Nonlinear diffusion of dislocation desities

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Dislocations

Dislocations are line defects in crystals whose typical length is $\sim 10^{-6}$ m and their thickness is $\sim 10^{-9}$ m.

When the material is submitted to shear stresses, these lines can move in the crystallographic planes and this dynamics can be observed using electron microscopy.

One possible (simplified) model of the dislocation dynamics is given by the system of ODEs

$$\dot{y}_i = F - V_0'(y_i) - \sum_{j \in \{1,\dots,N\} \setminus \{i\}} V'(y_i - y_j) \quad \text{for} \quad i = 1, \dots, N,$$

where F is a given constant force, V_0 is a given potential and V is a potential of two-body interactions.

Eikonal equation

The rescaled "cumulative distribution function"

$$\rho^\varepsilon(x,t) = \varepsilon\left(-\frac{1}{2} + \sum_{i=1}^{N} H\left(x - \varepsilon y_i\left(\frac{t}{\varepsilon}\right)\right)\right)$$

(where H is the Heaviside function) satisfies (as a discontinuous viscosity solution) the following nonlocal eikonal equation

$$\rho_t^\varepsilon(x,t) = \left(c\left(\frac{x}{\varepsilon}\right) + M^\varepsilon\left(\frac{\rho^\varepsilon(\cdot,t)}{\varepsilon}\right)(x)\right)|\rho_x^\varepsilon(x,t)|,$$

with $c(y) = V_0'(y) - F$.

Here, M^ε is the nonlocal operator defined by

$$M^\varepsilon(U)(x) = \int_{\mathbb{R}} J(z)\, E\big(U(x+\varepsilon z) - U(x)\big)\,\mathrm{d}z, \qquad (1)$$

where $J(z) = V''(z)$ on $\mathbb{R}\setminus\{0\}$ and E is the modification of the integer part: $E(r) = k + 1/2$ if $k \leq r < k+1$.

N. Forcadel, C. Imbert, R. Monneau, Discrete Contin. Dyn. Syst. Ser. A (2009)

Continuous model

If the kernel J is a sufficiently smooth, even, nonnegative function, and

$$J(z) = \frac{1}{|z|^2} \quad \text{for all} \quad |z| \geq R_0$$

and for some $R_0 > 0$, the rescaled cumulative distribution function ρ^ε converges towards a solution of the nonlinear equation

$$u_t = \widetilde{H}(-\Lambda u, u_x),$$

where the Hamiltonian $\widetilde{H}$ is a continuous function and Λ is defined for any function $U \in C_b^2(\mathbb{R})$ and for $r > 0$ by the formula

$$-\Lambda U(x) = C(1) \int_{\mathbb{R}} \Big(U(x+z) - U(x) - zU'(x) \mathbf{1}_{\{|z| \leq 1\}} \Big) \frac{1}{|z|^2} \, \mathrm{d}z$$

with a constant $C(1) > 0$.

Continuous model

In the particular case of

$$c = V_0'(y) - F \equiv 0$$

we have $\widetilde{H}(L, p) = L|p|$ which allows us to rewrite equation in the form

$$u_t + |u_x|\Lambda u = 0.$$

For suitably chosen $C(1)$, the operator

$$-\Lambda U(x) = C(1) \int_{\mathbb{R}} \Big(U(x+z) - U(x) - zU'(x)\mathbf{1}_{\{|z|\leq 1\}} \Big) \frac{1}{|z|^2} \, \mathrm{d}z$$

satisfies

$$\Lambda = \Lambda^1 = \left(-\partial^2/\partial x^2\right)^{1/2}.$$

Hence, this is the pseudodifferential operator defined in the Fourier variables by $\widehat{(\Lambda^1 w)}(\xi) = |\xi|\widehat{w}(\xi)$.

Equivalent formulation

Denoting $v = u_x$ we may rewrite equation

$$u_t + |u_x|\Lambda u = 0.$$

as follows

$$v_t + (|v|\mathcal{H}v)_x = 0,$$

where $\mathcal{H}$ is the Hilbert transform defined in the Fourier variables by

$$\widehat{(\mathcal{H}v)}(\xi) = -i\ \mathrm{sgn}(\xi)\ \widehat{v}(\xi).$$

Well known properties of this transform:

$$\mathcal{H}v(x) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{v(y)}{x-y}\,\mathrm{d}y \quad \text{and} \quad \Lambda^1 v = \mathcal{H}v_x.$$

Quasi-geostrophic equations

The 2D quasi-geostrophic equations, modeling the dynamics of the mixture of cold and hot air in a thin layer and the fronts between them, are of the form

$$\theta_t + (u \cdot \nabla)\theta = 0, \quad u = \nabla^{\perp}\psi, \quad \theta = -(-\Delta)^{1/2}\psi$$

for $x \in \mathbb{R}^2$ and $t > 0$, where $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$.

Here, $\theta(x, t)$ represents the air temperature.

D. Chae, A. Córdoba, D. Córdoba, M. A. Fontelos (Adv. Math. (2005)) studied the one dimesional counterpart of QG equation for the unknown function $\theta = \theta(x, t)$ for $x \in \mathbb{R}$ and $t > 0$:

$$\theta_t + (\theta \mathcal{H}\theta)_x = 0$$

for $x \in \mathbb{R}$ and $t > 0$.
They proved the existence, the regularity and the blow up in finite time of solutions this equation.

Nonlinear and nonlocal equation

Generalized initial value problem

$$\begin{aligned} &u_t = -|u_x|\ \Lambda^\alpha u \quad \text{on} \quad \mathbb{R}\times(0,+\infty),\\ &u(x,0) = u_0(x) \end{aligned}$$

Here, for $\alpha \in (0,2)$,

$$\Lambda^\alpha = \left(-\partial^2/\partial x^2\right)^{\alpha/2}$$

is the pseudodifferential operator defined via the Fourier transform $\widehat{(\Lambda^\alpha w)}(\xi) = |\xi|^\alpha \widehat{w}(\xi)$ with the Lévy–Khintchine integral representation for every $\alpha \in (0,2)$

$$-\Lambda^\alpha w(x) = C(\alpha)\int_{\mathbb{R}} \left(w(x+z) - w(x) - zw'(x)\mathbf{1}_{\{|z|\le 1\}}\right)\frac{\mathrm{d}z}{|z|^{1+\alpha}},$$

where $C(\alpha) > 0$ is a constant.

Self-similar solutions

First note that equation

$$u_t = -|u_x|\ \Lambda^\alpha u$$

is invariant under the scaling

$$u^\lambda(x,t) = u(\lambda x, \lambda^{\alpha+1} t)$$

for each $\lambda > 0$.
We found **explicit** self-similar solutions of this equation in the following form

$$u_\alpha(x,t) = \Phi_\alpha(y) \quad \text{with} \quad y = \frac{x}{t^{1/(\alpha+1)}},$$

where the self-similar profile Φ_α has to satisfy the following equation

$$-(\alpha+1)^{-1}\ y\ \Phi_\alpha'(y) = -(\Lambda^\alpha \Phi_\alpha(y))\ \Phi_\alpha'(y) \quad \text{for all} \quad y \in \mathbb{R}.$$

Self-similar solutions

$$-(\alpha+1)^{-1}\ y\ \Phi'_\alpha(y) = -(\Lambda^\alpha \Phi_\alpha(y))\ \Phi'_\alpha(y) \quad \text{for all} \quad y \in \mathbb{R}.$$

THEOREM (Existence of self-similar profile)
Let $\alpha \in (0,2)$. There exists a nondecreasing function Φ_α of the regularity $C^{1+\alpha/2}$ at each point and analytic on the interval $(-y_\alpha, y_\alpha)$ for some $y_\alpha > 0$, which satisfies

$$\Phi_\alpha = \begin{cases} 0 & \text{on} \quad (-\infty, -y_\alpha), \\ 1 & \text{on} \quad (y_\alpha, +\infty), \end{cases}$$

and

$$(\Lambda^\alpha \Phi_\alpha)(y) = \frac{y}{\alpha+1} \quad \text{for all} \quad y \in (-y_\alpha, y_\alpha).$$

Self-similar solutions

Proof

The crucial role is played by the function

$$v(x) = \begin{cases} K(\alpha)\left(1-|x|^2\right)^{\alpha/2} & \text{for} \quad |x|<1, \\ 0 & \text{for} \quad |x|\geq 1, \end{cases}$$

with $K(\alpha) = \Gamma(1/2)\left[2^\alpha \Gamma(1+\alpha/2)\Gamma((1+\alpha)/2)\right]^{-1}$.
It was computed by Getoor (1961) using a purely analytical argument (based on properties of the Fourier transform) that $\Lambda^\alpha v \in L^1(\mathbb{R})$ and

$$\Lambda^\alpha v(x) = 1 \quad \text{for} \quad |x|<1.$$

Now, we put

$$u(x) = \int_0^x v(y)\,\mathrm{d}y$$

which satisfies $u(x) = M(\alpha)$ for all $x \geq 1$ and $u(x) = -M(\alpha)$ for $x \leq -1$ with $M(\alpha) = K(\alpha)\int_0^1 \left(1-|y|^2\right)^{\alpha/2}\,\mathrm{d}y = \frac{\pi}{2^\alpha(\alpha+1)\Gamma\left(\frac{1+\alpha}{2}\right)^2}$.

Finally, we define

$$\Phi_\alpha(y) = \frac{\gamma}{\alpha+1}\left\{u\left(\gamma^{-1/(\alpha+1)}y\right) + M(\alpha)\right\} \quad \text{with} \quad \gamma^{-1} = \frac{2M(\alpha)}{\alpha+1}.$$

Uniqueness of self-similar solutions

At least formally, the function

$$u_\alpha(x,t) = \Phi_\alpha(y) \quad \text{with} \quad y = \frac{x}{t^{1/(\alpha+1)}},$$

is a solution of the equation with the initial datum being the Heaviside function

$$u_0(x) = H(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1 & \text{if} \quad x > 0. \end{cases} \tag{2}$$

THEOREM (Uniqueness of self-similar solution)
Let $\alpha \in (0,2)$. The function u_α with the profile Φ_α is the unique viscosity solution of equation with the initial datum (2).

Viscosity solutions

THEOREM
Consider $u_0 \in BUC(\mathbb{R})$. Then there exists the unique bounded continuous viscosity solution u of the initial value problem

$$
\begin{aligned}
&u_t = -|u_x| \, \Lambda^\alpha u \quad \text{on} \quad \mathbb{R} \times (0, +\infty), \\
&u(x, 0) = u_0(x)
\end{aligned}
$$

REMARK
The function $u_\alpha(x,t)$ at $t = 0$ does not belong to $BUC(\mathbb{R})$.

Stability

THEOREM (Stability of the self-similar solution)
Let $\alpha \in (0,2)$. For any initial data $u_0 \in BUC(\mathbb{R})$ satisfying

$$\lim_{x \to -\infty} u_0(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} u_0(x) = 1,$$

let us consider the unique viscosity solution $u = u(x,t)$ and, for each $\lambda > 1$, its rescaled version

$$u^\lambda(x,t) = u(\lambda x, \lambda^{\alpha+1} t).$$

Then, for any compact set $K \subset (\mathbb{R} \times [0,+\infty)) \setminus \{(0,0)\}$, we have

$$u^\lambda(x,t) \to \Phi_\alpha\left(\frac{x}{t^{1/(\alpha+1)}}\right) \quad \text{in} \quad L^\infty(K) \quad \text{as} \quad \lambda \to +\infty.$$

Stability

We have

$$u(\lambda x, \lambda^{\alpha+1} t) = u^{\lambda}(x,t) \to \Phi_\alpha\left(\frac{x}{t^{1/(\alpha+1)}}\right) \quad \text{in} \quad L^\infty(K) \quad \text{as} \quad \lambda \to +\infty.$$

COROLLARY (Large time asymptotics of solutions)

Substituting, first $t = 1$, next $\lambda = t^{1/(\alpha+1)}$ we obtain the convergence:

$$u\left(xt^{1/(\alpha+1)}, t\right) \to \Phi_\alpha(x) \quad \text{as} \quad t \to \infty.$$

Lévy operator

We define

$$\widehat{\mathcal{L}u}(\xi) = a(\xi)\widehat{u}(\xi)$$

with

$$a(\xi) = ib \cdot \xi + q(\xi) + \int_{\mathbb{R}^n} \left(1 - e^{-i\eta\xi} - i\eta\xi \mathbb{1}_{\{|\eta|<1\}}(\eta)\right) \Pi(d\eta).$$

Inverting the Fourier transform we obtain

$$\begin{aligned} \mathcal{L}u(x) &= b \cdot \nabla u(x) - \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} \\ &\quad - \int_{\mathbb{R}^n} \left(u(x-\eta) - u(x) - \eta \cdot \nabla u(x) \mathbb{1}_{\{|\eta|<1\}}(\eta)\right) \Pi(d\eta). \end{aligned}$$

Important example: fractional Laplacian

Let

$$\Pi(d\eta) = \frac{C(\alpha)}{|\eta|^{n+\alpha}} \quad \text{with} \quad \alpha \in (0,2)$$

in

$$\mathcal{L}u(x) = -\int_{\mathbb{R}^n} \big(u(x-\eta) - u(x) - \eta \cdot \nabla u(x) \mathbb{1}_{\{|\eta|<1\}}(\eta)\big)\, \Pi(d\eta).$$

In this case, we obtain the α-stable anomalous diffusion:

$$\mathcal{L} = (-\Delta)^{\alpha/2} \quad \text{and} \quad a(\xi) = |\xi|^{\alpha} \quad \text{for} \quad 0 < \alpha \leq 2.$$

Using symmetry of the Lévy measure, we can simplify:

$$(-\Delta)^{\alpha/2} u(x) = -C(\alpha)\, PV \int_{\mathbb{R}^n} \frac{u(x-\eta) - u(x)}{|\eta|^{n+\alpha}}\, \Pi(d\eta).$$

Maximum principle

Definition
The operator A satisfies the **positive maximum principle** if for any $\varphi \in D(A)$ the fact

$$0 \leq \varphi(x_0) = \sup_{x \in \mathbb{R}^n} \varphi(x) \quad \text{for some} \quad x_0 \in \mathbb{R}^n$$

implies

$$A\varphi(x_0) \leq 0.$$

□

REMARK
$A\varphi = \varphi''$ or, more generally, $A\varphi = \Delta\varphi$ satisfies the positive maximum principle.

Maximum principle

THEOREM
Denote by $\mathcal{L}$ the Lévy diffusion operator. Then $A = -\mathcal{L}$ satisfies the positive maximum principle.

Proof.
Assume that $0 \leq \varphi(x_0) = \sup_{x \in \mathbb{R}^n} \varphi(x)$. Then

$$\begin{aligned} & -\mathcal{L}\varphi(x_0) \\ = \quad & -b \cdot \nabla\varphi(x_0) + \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 \varphi(x_0)}{\partial x_j \partial x_k} \\ & + \int_{\mathbb{R}^n} \left(\varphi(x_0 - \eta) - \varphi(x_0) - \sum_{j=1}^{n} \eta_j \frac{\partial \varphi(x_0)}{\partial x_j} \mathbb{1}_{\{|\eta|<1\}}(\eta) \right) \Pi(d\eta) \leq 0. \end{aligned}$$

□

Kato inequality for Lévy operator

THEOREM

For every $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (\mathcal{L}\varphi) \operatorname{sgn} \varphi \, dx \geq 0.$$

Strook-Varopoulos inequality

THEOREM

Assume that $\mathcal{L}$ is a Lévy operator.

For every $p \in (1, \infty)$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi \geq 0$ we have

$$4\frac{p-1}{p^2}\int_{\mathbb{R}^n} (\mathcal{L}\varphi^{p/2})\,\varphi^{p/2}\,dx \leq \int_{\mathbb{R}^n} (\mathcal{L}\varphi)\,\varphi^{p-1}\,dx.$$

REMARK

For $\mathcal{L} = b \cdot \nabla$, both sides of the Strook-Varopoulos inequality are equal to 0.

REMARK

For $\mathcal{L} = -\Delta$ we integrate by parts to obtain **the equality**

$$\begin{aligned}
\int_{\mathbb{R}^n} (-\Delta\varphi)\,\varphi^{p-1}\,dx &= (p-1)\int_{\mathbb{R}^n} |\nabla\varphi|^2\,\varphi^{p-2}\,dx \\
&= (p-1)\int_{\mathbb{R}^n} |\nabla\varphi\,\varphi^{p/2-1}|^2\,dx \\
&= 4\frac{p-1}{p^2}\int_{\mathbb{R}^n} |\nabla\varphi^{p/2}|^2\,dx.
\end{aligned}$$

Decay estimates

THEOREM (Optimal decay estimates)
Let $\alpha \in (0,1]$. For any initial condition $u_0 \in BUC(\mathbb{R})$ such that $u_{0,x} \in L^1(\mathbb{R})$, the unique viscosity solution u satisfies

$$\|u(\cdot,t)\|_\infty \le \|u_0\|_\infty \quad \text{and} \quad \|u_x(\cdot,t)\|_\infty \le \|u_{0,x}\|_\infty \quad \text{for any} \quad t>0.$$

Moreover, for every $p \in [1,+\infty)$ we have

$$\|u_x(\cdot,t)\|_p \le C_{p,\alpha} \|u_{0,x}\|_1^{\frac{p\alpha+1}{p(\alpha+1)}} \; t^{-\frac{(p-1)}{p(\alpha+1)}} \quad \text{for any} \quad t>0,$$

with some constant $C_{p,\alpha} > 0$ depending only on p and α.

Proof of the decay estimates

For $v = u_x$, we consider the regularized problem

$$v_t = \varepsilon v_{xx} - (|v|\Lambda^{\alpha-1}\mathcal{H}v)_x \quad \text{on} \quad \mathbb{R}\times(0,+\infty), \tag{3}$$

$$v(\cdot,0) = v_0 = u_{0,x} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \tag{4}$$

We multiply this equation by $|v|^{p-2}v$ with $p > 1$ to get

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}} |v|^p \, dx = \varepsilon \int_{\mathbb{R}} v_{xx}|v|^{p-2}v \, dx - \int_{\mathbb{R}} \left((\Lambda^{\alpha-1}\mathcal{H}v)|v|\right)_x |v|^{p-2}v \, dx.$$

We drop the first term on the right hand side, because it is nonpositive. Integrating by parts and using the elementary identity

$$|v|\left(|v|^{p-2}v\right)_x = \frac{p-1}{p}\left(|v|^{p-1}v\right)_x,$$

we transform the second quantity on the right hand side as follows

$$-\int_{\mathbb{R}} \left((\Lambda^{\alpha-1}\mathcal{H}v)|v|\right)_x |v|^{p-2}v \, dx = -\frac{p-1}{p}\int_{\mathbb{R}} (\Lambda^\alpha v)|v|^{p-1}v \, dx.$$

Consequently, we can apply the Stroock–Varopoulos inequality.

Nonlocal porous medium equation

The equation satisfied by $v = u_x$

$$v_t = -(|v|\Lambda^{\alpha-1}\mathcal{H}v)_x \tag{5}$$

(with the Hilbert transform $\mathcal{H}$) can be treated as a *nonlocal counterpart of the porous medium equation*. For $\alpha = 2$ and for nonnegative v, this equation reduces to

$$v_t = (vv_x)_x = \left(v^2/2\right)_{xx}.$$

The L^p-decay estimates of u_x show a regularizing effect created by the equation, even for the anomalous diffusion:

$$\text{if} \quad v_0 \in L^1(\mathbb{R}) \quad \text{then} \quad v \in L^p(\mathbb{R}) \quad \text{for each} \quad p > 1.$$

Equation (5) has compactly supported self-similar solution $v(x,t) = t^{-\frac{1}{\alpha+1}}\Phi'_\alpha\left(x/t^{\frac{1}{\alpha+1}}\right)$, where the profile Φ_α was constructed above. *This function for $\alpha = 2$ corresponds to the well known Barenblatt–Prattle solution of the porous medium equation.*

Nonlocal porous medium equation

Multidimensional case

$$\partial_t u = \nabla \cdot (|u|^{m-1} \nabla^{\alpha-1} u), \quad t > 0, \; x \in \mathbb{R}^N$$

where $m > 1$ and $\nabla^{\alpha-1}$ is a singular integral operator which coincides with the classical gradient when $\alpha = 2$.

One possible definition: this is the Fourier multiplier with symbol

$$|\xi|^{\alpha-2} i\xi.$$

Work in progress – jointly with **Piotr Biler and Cyril Imbert**.

Recent works by Caffarelli and Vazquez !

Thank you !