \geq 0$$

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$\forall$ convex entropies η, entropy-fluxes q, r, $q' = \eta' f, r' = \eta' a$

$$\partial_t \eta(u) + \partial_x q(u) \leq \partial_x^2 r(u) - \eta''(u)a(u)(\partial_x u)^2 + \mathcal{L}[\eta(u)] - m^{u,\eta}$$

$$m^{u,\eta} = \int_{\mathbb{R}\setminus\{0\}} \overline{\eta''}(u;z)\,\big(u(t,x+z) - u(t,x)\big)^2\,\pi(dz),$$

(Chen-Perthame, Bendahmane-K, Ulusoy-K)

Uniqueness, stability, continuous dependence (Ulusoy-K) ...

$$\partial_t u + \operatorname{div} f(u) = \operatorname{div}(a(u)\nabla u) + \mathcal{L}[u], \quad u|_{t=0} = u_0$$

$$\partial_t v + \operatorname{div} \tilde{f}(v) = \operatorname{div}(\tilde{a}(v)\nabla v) + \tilde{\mathcal{L}}[v], \quad v|_{t=0} = v_0$$

$$\mathcal{L}[u] = \int \left[u(t,x+z) - u(t,x) - z\cdot\nabla_x u\,\mathbf{1}_{|z|<1}\right]\,\pi(dz), \quad \pi(dz) = m(z)\,dz$$

$$\tilde{\mathcal{L}}[v] = \int \left[v(t,x+z) - v(t,x) - z\cdot\nabla_x v\,\mathbf{1}_{|z|<1}\right]\,\tilde{\pi}(dz), \quad \tilde{\pi}(dz) = \tilde{m}(z)\,dz$$

Assume: $u \in L^\infty(0,T;BV(\mathbb{R}^d))$ entropy solution

with BV data $u_0 \in L^1 \cap L^\infty \cap BV$

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Assume: $u \in L^\infty(0,T;BV(\mathbb{R}^d))$ entropy solution with BV data $u_0 \in L^1 \cap L^\infty \cap BV$

For any $t \in (0,T)$,

$$\begin{aligned}
&\|u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbb{R}^d)} \\
&\qquad \le \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + C_1 t \left\| f - \tilde{f} \right\|_{W^{1,\infty}(I);\mathbb{R}^d)} \\
&\qquad\quad + C_2\sqrt{t} \left\| \sqrt{a} - \sqrt{\tilde{a}} \right\|_{L^\infty(I;\mathbb{R}^{d\times K})} \\
&\qquad\quad + C_3\sqrt{t} \sqrt{\left(\int_{|z|<1} |z|^2 \left| m(z) - \tilde{m}(z) \right| dz \right)} \\
&\qquad\quad + C_4 t \int_{|z|\ge 1} |z| \left| m(z) - \tilde{m}(z) \right| dz,
\end{aligned}$$

1D entropy solutions in BV class

$$u_t + f(u)_x = (a(u)u_x)_x + \mathcal{L}[u], \qquad a \geq 0$$

$$\mathcal{L}[u(x,t)] = c_\lambda \int_{|z|>0} \frac{u(x+z,t) - u(x,t)}{|z|^{1+\lambda}}\, dz, \quad \lambda \in (0,1)$$

$u \in L^\infty(Q_T)$ is BV entropy solution if $\quad (Q_T = (0,T) \times \mathbb{R})$

- $u \in L^\infty(0,T; L^1(\mathbb{R})) \cap BV(Q_T);$
- $A(u) \in C^{1,\frac{1}{2}}(Q_T); \qquad A = \int^u a;$
- Kruzkov-type entropy condition

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$$\forall 0 \leq \varphi \in C_c^\infty(\mathbb{R} \times [0,T)),\ \forall k \in \mathbb{R},$$

$$\iint_{Q_T} |u-k|\,\varphi_t + q_k(u)\varphi_x + r_k(u)\varphi_{xx}$$

$$+\operatorname{sgn}(u-k)\mathcal{L}[u]\varphi\, dxdt$$

$$+\int_{\mathbb{R}} |u_0 - k|\,\varphi(0,x)\, dx \geq 0$$

$$q_k(u) = \operatorname{sgn}(u-k)(f(u) - f(k)) \qquad r_k(u) = |A(u) - A(k)|$$

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$$\boxed{+\operatorname{sgn}(u-k)\mathcal{L}[u]\varphi}\,dxdt$$

Finite since u is BV

$$+\int_{\mathbb{R}} |u_0 - k|\,\varphi(0,x)\,dx \ge 0$$

$$q_k(u) = \operatorname{sgn}(u-k)(f(u)-f(k)) \qquad r_k(u) = |A(u) - A(k)|$$

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Numerical approximation

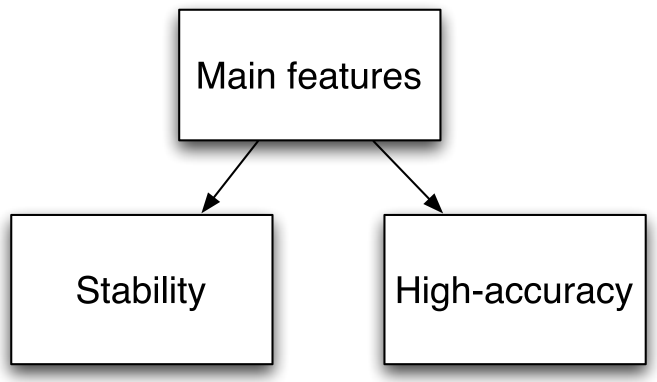
- Vast literature for nonlocal linear equations (finance applications)
- Nonlinear convection of compressible radiating fluids (Dedner-Rohde)
- Fractional conservation laws (Droniou)
  – convergence of monotone FV method
- Fractional conservation laws / convection-diffusion (Cifani-Jakobsen-K)
  – discontinuous Galerkin methods

Discontinuous Galerkin (DG) methods

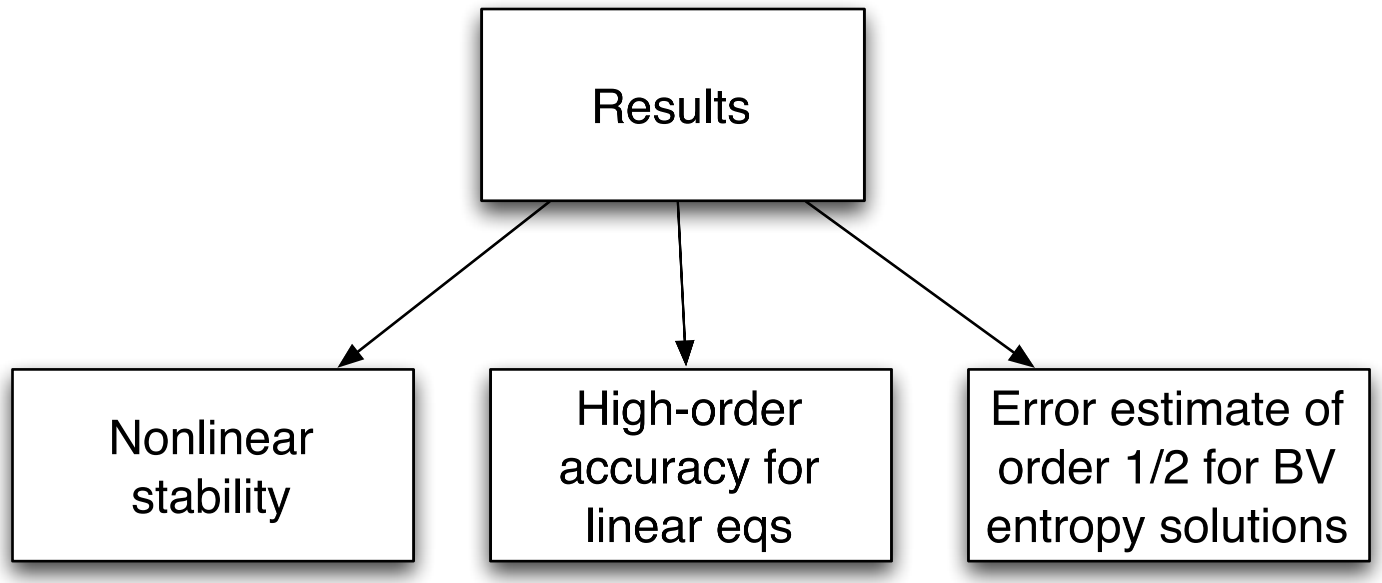
- Well established for the pure conservation law

  $$\partial_t u + \partial_x f(u) = 0$$

  (Cockburn-Shu + many other names)



- Aim is to extend to fractional CL / conv-diff



A semi-discrete DG method

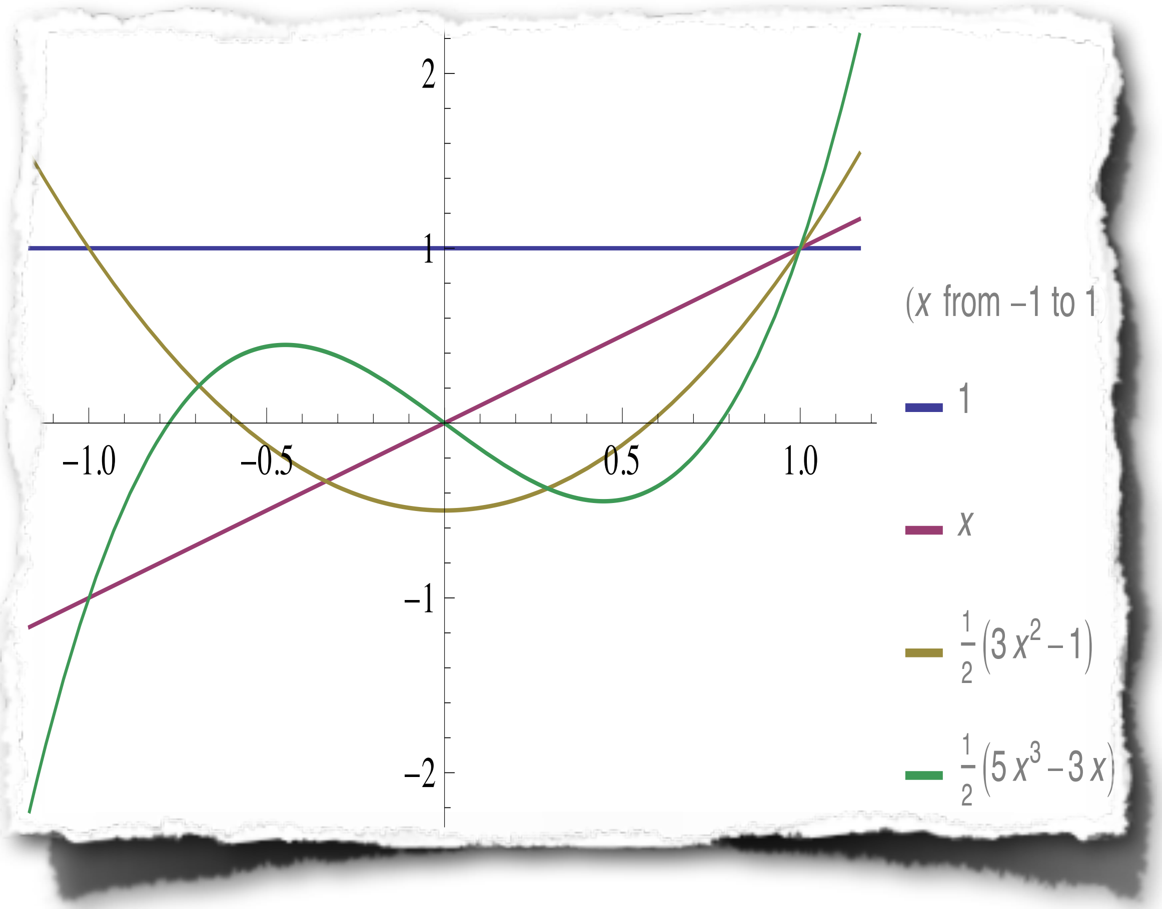
- Spatial grid $x_i = i\Delta x$, $i \in \mathbb{Z}$; $\quad I_i = (x_i, x_{i+1})$

- $P^k(I_i)$ polynomials of degree k

  Orthogonal basis - Legendre polynomials

  $\{\varphi_{0,i}, \varphi_{1,i}, \ldots, \varphi_{k,i}\}$,

  $\varphi_{j,i} \in P^j(I_i)$ $j = 0, \ldots, k$



- Multiply $u_t + f(u)_x = \mathcal{L}[u]$ by a $\varphi \in P^k(I_i)$,

  $\int$ over the interval I_i and integrate by parts

  Replace f by (consistent, monotone) numerical flux F

$$\int_{I_i} u_t \varphi - \int_{I_i} f(u)\varphi_x + F(u_{i+1})\varphi(x_{i+1}^-) - F(u_i)\varphi(x_i^+) = \int_{I_i} \mathcal{L}[u]\varphi.$$

- Determine $\tilde{u}(x,t) = \sum_{i\in\mathbb{Z}} \sum_{p=0}^{k} U_{p,i}(t)\varphi_{p,i}(x)$,

  such that equation holds $\forall \varphi \in P^k(I_i)$, $i \in \mathbb{Z}$.

- Mul...

$\int$ o...

Rep... flux F

$\int_{I_i} u$ ... (x_i^+)

(i) The Godunov flux:

$$\hat{f}^G(a,b) = \begin{cases} \min\limits_{a \leqslant u \leqslant b} f(u), & \text{if} \quad a \leqslant b \\ \max\limits_{b \leqslant u \leqslant a} f(u), & \text{otherwise} \end{cases}$$

(ii) The Engquist–Osher flux:

$$\hat{f}^{EO}(a,b) = \int_0^b \min(f'(s),0)\,ds + \int_0^a \max(f'(s),0)\,ds + f(0)$$

(iii) The Lax–Friedrichs flux:

$$\hat{f}^{LF}(a,b) = \tfrac{1}{2}\left[f(a) + f(b) - C(b-a)\right]$$

$$C = \max_{\inf u^0(x) \leqslant s \leqslant \sup u^0(x)} |f'(s)|$$

- Det...

such that equation holds $\forall \varphi \in P^k(I_i)$, $i \in \mathbb{Z}$.

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> Lemma: For $\varphi, \phi \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$.
>
> $$\int_{\mathbb{R}} \varphi \mathcal{L}[\phi] dx = \int_{\mathbb{R}} \mathcal{L}[\varphi] \phi dx$$

$$\int_{I_i} u_t \varphi - \int_{I_i} f(u)\varphi_x + F(u_{i+1})\varphi(x_{i+1}^-) - F(u_i)\varphi(x_i^+) = \int_{I_i} \mathcal{L}[u]\varphi.$$

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- Properties of Legendre polynomials $\Longrightarrow$

  $\forall q = 0, \dots, k$ and $i \in \mathbb{Z}$,

$$\frac{\Delta x}{2q+1} \frac{d}{dt} U_{q,i} = \int_{I_i} f(\tilde{u}) \frac{d}{dx} \varphi_{q,i} + (-1)^q F(\tilde{u}_i) - F(\tilde{u}_{i+1})$$

$$+ \int_{I_i} \mathcal{L}[\tilde{u}] \varphi_{q,i},$$

$$U_{q,i}(0) = \frac{2q+1}{\Delta x} \int_{I_i} u_0(x) \varphi_{q,i}(x) dx.$$

Time discretizations: Explicit, Runge–Kutta methods

Existence/uniqueness of solutions

in $C^1([0,T]; V^k \cap L^2(\mathbb{R}))$

by Picard-Cauchy-Lipschitz theorem.

- Properties of

$\forall q = 0, \ldots, k$

$$\frac{\Delta x}{2q+1}\frac{d}{dt}U_{q,i} = \int_{I_i} f(\tilde{u})\frac{d}{dx}\varphi_{q,i} + (-1)^q F(\tilde{u}_i) - F(\tilde{u}_{i+1}) + \int_{I_i} \mathcal{L}[\tilde{u}]\varphi_{q,i},$$

$$U_{q,i}(0) = \frac{2q+1}{\Delta x}\int_{I_i} u_0(x)\varphi_{q,i}(x)dx.$$

Time discretizations: Explicit, Runge–Kutta methods

Nonlinear stability

$V^k := \{u : u|_{I_i} \in P^k(I_i) \text{ for all } i \in \mathbb{Z}\}$ piecewise polynomials

$H^{\lambda/2}(\mathbb{R})$ fractional Sobolev space

$$\|u\|^2_{H^{\lambda/2}(\mathbb{R})} := \|u\|^2_{L^2(\mathbb{R})} + |u|^2_{H^{\lambda/2}(\mathbb{R})} < \infty$$

$$|u|^2_{H^{\lambda/2}(\mathbb{R})} := \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{[u(z) - u(x)]^2}{|z-x|^{1+\lambda}} dz dx.$$

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$$V^k \cap L^2(\mathbb{R}) \subseteq H^{\lambda/2}(\mathbb{R})$$

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$$|u|^2_{H^{\lambda/2}(\mathbb{R})} := \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{[u(z) - u(x)]^2}{|z - x|^{1+\lambda}} \, dz \, dx$$

If $\phi \in V^k \cap L^2(\mathbb{R})$, then for all $\lambda \in (0,1)$,

$$\|\phi\|^2_{H^{\frac{\lambda}{2}}(\mathbb{R})} \leq \frac{C}{\Delta x} \|\phi\|^2_{L^2(\mathbb{R})}.$$

$$V^k \cap L^2(\mathbb{R}) \subseteq H^{\lambda/2}(\mathbb{R})$$

Nonlinear stability

$V^k := \{u : u|_{I_i} \in P^k(I_i) \text{ for all } i \in \mathbb{Z}\}$ piecewise polynomials

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Theorem (Stability). $u_{\Delta x}$ numerical solution.

$$\|u_{\Delta x}(\cdot, T)\|^2_{L^2(\mathbb{R})} + c_\lambda \int_0^T |u_{\Delta x}(\cdot, t)|^2_{H^{\lambda/2}(\mathbb{R})} dt \leq \|u_0\|^2_{L^2(\mathbb{R})}.$$

Nonlinear stability

Proof.

Choose test function $\varphi = u_{\Delta x}(\cdot, t)$ in the DG scheme

Sum over $i \in \mathbb{Z}$, rearrange terms + integrate in time

$$\iint (u_{\Delta x})_t u_{\Delta x} = \int_0^T \sum_{i\in\mathbb{Z}} \Big[F(u_i)(u_{\Delta x}(x_i^+) - u_{\Delta x}(x_i^-)) - \int_{u_{\Delta x}(x_i^-)}^{u_{\Delta x}(x_i^+)} f(\xi)\,d\xi \Big] + \iint \mathcal{L}[\tilde{u}]\tilde{u}.$$

$$\boxed{F(u_i) = F(u_{\Delta x}(x_i^-), u_{\Delta x}(x_i^+))}$$

Nonlinear stability

Proof.

Choose test function $\varphi = u_{\Delta x}(\cdot, t)$ in the DG scheme

Sum over $i \in \mathbb{Z}$, rearrange terms + integrate in time

E-flux

$$F(u_i)(u_{\Delta x}(x_i^+) - u_{\Delta x}(x_i^-)) - \int_{u_{\Delta x}(x_i^-)}^{u_{\Delta x}(x_i^+)} f(\xi)d\xi \leq 0$$

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Nonlinear stability

Proof.

Choose test function $\varphi = u_{\Delta x}(\cdot, t)$ in the DG scheme

Sum over $i \in \mathbb{Z}$, rearrange terms + integrate in time

"Integration-by-parts"

$$\int_{\mathbb{R}} \varphi g[\varphi] dx = -\frac{c_\lambda}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\varphi(z) - \varphi(x))^2}{|z-x|^{1+\lambda}} dz dx,$$

for $\varphi \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$

$$+ \iint \mathcal{L}[\tilde{u}]\tilde{u}.$$

$$\boxed{F(u_i) = F(u_{\Delta x}(x_i^-), u_{\Delta x}(x_i^+))}$$

Higher-order accuracy (linear equations)

Exists a unique $H^{k+1}(Q_T)$ solution to the linear equation

$$\partial_t u + c\partial_x u = \mathcal{L}[u], \quad u(0,x) = u_0 \in H^k(\mathbb{R}), \quad (k \geq 0)$$

Theorem (error estimate). For any $T > 0$,

$$\|u(\cdot,T) - u_{\Delta x}(\cdot,T)\|_{L^2(\mathbb{R})} \leq c_{k,T}\Delta x^{k+\frac{1}{2}}.$$

Proof. • Error $e := u - u_{\Delta x} \in H^{\lambda/2}(\mathbb{R})$ obeys

$$B(e,\varphi) := \int_{\mathbb{R}} e_t \varphi + \sum_{i\in\mathbb{Z}} \left[F(e_i)(\varphi(x_i^-) - \varphi(x_i^+)) - \int_{I_i} ce\varphi_x \right] - \int_{\mathbb{R}} g[e]\varphi = 0$$

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- Let $\mathbf{u}$ be L^2-projection of u into V^k and set $\mathbf{e} := \mathbf{u} - u_{\Delta x} \in H^{\lambda/2}(\mathbb{R})$. Then

$$\int_0^T \int_{\mathbb{R}} \mathbf{e}_t \mathbf{e} = \int_0^T \int_{\mathbb{R}} (\mathbf{u} - u)_t \mathbf{e} - \int_0^T \sum_{i\in\mathbb{Z}} \Big[F(\mathbf{e}_i)(\mathbf{e}(x_i^-) - \mathbf{e}(x_i^+)) - \int_{I_i} c\mathbf{e}\mathbf{e}_x \Big]$$
$$+ \int_0^T \sum_{i\in\mathbb{Z}} \Big[F((\mathbf{u}-u)_i)(\mathbf{e}(x_i^-) - \mathbf{e}(x_i^+)) - \int_{I_i} c(\mathbf{u}-u)\mathbf{e}_x \Big]$$
$$+ \int_0^T \int_{\mathbb{R}} g[\mathbf{e}]\mathbf{e} - \int_0^T \int_{\mathbb{R}} g[\mathbf{e} - e]\mathbf{e}.$$

Proof. • Error $e := u - u_{\Delta x} \in H^{\lambda/2}(\mathbb{R})$ obeys

$$B(e,\varphi) := \int_{\mathbb{R}} e_t \varphi + \sum_{i\in\mathbb{Z}} \Big[F(e_i)(\varphi(x_i^-) - \varphi(x_i^+)) - \int_{I_i} c e \varphi_x \Big]$$
$$- \int_{\mathbb{R}} g[e]\varphi = 0$$

Bounded by by $c_{k,T}\Delta x^{2k+1}$

- Let $\mathbf{u}$ be L^2-projection of u into V^k and set $\mathbf{e} := \mathbf{u} - u_{\Delta x} \in H^{\lambda/2}(\mathbb{R})$. Then

$$\int_0^T \int_{\mathbb{R}} \mathbf{e}_t \mathbf{e} = \boxed{\int_0^T \int_{\mathbb{R}} (\mathbf{u} - u)_t \mathbf{e} - \int_0^T \sum_{i\in\mathbb{Z}} \Big[F(\mathbf{e}_i)(\mathbf{e}(x_i^-) - \mathbf{e}(x_i^+)) - \int_{I_i} c\mathbf{e}\mathbf{e}_x \Big] + \int_0^T \sum_{i\in\mathbb{Z}} \Big[F((\mathbf{u}-u)_i)(\mathbf{e}(x_i^-) - \mathbf{e}(x_i^+)) - \int_{I_i} c(\mathbf{u}-u)\mathbf{e}_x \Big]}$$
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- Let $\mathbf{u}$ be L^2-projection of u into V^k and set $\mathbf{e} := \mathbf{u} - u_{\Delta x} \in H^{\lambda/2}(\mathbb{R})$. Then

$$\int_0^T \int_{\mathbb{R}} \mathbf{e}_t \mathbf{e} = \int_0^T \int_{\mathbb{R}} (\mathbf{u} - u)_t \mathbf{e} - \int_0^T \sum_{i\in\mathbb{Z}} \Big[F(\mathbf{e}_i)(\mathbf{e}(x_i^-) - \mathbf{e}(x_i^+)) - \int_{I_i} c \mathbf{e}\mathbf{e}_x \Big]$$

$$+ \int_0^T \sum_{i\in\mathbb{Z}} \Big[F((\mathbf{u} - u)_i)(\mathbf{e}(x_i^-) - \mathbf{e}(x_i^+)) - \int_{I_i} c(\mathbf{u} - u)\mathbf{e}_x \Big]$$

$$+ \int_0^T \int_{\mathbb{R}} g[\mathbf{e}]\mathbf{e} - \int_0^T \int_{\mathbb{R}} g[\mathbf{e} - e]\mathbf{e}.$$

Remains to estimate this term

Proof. • Error $e := u - u_{\Delta x} \in H^{\lambda/2}(\mathbb{R})$ obeys

• $$\int_0^T \int_{\mathbb{R}} g[\mathbf{e}]\mathbf{e} - \int_0^T \int_{\mathbb{R}} g[\mathbf{e} - e]\mathbf{e}$$

$$= \frac{1}{2}\int_0^T \int_{\mathbb{R}} g[\mathbf{e}]\mathbf{e} + \frac{1}{2}\int_0^T \int_{\mathbb{R}} g[e]e - \frac{1}{2}\int_0^T \int_{\mathbb{R}} g[\mathbf{e} - e](\mathbf{e} - e)$$

$$\leq \int_0^T \|(u - \mathbf{u})(\cdot, t)\|^2_{H^{\lambda/2}(\mathbb{R})} dt$$

and moreover

$$\|(u - \mathbf{u})(\cdot, t)\|^2_{H^{\lambda/2}(\mathbb{R})} \leq c_k \|u(\cdot, t)\|^2_{H^{k+1}(\mathbb{R})} \Delta x^{2k+2-\lambda}.$$

$$\leq C_k \Delta x^{2k+1+\varepsilon}, \quad \varepsilon := 1 - \lambda \in (0, 1).$$

$$+ \int_0^T \int_{\mathbb{R}} g[\mathbf{e}]\mathbf{e} - \int_0^T \int_{\mathbb{R}} g[\mathbf{e} - e]\mathbf{e}.$$

Convergence / error estimate in nonlinear case

- Restrict to piecewise constant elements ($k = 0$):

$$\{\varphi_{0,i}, \varphi_{1,i}, \dots, \varphi_{k,i}\} = \{\varphi_{0,i}\}, \quad \varphi_{0,i} = \mathbf{1}_{I_i}$$

- Implicit-explicit method

$$\begin{cases} U_i^{n+1} = U_i^n - \Delta t D_- F(U_i^n, U_{i+1}^n) + \Delta t \mathcal{L}\langle U^{n+1}\rangle_i, \\ U_i^0 = \dfrac{1}{\Delta x}\displaystyle\int_{I_i} u_0(x)dx \end{cases}$$

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$$\begin{cases} U_i^{n+1} = U_i^n - \Delta t D_- F(U_i^n, U_{i+1}^n) + \Delta t \mathcal{L}\langle U^{n+1}\rangle_i, \\ U_i^0 = \dfrac{1}{\Delta x}\displaystyle\int_{I_i} u_0(x)dx \end{cases}$$

- Nonlocal operator

$$\mathcal{L}\langle U^n\rangle_i := \frac{1}{\Delta x}\int_{I_i} \mathcal{L}[U_{\Delta x}^n]dx = \frac{1}{\Delta x}\sum_{j\in\mathbb{Z}} G_j^i U_j^n,$$

$$G_j^i := \int_{I_i} \mathcal{L}[\mathbf{1}_{I_j}]dx$$

Convergence / error estimate in nonlinear case

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Lemma.

For all $(i,j) \in \mathbb{Z} \times \mathbb{Z}$,

$$\sum_{k\in\mathbb{Z}} |G_k^i| < \infty, \quad \sum_{k\in\mathbb{Z}} G_k^i = 0, \quad G_j^i = G_i^j, \quad G_{j+1}^{i+1} = G_j^i.$$

Moreover, $G_j^i \geq 0$ whenever $i \neq j$, while

$$G_i^i = -d_\lambda \Delta x^{1-\lambda}, \text{ where } d_\lambda := c_\lambda \left(\int_{|z|<1} \frac{dz}{|z|^\lambda} + \int_{|z|>1} \frac{dz}{|z|^{1+\lambda}} \right) > 0.$$

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Convergence / error estimate in nonlinear case

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$$G_j^i := \int_{I_i} \mathcal{L}[\mathbf{1}_{I_j}]dx$$

Lemma. Suppose $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$.

Let $u_{\Delta x}$ be piecewise constant numerical solution.

Then for all $t \geq 0$,

$$i)\ \|u_{\Delta x}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})},$$

$$ii)\ \|u_{\Delta x}(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})},$$

$$iii)\ |u_{\Delta x}(\cdot,t)|_{BV(\mathbb{R})} \leq |u_0|_{BV(\mathbb{R})}.$$

$$iv)\ \|\tilde{u}(\cdot,s) - \tilde{u}(\cdot,t)\|_{L^1(\mathbb{R})} \leq c(|s-t| + \Delta x).$$

Theorem (convergence/error estimate).

Suppose $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$.

Let u be BV entropy solution of fractional CL.

Let $u_{\Delta x}$ be numerical solution.

The there exists a constant $C_T > 0$ such that

$$\|u(\cdot,T) - u_{\Delta x}(\cdot,T)\|_{L^1(\mathbb{R})} \leq C_T \sqrt{\Delta x}.$$

Proof based on adaption of Kuznetsov's lemma

Lemma (Kuznetsov)

Let u be the exact BV entropy solution

Let $\tilde{u}$ be an approximate solution

For any $\epsilon > 0$, $0 < \delta < T$,

$$\|u(\cdot, T) - \tilde{u}(\cdot, T)\|_{L^1(\mathbb{R})} \leq c(\epsilon + \delta + \Delta x) - \Lambda_{\epsilon,\delta}[\tilde{u}, u].$$

Kruzkov form:

$$\Lambda[u, \varphi, k] := \iint |u - k| \, \varphi_t + q_k(u)\varphi_x + \operatorname{sgn}(u - k)\mathcal{L}[u]\varphi dx dt$$
$$+ \int_{\mathbb{R}} |u_0(x) - k| \, \varphi(x, 0) dx - \int_{\mathbb{R}} |u(x, T) - k| \, \varphi(x, T) dx$$

Proof based on adaption of Kuznetsov's lemma

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Let u be the exact BV entropy solution

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For any $\epsilon > 0$, $0 < \delta < T$

$$\|u(\cdot, T) - \tilde{u}(\cdot, T)\|_{L^1(\mathbb{R})} \leq c(\epsilon + \delta + \Delta x) - \Lambda_{\epsilon,\delta}[\tilde{u}, u]$$

$$q_k(u) = \operatorname{sgn}(u-k)(f(u) - f(k))$$

$$\varphi(x, y, t, s) = \omega_\epsilon(x-y)\omega_\delta(t-s)$$

$\omega_\alpha \in C_c^\infty(\mathbb{R})$, $\alpha > 0$, is an approximate delta function.

Kruzkov form:

$$\Lambda[u, \varphi, k] := \iint |u-k|\,\varphi_t + q_k(u)\varphi_x + \operatorname{sgn}(u-k)\mathcal{L}[u]\varphi\, dx dt$$
$$+ \int_{\mathbb{R}} |u_0(x) - k|\,\varphi(x, 0) dx - \int_{\mathbb{R}} |u(x, T) - k|\,\varphi(x, T) dx$$

Convection / diffusion / fractional diffusion

- DG for diffusion equation $u_t - u_{xx} = 0$

$$\int_{I_j} u_t v + \int_{I_j} u_x v_x dx - \widehat{(u_x)}_{j+1/2} v^-_{j+1/2} + \widehat{(u_x)}_{j-1/2} v^+_{j-1/2} = 0,$$

u, v piecewise polynomials

with respect to cells $I_j = (x_{j-1/2}, x_{j+1/2})$

Convection / diffusion / fractional diffusion

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Numerical fluxes

Convection / diffusion / fractional diffusion

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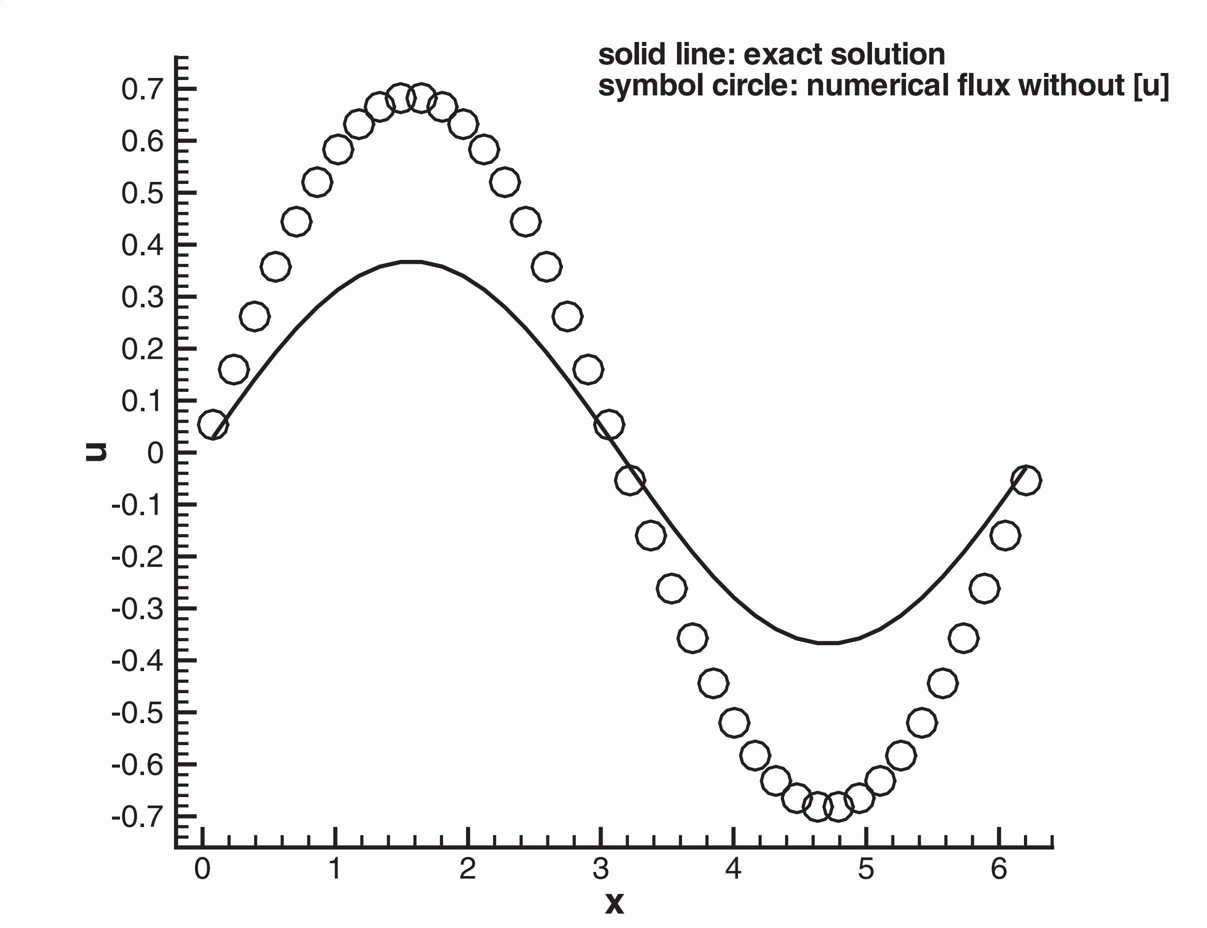
with respect to cells $I_j = (x_{j-1/2}, x_{j+1/2})$

Naive choice does not work
(not consistent):

$$\widehat{(u_x)}_{j+1/2} = ((u_x)^-_{j+1/2} + (u_x)^+_{j+1/2})/2$$

Conve... diffusion



Liu-Yan. Subtle inconsistency (Shu)

- DG f... $= 0$

$$\int_{I_j} u_t v \ldots \widehat{(u_x)}_{j-1/2} v^+_{j-1/2} = 0,$$

u, v p...

with ... $_{j+1/2})$

Naive choice does not work
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- Local DG method (LDG) of Cockburn & Shu

Rewrite heat equation $u_t - u_{xx} = 0$ as 1st order system

$$u_t - q_x = 0, \qquad q - u_x = 0.$$

Apply DG method to system.

So both solution u and flux q are evolved in each cell !

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Based on the standard weak formulation

+ convenient choice of numerical flux.

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Apply DG method to system.

> Piecewise constant approximation ($k = 0$):
>
> $$\widehat{u_x} = \frac{u_+ - u_-}{\Delta x}$$
>
> yields standard central differencing.

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$$u_t - q_x = 0, \quad q - u_x = 0.$$

Apply DG method to system.

Piecewise linear approximation ($k = 1$):

$$\widehat{u_x} = \frac{u_+ - u_-}{\Delta x} + \frac{1}{2}\left((u_x)^+ + (u_x)^-\right).$$

yields a second order approximation.

So both solution u and flux q are evolved in each cell !

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Based on the standard weak formulation

+ convenient choice of numerical flux.

$$\int_{I_j} u_t v + \int_{I_j} u_x v_x dx - \widehat{(u_x)}_{j+1/2} v^-_{j+1/2} + \widehat{(u_x)}_{j-1/2} v^+_{j-1/2} = 0$$

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Apply DG method to system

So b... ell !

- Direc...

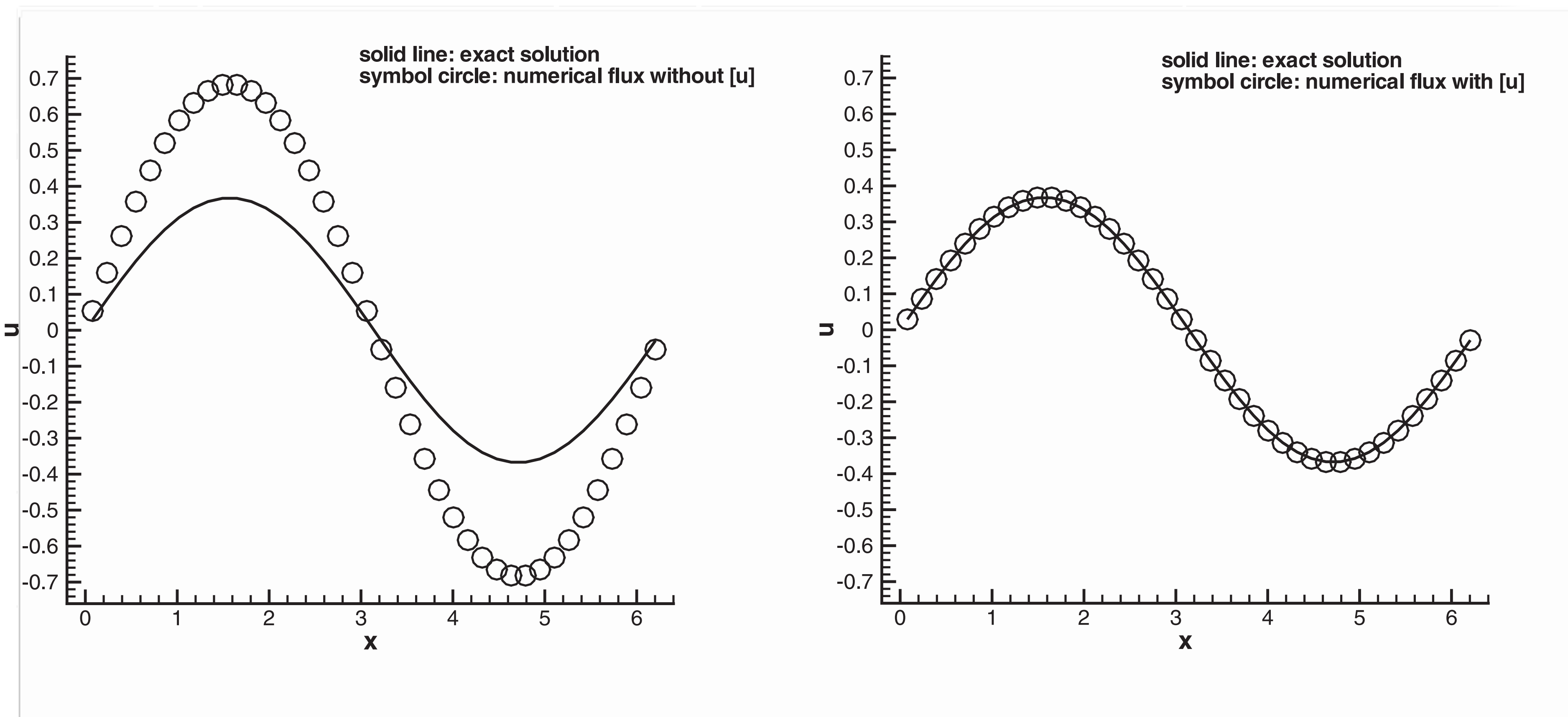
Base...

+ co...

$$\int_{I_j} u \ldots \int_{I_j} \ldots 0$$

Piecewise linear approximation ($k = 1$):

$$\widehat{u_x} = \frac{u_+ - u_-}{\Delta x} + \frac{1}{2}\left((u_x)^+ + (u_x)^-\right).$$



(Liu-Yan)

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Solution of heat equation

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4t)} g(y) dy$$

with data g having a discontinuity at $x = 0$.

Formula for gradient of solution to heat equation: (Liu-Yan)

$$u_x(0,t) = \sum \frac{2^{m-1}}{(2m-1)!!} t^m \left[\partial_x^{2m} g\right] / \sqrt{\pi t} + \sum \frac{2^m}{(2m)!!} t^m \overline{\partial_x^{2m+1} g}$$
$$= \frac{1}{\sqrt{4\pi t}}[g] + \overline{\partial_x g} + \sqrt{\frac{t}{\pi}} \left[\partial_x^2 g\right] + t\overline{\partial_x^3 g} + \cdots,$$

$$\int_{I_j} u_t v + \int_{I_j} u_x v_x dx - \widehat{(u_x)}_{j+1/2} v_{j+1/2}^- + \widehat{(u_x)}_{j-1/2} v_{j-1/2}^+ = 0$$

Solution of heat equation

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4t)} g(y)\,dy$$

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$$= \frac{1}{\sqrt{4\pi t}} [g] + \overline{\partial_x g} + \sqrt{\frac{t}{\pi}} \left[\partial_x^2 g\right] + t\overline{\partial_x^3 g} + \cdots ,$$

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Jump

Solution of heat equation

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4t)} g(y)dy$$

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$$= \frac{1}{\sqrt{4\pi t}} [g] + \boxed{\overline{\partial_x g}} + \sqrt{\frac{t}{\pi}} \left[\partial_x^2 g\right] + t\overline{\partial_x^3 g} + \cdots,$$

Average

Local Discontinuous Galerkin (LDG) method

$$\partial_t u + \partial_x f(u) = \partial_x (a(u)\partial_x u) + c_\lambda \int_{|z|>0} \frac{u(t,x+z) - u(t,x)}{|z|^{1+\lambda}} dz$$

Rewrite as a system

$$u_t + (f(u) - \sqrt{a(u)}q)_x = b\mathcal{L}[u]$$

$$q - g(u)_x = 0, \qquad g = \int \sqrt{a}$$

Apply DG method

Local Discontinuous Galerkin (LDG) method

$$\partial_t u + \partial_x f(u) = \partial_x(a(u)\partial_x u) + c_\lambda \int_{|z|>0} \frac{u(t,x+z)-u(t,x)}{|z|^{1+\lambda}}\, dz$$

$$\mathbf{h}(\mathbf{w}) = \mathbf{h}(u,q) = \begin{pmatrix} h_u(\mathbf{w}) \\ h_q(u) \end{pmatrix} = \begin{pmatrix} f(u) - \sqrt{a(u)}q \\ -g(u) \end{pmatrix}.$$

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$$\partial_t u + \partial_x f(u) = \partial_x(a(u)\partial_x u) + c_\lambda \int_{|z|>0} \frac{u(t,x+z)-u(t,x)}{|z|^{1+\lambda}}\, dz$$

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Rewrite as a system

$$u_t + (f(u) - \sqrt{a(u)}q)_x = b\mathcal{L}[u]$$

$$q - g(u)_x = 0, \qquad g = \int \sqrt{a}$$

Variational form For all $v_u, v_q \in P^k(I_i)$

$$\int_{I_i} \partial_t u v_u - \int_{I_i} h_u(\mathbf{w})\partial_x v_u + h_u(\mathbf{w}_{i+1})v^-_{u,i+1} - h_u(\mathbf{w}_i)v^+_{u,i} = \int_{I_i} \mathcal{L}[u]v_u,$$

$$\int_{I_i} q v_q - \int_{I_i} h_q(u)\partial_x v_q + h_q(u_{i+1})v^-_{q,i+1} - h_q(u_i)v^+_{q,i} = 0,$$

Apply DG method

Local Discontinuous Galerkin (LDG) method

$$\partial_t u + \partial_x f(u) = \partial_x(a(u)\partial_x u) + c_\lambda \int_{|z|>0} \frac{u(t,x+z) - u(t,x)}{|z|^{1+\lambda}} dz$$

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Rewrite as a system

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Variational form   For all $v_u, v_q \in P^k(I_i)$

$$\int_{I_i} \partial_t u v_u - \int_{I_i} h_u(\mathbf{w})\partial_x v_u + \boxed{h_u(\mathbf{w}_{i+1})} v^-_{u,i+1} - \boxed{h_u(\mathbf{w}_i)} v^+_{u,i} = \int_{I_i} \mathcal{L}[u] v_u,$$

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Apply DG method

Replace h_u, h_q by numerical fluxes ala Cockburn & Shu.

Local Discontinuous Galerkin (LDG) method

$$\partial_t u + \partial_x f(u) = \partial_x(a(u)\partial_x u) + c_\lambda \int_{|z|>0} \frac{u(t,x+z)-u(t,x)}{|z|^{1+\lambda}}\,dz$$

$$\mathbf{h}(\mathbf{w}) = \mathbf{h}(u,q) = \begin{pmatrix} h_u(\mathbf{w}) \\ h_q(u) \end{pmatrix} = \begin{pmatrix} f(u) - \sqrt{a(u)}q \\ -g(u) \end{pmatrix}.$$

Local DG method

$$\int_{I_i} \partial_t u v_u - \int_{I_i} h_u(\mathbf{w})\partial_x v_u + \hat{h}_u(\mathbf{w}_{i+1})v^-_{u,i+1} - \hat{h}_u(\mathbf{w}_i)v^+_{u,i} = b\int_{I_i} \mathcal{L}[u]v_u,$$

$$\int_{I_i} q v_q - \int_{I_i} h_q(u)\partial_x v_q + \hat{h}_q(u_{i+1})v^-_{q,i+1} - \hat{h}_q(u_i)v^+_{q,i} = 0,$$

for all $v_u, v_q \in P^k(I_i)$, $i \in \mathbb{Z}$

Theorem (nonlinear stability).

Let $\tilde{\mathbf{w}} = (\tilde{u}, \tilde{q})$ be a solution of LDG method. Then

$$\|\tilde{u}(\cdot, T)\|^2_{L^2(\mathbb{R})} + 2\|\tilde{q}\|^2_{L^2(Q_T)} + 2\Theta_T(\tilde{\mathbf{w}}) + c_\lambda \int_0^T |\tilde{u}(\cdot, t)|^2_{H^{\lambda/2}(\mathbb{R})}\, dt \leq \|u_0\|^2_{L^2(\mathbb{R})},$$

where

$$\Theta_T[\mathbf{w}] = \int_0^T \sum_{i\in\mathbb{Z}} [\mathbf{w}_i]' \mathbb{C} [\mathbf{w}_i] \; (\geq 0).$$

(the matrix $\mathbb{C}$ is semipositive definite)

Theorem (error estimate linear eqs).

Let $\tilde{\mathbf{w}} = (\tilde{u}, \tilde{q})$ be a solution of LDG method.

With $e_u = u - \tilde{u}$ and $e_q = q - \tilde{q}$,

$$\int_{\mathbb{R}} e_u^2(x,T) + \int_0^T \int_{\mathbb{R}} e_q^2 + \Theta_T[\mathbf{e}] + c_\lambda \int_0^T |e_u|^2_{H^{\lambda/2}(\mathbb{R})} = \mathcal{O}(1)\Delta x^{2k}.$$

NB! Error estimate is optimal without fractional diffusion.

Direct Discontinuous Galerkin (LDG) method

$$\partial_t u + \partial_x f(u) = \partial_x(a(u)\partial_x u) + c_\lambda \int_{|z|>0} \frac{u(t,x+z) - u(t,x)}{|z|^{1+\lambda}} dz$$

For an arbitrary $v \in P^k(I_i)$,

$$\int_{I_i} u_t v \; - \int_{I_i} f(u)v_x + f(u_{i+1})v_{i+1}^- - f(u_i)v_i^+$$

$$+ \int_{I_i} a(u)u_x v_x - h(u_{i+1}, u_{x,i+1})v_{i+1}^- + h(u_i, u_{x,i})v_i^+$$

$$= \int_{I_i} \mathcal{L}[u]v, \qquad (h(u,u_x) = a(u)u_x)$$

Direct Discontinuous Galerkin (LDG) method

$$\partial_t u + \partial_x f(u) = \partial_x(a(u)\partial_x u) + c_\lambda \int_{|z|>0} \frac{u(t,x+z) - u(t,x)}{|z|^{1+\lambda}} dz$$

For an arbitrary $v \in P^k(I_i)$,

Numerical fluxes

$$\int_{I_i} u_t v \; - \int_{I_i} f(u) v_x + \boxed{f(u_{i+1})} v_{i+1}^- - \boxed{f(u_i)} v_i^+$$

$$+ \int_{I_i} a(u) u_x v_x - \boxed{h(u_{i+1}, u_{x,i+1})} v_{i+1}^- + \boxed{h(u_i, u_{x,i})} v_i^+$$

Numerical fluxes

$$= \int_{I_i} \mathcal{L}[u] v, \qquad (h(u, u_x) = a(u) u_x)$$

Direct Discontinuous Galerkin (LDG) method

$$\partial_t u + \partial_x f(u) = \partial_x(a(u)\partial_x u) + c_\lambda \int_{|z|>0} \frac{u(t,x+z) - u(t,x)}{|z|^{1+\lambda}} dz$$

For convection, choose any consistent and monotone flux

$$\hat{f}(u_i) = \hat{f}(u(x_i^-), u(x_i^+)).$$

For an arbitrary $v \in P^k(I_i)$,

Numerical fluxes

$$\int_{I_i} u_t v - \int_{I_i} f(u)v_x + \boxed{f(u_{i+1})} v_{i+1}^- - \boxed{f(u_i)} v_i^+$$

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Numerical fluxes

$$= \int_{I_i} \mathcal{L}[u]v, \qquad (h(u,u_x) = a(u)u_x)$$

Direct Discontinuous Galerkin (LDG) method

$$\partial_t u + \partial_x f(u) = \partial_x(a(u)\partial_x u) + c_\lambda \int_{|z|>0} \frac{u(t,x+z) - u(t,x)}{|z|^{1+\lambda}} dz$$

For an arbitrary $v \in P^k(I_i)$,

Numerical fluxes

$$\int_{I_i} u_t v \; - \int_{I_i} f(u) v_x + \boxed{f(u_{i+1})} v_{i+1}^- - \boxed{f(u_i)} v_i^+$$

$$+ \int_{I_i} a(u) u_x v_x - \boxed{h(u_{i+1}, u_{x,i+1})} v_{i+1}^- + \boxed{h(u_i, u_{x,i})} v_i^+$$

Numerical fluxes

$$= \int_{I_i} \mathcal{L}[u] v, \qquad (h(u, u_x) = a(u) u_x)$$

For diffusion flux $h = a(u)u_x = A(u)_x$, we follow Liu-Yan

$$\hat{h}(u_i) = \hat{h}(u(x_i^-), \ldots, \partial_x^k u(x_i^-), u(x_i^+), \ldots, \partial_x^k u(x_i^+))$$

$$= \beta_0 \frac{[A(u_i)]}{\Delta x} + \overline{A(u_i)_x} + \sum_{m=1}^{\lfloor k/2 \rfloor} \beta_m \Delta x^{2m-1} [\partial_x^{2m} A(u_i)],$$

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$\{\beta_0, \dots, \beta_{\lfloor k/2 \rfloor}\}$ satisfy for some $\gamma \in (0,1)$ and $\alpha \geq 0$

$$\sum_{i \in \mathbb{Z}} \hat{h}(u_i)[u_i] \geq \alpha \sum_{i \in \mathbb{Z}} \frac{[A(u_i)]}{\Delta x}[u_i] - \gamma \sum_{i \in \mathbb{Z}} \int_{I_i} a(u)(u_x)^2.$$

For diffusion flux $h = a(u)u_x = A(u)_x$, we follow Liu-Yan

$$\hat{h}(u_i) = \hat{h}(u(x_i^-), \ldots, \partial_x^k u(x_i^-), u(x_i^+), \ldots, \partial_x^k u(x_i^+))$$

$$= \beta_0 \frac{[A(u_i)]}{\Delta x} + A(u_i)_x + \sum_{m=1}^{\lfloor k/2 \rfloor} \beta_m \Delta x^{2m-1} [\partial_x^{2m} A(u_i)],$$

For example, if $k = 0$ and $\beta_0 = 1$,

$$\hat{h}(u_i) = \frac{1}{\Delta x}[A(u_i)] = \frac{A(u(x_i^+)) - A(u(x_i^-))}{\Delta x}.$$

$\{\beta_0, \ldots, \beta_{\lfloor k/2 \rfloor}\}$ satisfy for some $\gamma \in (0,1)$ and $\alpha \geq 0$

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With $k = 2$,

$$\sum_{m=1}^{\lfloor k/2 \rfloor} \beta_m \Delta x^{2m-1} [\partial_x^{2m} A(u_i)]$$

$$= \beta_1 \Delta x [\partial_x^2 A(u_i)]$$

$$= \beta_1 \Delta x [a'(u_i)(\partial_x u_i)^2 + a(u_i)\partial_x^2 u_i].$$

Theorem (nonlinear stability).

Let $\hat{u}$ be DDG solution. Then

$$\|\hat{u}(\cdot,T)\|_{L^2(\mathbb{R})}^2 + 2\Gamma_T[\hat{u}] + c_\lambda \int_0^T |\hat{u}(\cdot,t)|_{H^{\lambda/2}(\mathbb{R})}^2 \, dt \leq \|u_0\|_{L^2}^2$$

$$\Gamma_T[u] = (1-\gamma) \int_0^T \sum_{i\in\mathbb{Z}} \int_{I_i} a(u)(u_x)^2 + \alpha \int_0^T \sum_{i\in\mathbb{Z}} \frac{[A(u_i)]}{\Delta x}[u_i].$$

Theorem (error estimate for linear eqs).

Let $u \in H^{k+1}(Q_T)$ be a solution of IPDE.

Let $\hat{u}$ be a DDG solution.

With $e = u - \hat{u}$,

$$\int_{\mathbb{R}} e^2(x,T) + \frac{|c|}{2}\int_0^T \sum_{i\in\mathbb{Z}} [e_i]^2 + (1-\gamma)\int_0^T \int_{\mathbb{R}} (e_x)^2$$

$$+\alpha \int_0^T \sum_{i\in\mathbb{Z}} \frac{[e_i]^2}{\Delta x}$$

$$+c_\lambda \int_0^T |e|^2_{H^{\lambda/2}(\mathbb{R})} = \mathcal{O}(1)\Delta x^{2k}.$$

Convergence of DDG method in nonlinear case

$$\partial_t u + \partial_x f(u) = \partial_x(a(u)\partial_x u) + c_\lambda \int_{|z|>0} \frac{u(t,x+z) - u(t,x)}{|z|^{1+\lambda}} dz$$

Piecewise constant elements ($k = 0$):

$$\Delta x \frac{d}{dt} U_i + \hat{f}(U_i, U_{i+1}) - \hat{f}(U_{i-1}, U_i)$$

$$- \frac{[A(U_{i+1})]}{\Delta x} + \frac{[A(U_i)]}{\Delta x} = \sum_{j \in \mathbb{Z}} U_j \int_{I_i} \mathcal{L}[\mathbf{1}_{I_j}].$$

Convergence of DDG method in nonlinear case

$$\partial_t u + \partial_x f(u) = \partial_x(a(u)\partial_x u) + c_\lambda \int_{|z|>0} \frac{u(t,x+z) - u(t,x)}{|z|^{1+\lambda}} dz$$

Explicit method

$$\begin{cases} \dfrac{U_i^{n+1} - U_i^n}{\Delta t} + D_-\Big[\hat{f}(U_i^n, U_{i+1}^n) - D_+ A(U_i^n)\Big] = \mathcal{L}\langle U^n \rangle_i, \\ U_i^0 = \dfrac{1}{\Delta x} \displaystyle\int_{I_i} u_0(x)\, dx. \end{cases}$$

Lemma (a priori estimates).

$$i)\ \ \|U^n\|_{L^1(\mathbb{Z})} \leq \|u_0\|_{L^1(\mathbb{R})},$$

$$ii)\ \ \|U^n\|_{L^\infty(\mathbb{Z})} \leq \|u_0\|_{L^\infty(\mathbb{R})},$$

$$iii)\ \ |U^n|_{BV(\mathbb{Z})} \leq |u_0|_{BV(\mathbb{R})}.$$

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$$iii)\ |U^n|_{BV(\mathbb{Z})} \leq |u_0|_{BV(\mathbb{R})}.$$

Moreover,

$$\left\| \hat{f}(U_i^n, U_{i+1}^n) \; - \; D_+A(U_i^n) - \sum_{k=-\infty}^{i} \sum_{j\in\mathbb{Z}} G_j^k U_j^n \right\|_{L^\infty(\mathbb{Z})}$$

$$\leq \left\| \hat{f}(U_i^0, U_{i+1}^0) - D_+A(U_i^0) - \sum_{k=-\infty}^{i} \sum_{j\in\mathbb{Z}} G_j^k U_j^0 \right\|_{L^\infty(\mathbb{Z})},$$

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$$\leq \left\|\hat{f}(U_i^0, U_{i+1}^0) - D_+A(U_i^0) - \sum_{k=-\infty}^{i} \sum_{j\in\mathbb{Z}} G_j^k U_j^0 \right\|_{BV(\mathbb{Z})},$$

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Consequently,

$$\sum_{i\in\mathbb{Z}} |U_i^m - U_i^n| \le C\frac{\Delta t}{\Delta x}|m-n|.$$

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Consequently,

$$\sum_{i\in\mathbb{Z}} |U_i^m - U_i^n| \leq C\frac{\Delta t}{\Delta x}|m-n|.$$

This implies strong compactness / convergence of DDG solution $u_{\Delta x}$.

Lemma (diffusion term)

There holds

$$|A(U_i^m) - A(U_j^n)| = \mathcal{O}(1)\left[|i-j|\Delta x + \sqrt{|m-n|\Delta t}\right].$$

Consequently, the limit obeys $A(u) \in C^{1/2,1}(Q_T)$.

Lemma (cell entropy inequality)

$$\eta_i^{n+1} - \eta_i^n + \Delta t D_- Q_i^n$$
$$- \Delta t D_- D_+ |A(U_i^n) - A(k)| \leq \Delta t \eta_k'(U_i^{n+1}) \mathcal{L}\langle U^n \rangle_i,$$

$$Q_i^n = \hat{f}(U_i^n \vee k, U_{i+1}^n \vee k) - \hat{f}(U_i^n \wedge k, U_{i+1}^n \wedge k)$$

Theorem (convergence).

Suppose $u_0 \in L^1 \cap BV$ is s.t. $|f(u_0) - \partial_x A(u_0)|_{BV} < \infty$.

Let $\hat{u}_{\Delta x}$ be explicit DGG solution.

Then $\{\hat{u}_{\Delta x}\}_{\Delta x > 0}$ converges in L^1_{loc}

to the BV entropy solution of the IPDE.

Numerical examples

- Implemented in the cases $k = 0, 1, 2$.
- Set our numerical solutions to zero outside the region

  $\Omega = \{(x,t) : |x| \leq 3/2, t \geq 0\}$.

Example. Pure fractional equation $\partial_t u = \mathcal{L}[u]$

$$\mathcal{L}[\varphi(x)] = c_\lambda \int_{|z|>0} \frac{\varphi(x+z) - \varphi(x)}{|z|^{1+\lambda}} dz$$

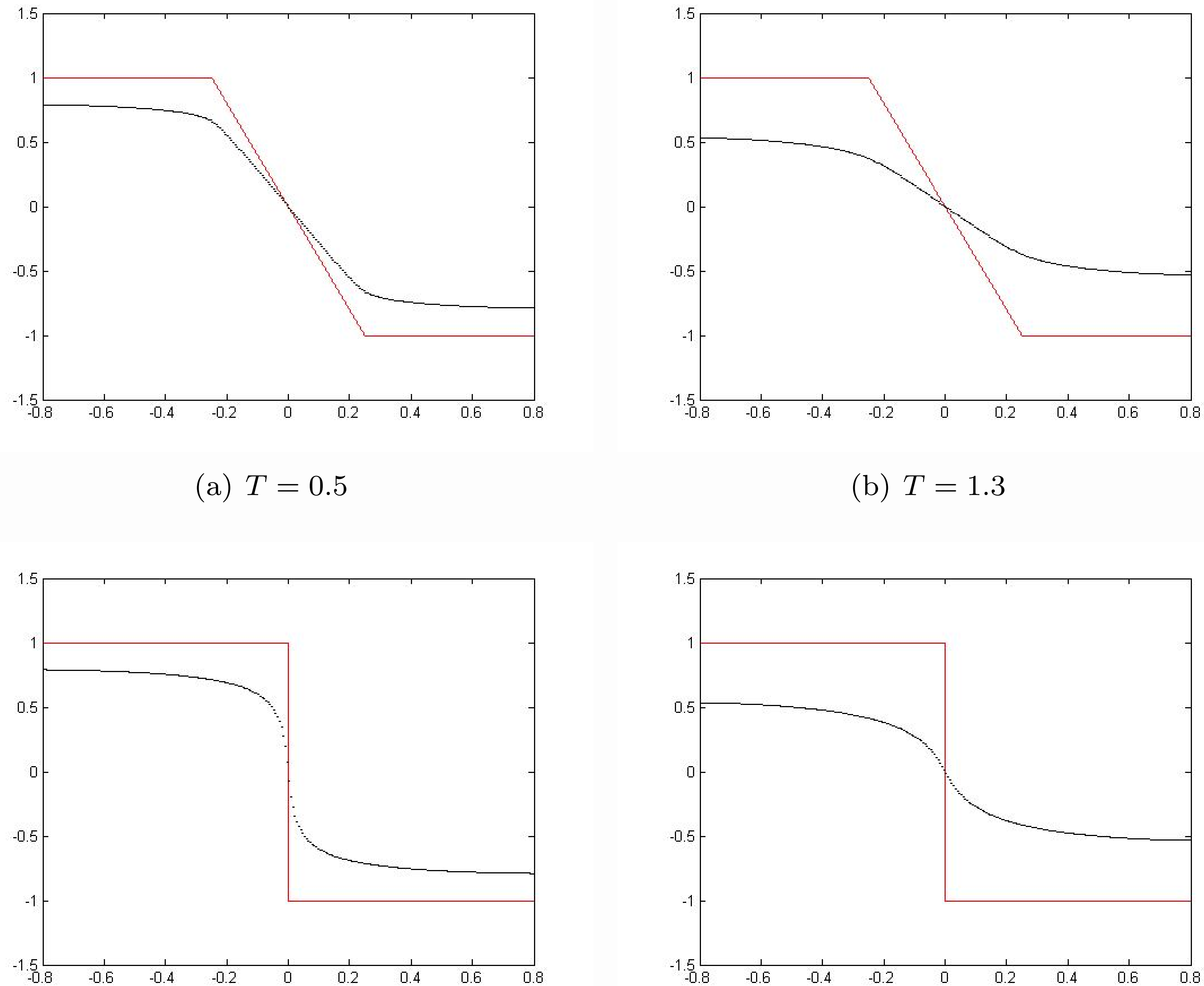
Numeric

- Imple
- Set c ... region

$\Omega =$

Examp

dz



(a) $T = 0.5$ (b) $T = 1.3$

(c) $T = 0.5$ (d) $T = 1.3$

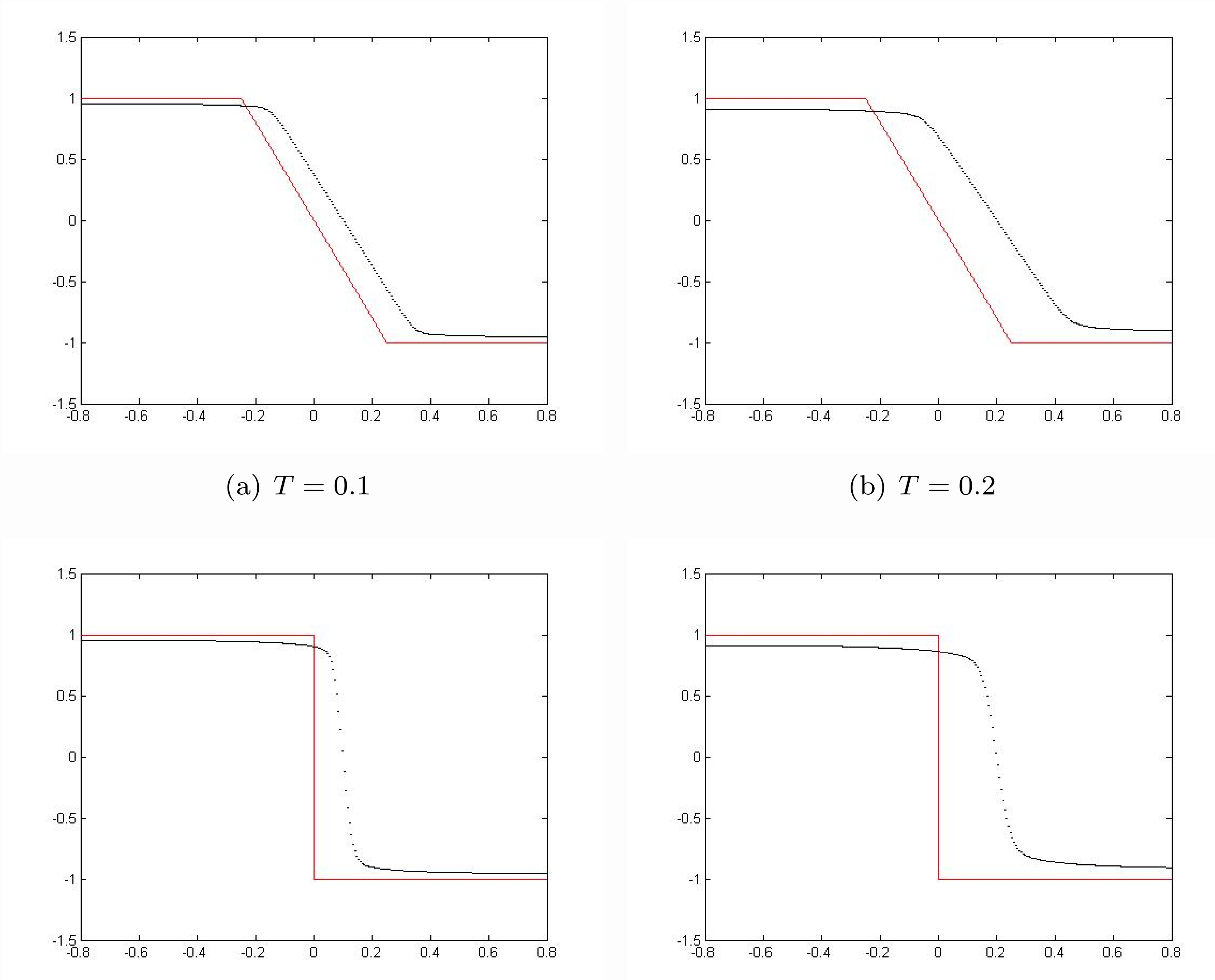
FIGURE 1. Initial data (piecewise linear) and solutions of the pure fractional equation ($\lambda = 0.5$) with $k = 0$ and $\Delta x = 1/160$.

Example. Fractional transport equation

$$\partial_t u + \partial_x u = c_\lambda \int_{|z|>0} \frac{u(t, x+z) - u(t, x)}{|z|^{1+\lambda}} dz$$

Solution is smooth.

Exam



(a) $T = 0.1$ (b) $T = 0.2$

(c) $T = 0.1$ (d) $T = 0.2$

FIGURE 2. Initial data (piecewise linear) and solutions of the fractional transport equation ($\lambda = 0.5$) with $k = 0$ and $\Delta x = 1/160$.

Example. Fractional Burgers' equation

$$\partial_t u + u\partial_x u = c_\lambda \int_{|z|>0} \frac{u(t,x+z)-u(t,x)}{|z|^{1+\lambda}} dz$$

Solution not necessarily smooth.

Accuracy improves with $k = 0, 1, 2$.

A third order Runge-Kutta (RK3) time discretization and slope limiters

Lax-Friedrichs flux $F(a,b) = \frac{1}{2}[f(a) + f(b) - c(b-a)]$,

$$c = \max\{|f'(a)| : |a| \leq \|u_0\|_{L^\infty(\mathbb{R})}\}.$$

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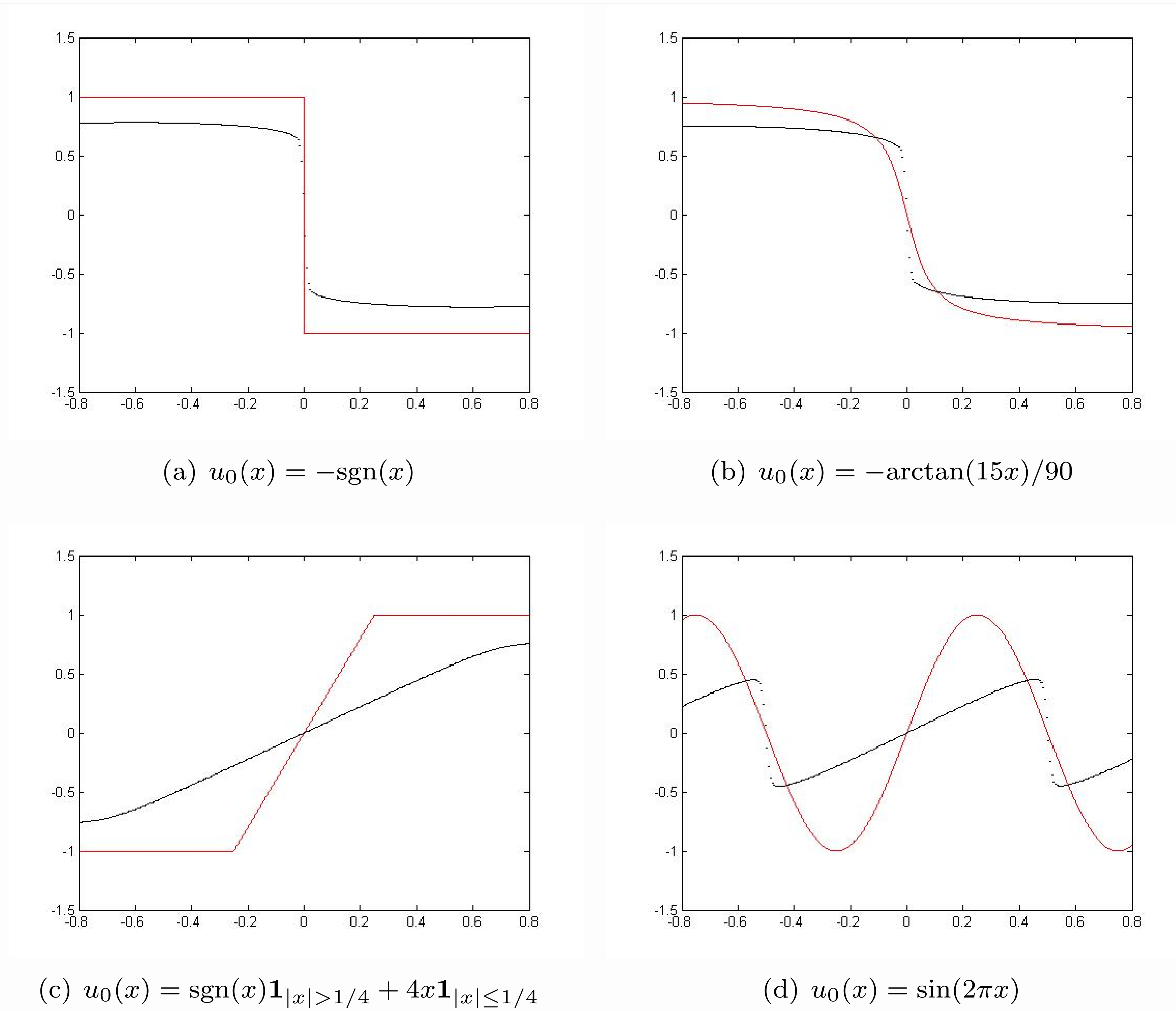

(a) $u_0(x) = -\mathrm{sgn}(x)$ (b) $u_0(x) = -\arctan(15x)/90$

(c) $u_0(x) = \mathrm{sgn}(x)\mathbf{1}_{|x|>1/4} + 4x\mathbf{1}_{|x|\le 1/4}$ (d) $u_0(x) = \sin(2\pi x)$

FIGURE 3. Initial data and solutions of the fractional Burgers' equation ($\lambda = 0.5$) using $k = 0$; $T = 0.5$ and $\Delta x = 1/160$.

Example. Fractional Burgers' equation

$$\partial_t u + u\partial_x u = c_\lambda \int_{|z|>0} \frac{u(t,x+z) - u(t,x)}{|z|^{1+\lambda}} dz$$

Solution not necessarily smooth.

Accuracy improves with $k = 0, 1, 2$.

A third order Runge-Kutta (RK3) time discretization and slope limiters

Lax-Friedrichs flux $\quad F(a,b) = \frac{1}{2}[f(a) + f(b) - c(b-a)],$

$$c = \max\{|f'(a)| : |a| \leq \|u_0\|_{L^\infty(\mathbb{R})}\}.$$

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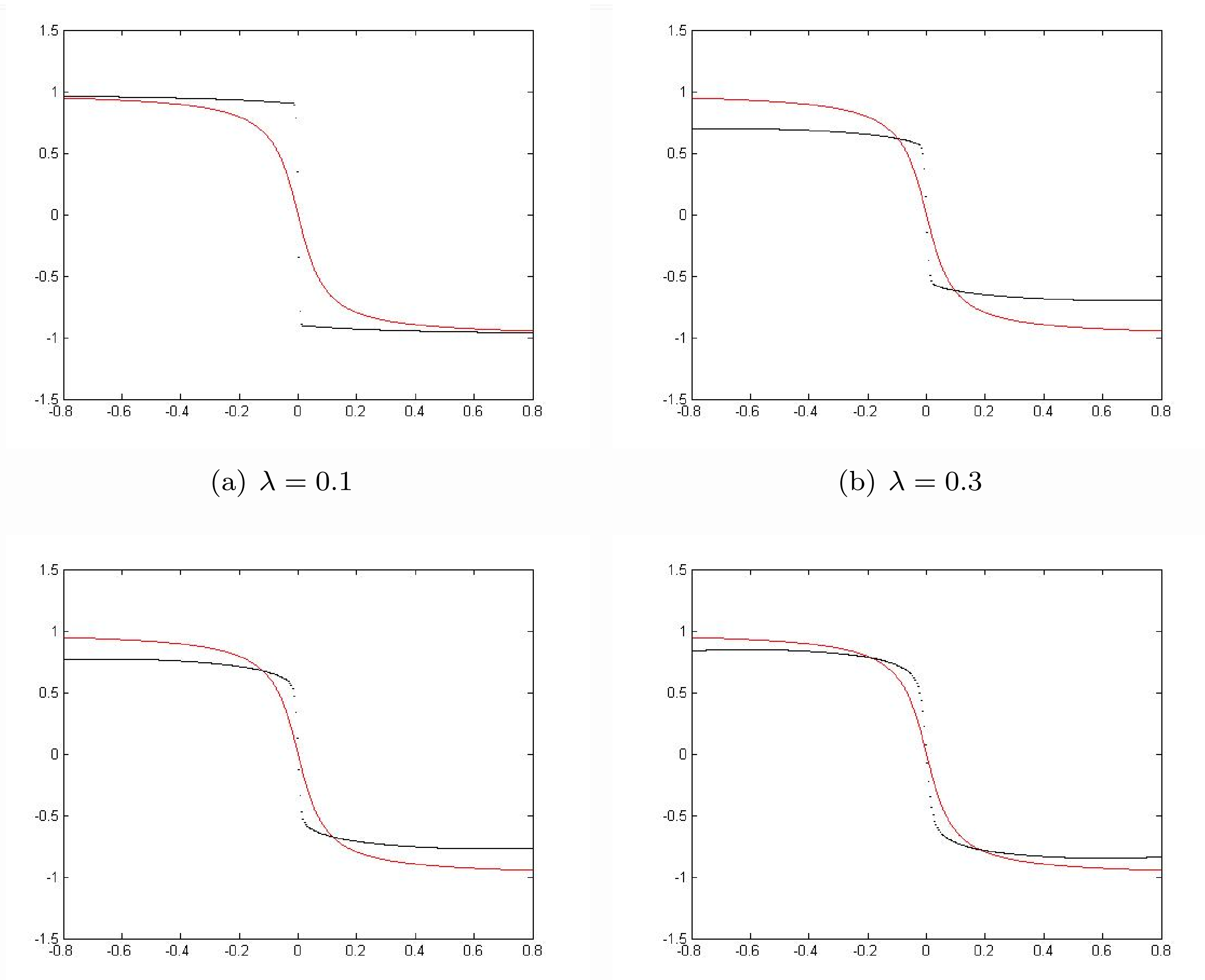

(a) $\lambda = 0.1$ (b) $\lambda = 0.3$

(c) $\lambda = 0.7$ (d) $\lambda = 0.99$

FIGURE 4. Initial data and solutions of the fractional Burgers' equation for different values of λ using $k = 0$; $T = 0.5$, $\Delta x = 1/200$, and $u_0(x) = -\arctan(15x)/90$.

Example. Fractional Burgers' equation

$$\partial_t u + u\partial_x u = c_\lambda \int_{|z|>0} \frac{u(t,x+z)-u(t,x)}{|z|^{1+\lambda}} dz$$

Solution not necessarily smooth.

Accuracy improves with $k = 0, 1, 2$.

A third order Runge-Kutta (RK3) time discretization and slope limiters

Lax-Friedrichs flux $\quad F(a,b) = \frac{1}{2}[f(a) + f(b) - c(b-a)],$

$$c = \max\{|f'(a)| : |a| \leq \|u_0\|_{L^\infty(\mathbb{R})}\}.$$

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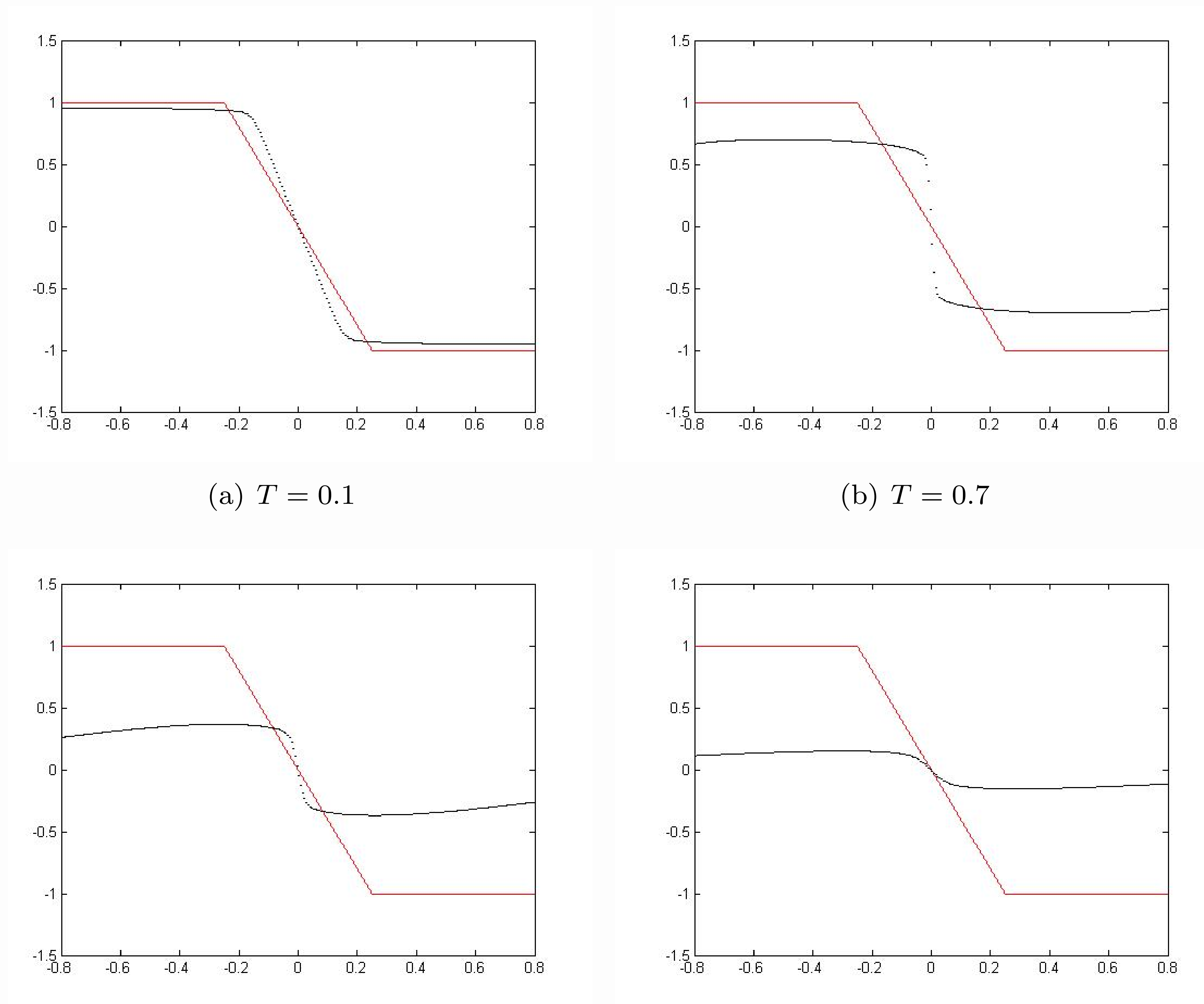

(a) $T = 0.1$ (b) $T = 0.7$

(c) $T = 1.7$ (d) $T = 2.9$

Figure 5. Initial data (piecewise linear) and solutions of the fractional Burgers' equation ($\lambda = 0.5$) at different times T using $k = 0$; $\Delta x = 1/200$.

Example. Fractional Burgers' equation

$$\partial_t u + u\partial_x u = c_\lambda \int_{|z|>0} \frac{u(t, x+z) - u(t,x)}{|z|^{1+\lambda}} dz$$

Solution not necessarily smooth.

Accuracy improves with $k = 0, 1, 2$.

A third order Runge-Kutta (RK3) time discretization and slope limiters

Lax-Friedrichs flux $\quad F(a,b) = \frac{1}{2}[f(a) + f(b) - c(b-a)],$

$$c = \max\{|f'(a)| : |a| \leq \|u_0\|_{L^\infty(\mathbb{R})}\}.$$

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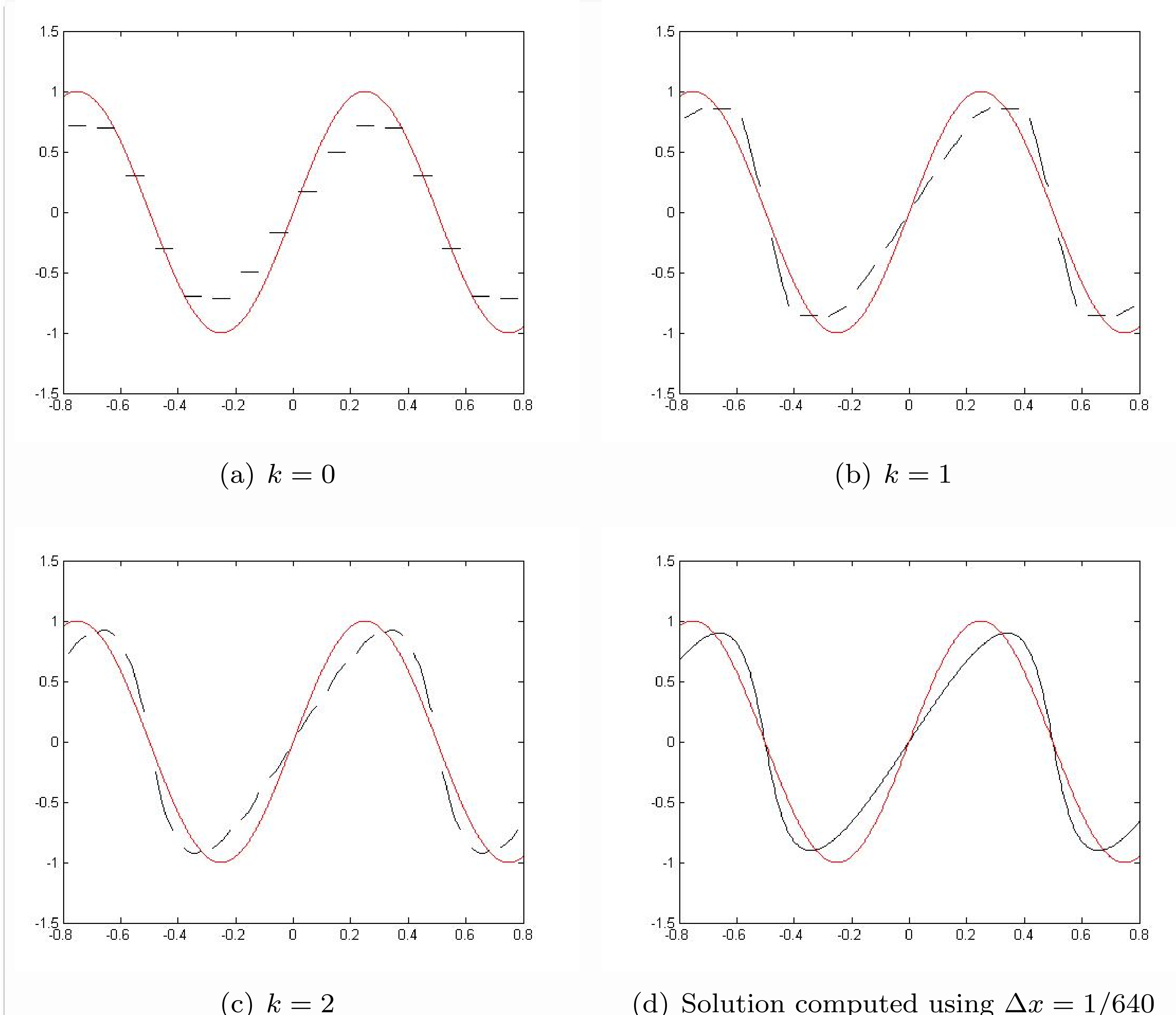


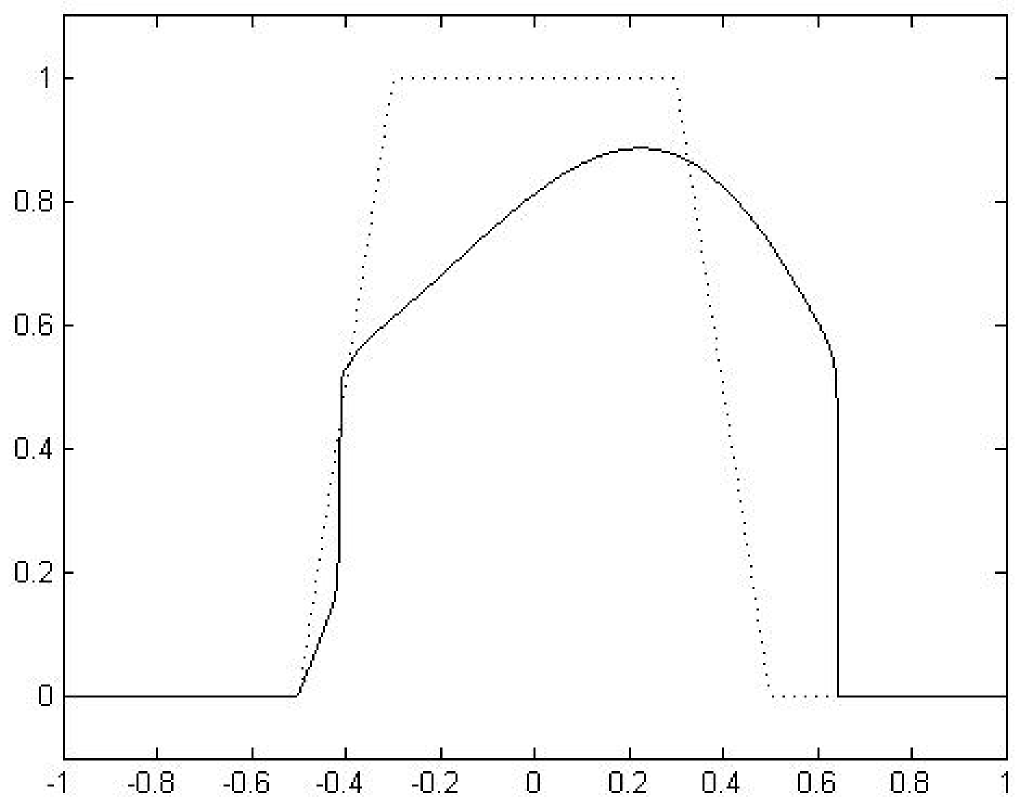
FIGURE 6. Initial data and solutions of the fractal Burgers' equation at $T = 1/10$ using different values of $k = 0, 1, 2$; $\Delta x = 1/10$, and $u_0(x) = \sin(2\pi x)$.

Example. Convection / diffusion / fractional diffusion

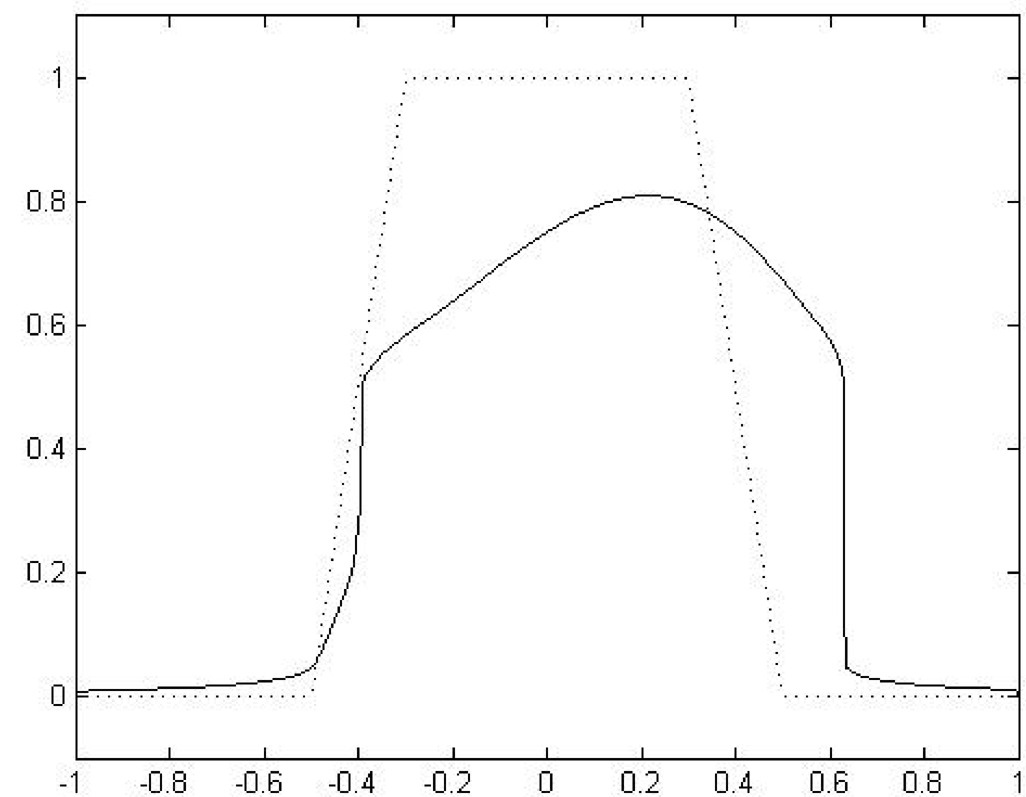
$$\partial_t u + \partial_x u^2 = \partial_x(a(u)\partial_x u) + c_\lambda \int_{|z|>0} \frac{u(t,x+z) - u(t,x)}{|z|^{1+\lambda}} dz$$

$$a(u) = \begin{cases} 0 & \text{for } u \le 0.5 \\ 2.5u - 1.25 & \text{for } 0.5 < u \le 0.6 \\ 0.25 & \text{for } u > 0.6, \end{cases}$$

Example. Convection / diffusion / fractional diffusion



(a) $u_t + f(u)_x = (a(u)u_x)_x$



(b) Equation (1.1) with $\lambda = 0.5$

FIGURE 1. (**Ex**.1): $T = 0.15$ and $\Delta x = 1/640$.