A fast marching method for the non monotone evolution of fronts and some applications to dislocation dynamics

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joint works with
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"Nonlocal aspects in PDEs and applications"
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1 The classical Fast Marching Method
2 A Generalized Fast Marching Method (GFMM)
3 Extension to dislocation dynamics
4 GFMM on unstructured grids

Propagation of front: level set approach

The curve

$$\Gamma_t = \{(x,y) \in \mathbb{R}^2, u(x,y,t) = 0\}$$

moves with normal velocity c, if the function u solves the PDE

$$\begin{cases} u_t = c(x,y,t)|\nabla u| & \mathbb{R}^2 \times (0,T) \\ u(x,y,0) = \frac{1}{2} dist(x,y,\Gamma_0)^2. \end{cases}$$

in the class of continuous viscosity solutions.
Ref. Crandall, Lions, Evans, Ishii, etc...

The Main Fast Marching schemes

- $c(x,y) > 0$
  Fast Marching Method
  (Tsitsiklis 95, Sethian 96)
- $c(x,y) \geq 0$
  Semi-Lagrangian Fast Marching Methods
  (F., Cristiani 05)
- Monotone evolution: $c(x,y,t) > 0$ (or $c(x,y,t) < 0$)
  Ordered Upwind Method
  (Sethian,Vladimirsky 01)
- unsigned $c(x,y,t)$
  Generalized Fast Marching Method
  (Carlini, F., Forcadel, Monneau 08)

The stationary problem for the monotone eikonal equation

$\Gamma_t = \{(x,y) \in \mathbb{R}^2 : u(x,y,t) = 0\} = \{(x,y) \in \mathbb{R}^2 : T(x,y) = t\}$
where $T(x,y)$ solves the minimum time problem.

$$\Omega = \{(x,y) \in \mathbb{R}^2 : u(x,y,0) \leq 0\}$$

- $c(x,y) > 0$ (see Osher (93), F.-Giorgi-Loreti (94))
$$\begin{cases} c(x,y)|\nabla T(x,y)| = 1 & \mathbb{R}^2 \setminus \Omega \\ T(x,y) = 0 & \Omega \end{cases}$$
- $c(x,y,t) > 0$ (see Sethian Vladimirsky 01)
$$\begin{cases} c(x,y,T(x,y))|\nabla T(x,y)| = 1 & \mathbb{R}^2 \setminus \Omega \\ T(x,y) = 0 & \Omega \end{cases}$$

The Finite Difference approximation

Let us write the equation as

$$T_x^2 + T_y^2 = \frac{1}{c^2(x,y)}$$

The standard up-wind FD approximation is

$$\begin{aligned} (1) \qquad & \max(0, T_{i,j} - T_{i-1,j}, T_{i,j} - T_{i+1,j})^2 + \\ & \max(0, T_{i,j} - T_{i,j-1}, T_{i,j} - T_{i,j+1})^2 = \left(\frac{\Delta x}{c_{i,j}}\right)^2 \end{aligned}$$

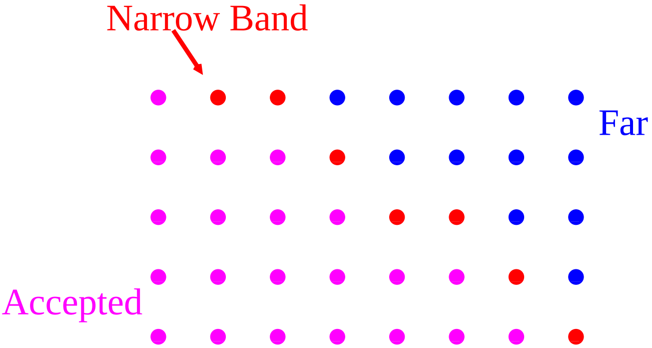
The scheme satisfies a specific (Causality property) :
the solution at each grid point depends only on the smallest adjacent value.

Properties of the FD scheme

The iterative method is

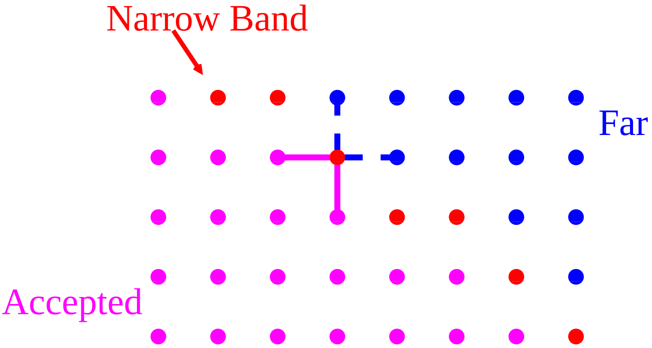
- consistent
- stable, provided a CFL condition is satisfied
- convergent
- expensive, since it globally works on *all* the grid values at every iteration

The Classical Fast Marching Method (FMM)



- $NB \equiv V(\Omega_\Delta) \backslash \Omega_\Delta$ where $\Omega_\Delta = \{(i,j) : (x_i, y_j) \in \Omega\}$ and $V(i,j) \equiv \{(l,m) : |(l,m)-(i,j)| = 1\}$.
- Ω, $T = 0$
- $T = \infty$

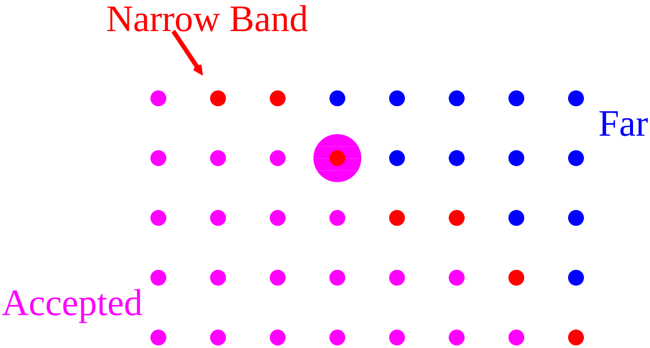
FMM at work



1 Compute the time $\tilde{T}_{i,j}$ in the NB with:

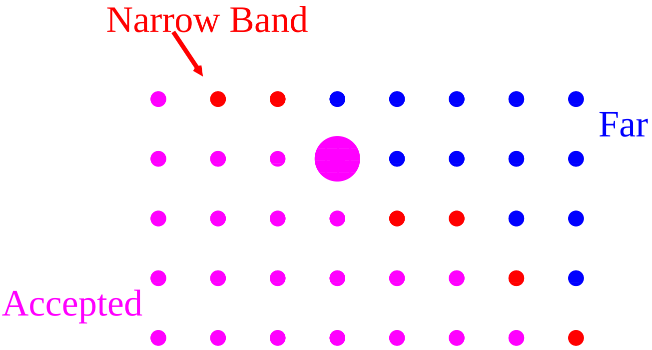
$$\max(0, \tilde{T}_{i,j} - T_{i-1,j}, \tilde{T}_{i,j} - T_{i+1,j})^2 + \max(0, \tilde{T}_{i,j} - T_{i,j-1}, \tilde{T}_{i,j} - T_{i,j+1})^2 = \left(\frac{\Delta x}{c_{i,j}}\right)^2$$

FMM at work



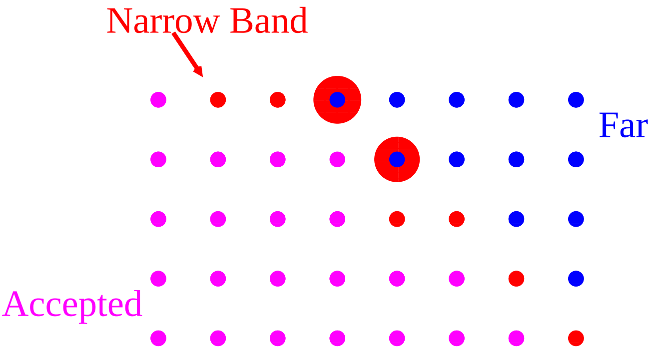
1. Compute the time $\tilde{T}_{i,j}$ in the NB
2. Call $T_{i,j}$ the minimal time on the NB

FMM at work



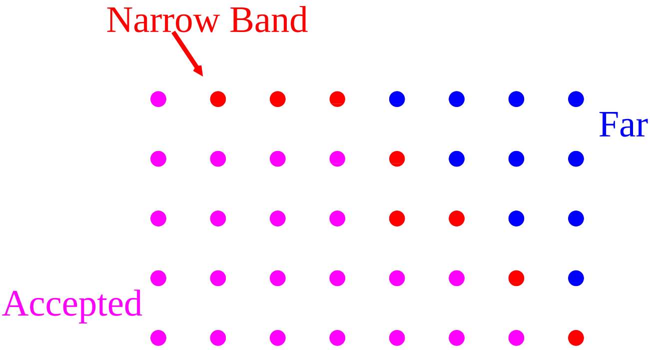
1. Compute the time $\tilde{T}_{i,j}$ in the NB
2. Call $T_{i,j}$ the minimal time on the NB and accept it .

FMM at work



1. Compute the time $\tilde{T}_{i,j}$ in the NB
2. Call $T_{i,j}$ the minimal time on the NB and accept it
3. The new NB is defined as the boundary of the new accepted region A

FMM at work



3. The new NB is defined as the boundary of the new accepted region A
4. Iterate until the NB is empty

Main properties of FMM

- The FMM method compute the time $T(i,j)$ the front reaches the point (i,j) with complexity $O(NlogN)$ and it has been proved:
- Convergence
  Theorem *in the case $c \geq 0$, the FM method is convergent to (1), in the sense that*

$$||T(x_i, y_j) - T_{i,j}||_\infty \to 0 \quad \textit{for} \quad \Delta x \to 0$$

Idea of the proof:
Show that the solution computed by the FM method is exactly the same that the one computed by the iterative scheme
Some references: Sethian(1996,1999), Cristiani-F. (2005).

A Generalized Fast Marching Method (GFMM)

AIM: to extend the FMM to the case $c(x, y, t)$ unsigned.

ADVANTAGE :

1. no need of techniques of reinitialization, in case of small gradient of the solution
2. complexity $O(NlogN)$ in case of smooth speed c

TOOL : an auxiliary discontinuous function $\theta(x, y, t)$ to track the front.

Non monotone evolution

If the speed function is NOT always positive then the crossing time $T(x,y)$ is NOT single-valued function.
We decide to use a discontinuous function to track the position of the front

$$\theta(x,y,t) = \begin{cases} 1 & \text{if} \quad x,y \in \Omega_t, \\ -1 & \text{if} \quad x,y \notin \Omega_t. \end{cases}$$

and to solve locally in time the stationary equation for the time evolution

$$\begin{cases} |c(x,y,t_n)||\nabla T(x,y)| = 1 & \mathbb{R}^2 \setminus \Omega_{t_n} \\ T(x,y) = U(x,y) & \Omega_{t_n} \end{cases}$$

GFMM method

We introduce an auxiliary discrete function

$$\theta_{i,j}^n = \begin{cases} 1 & \text{if } (x_i, y_j) \in \Omega_n \\ -1 & \text{otherwise.} \end{cases}$$

We define the two phases

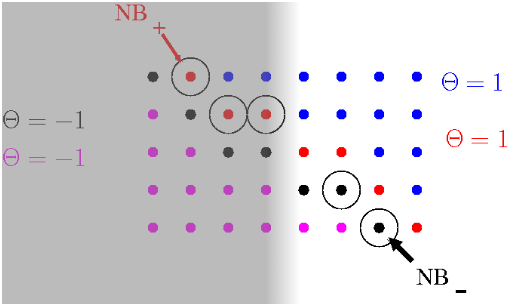
$$\Theta_{\pm}^n \equiv \{i, j : \theta_{i,j}^n = \pm 1\},$$

and the fronts

$$F_+^n \equiv V(\Theta_-^n) \backslash \Theta_-^n, \quad F_-^n \equiv V(\Theta_+^n) \backslash \Theta_+^n$$

where $V(D)$ represents the set of first neighbours to the nodes in D.

GFMM method

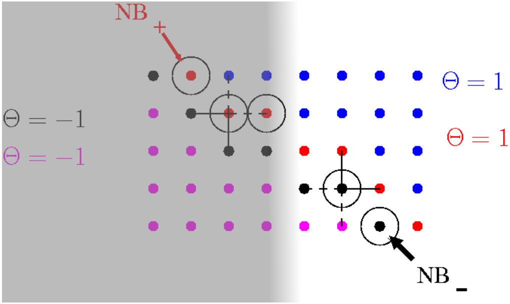


Def. We define two different **narrow bands**:

$$NB_+^n = F_+^n \cap \{(i,j), c_{i,j}^n < 0\}, \quad NB_-^n = F_-^n \cap \{(i,j), c_{i,j}^n > 0\}.$$

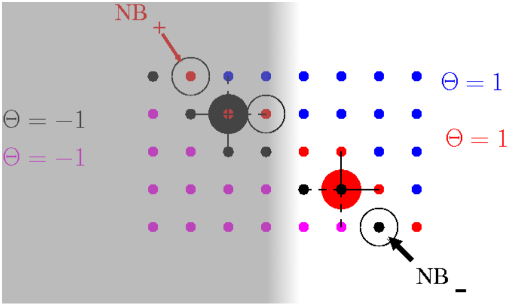
- F_-^n • F_+^n

GFMM



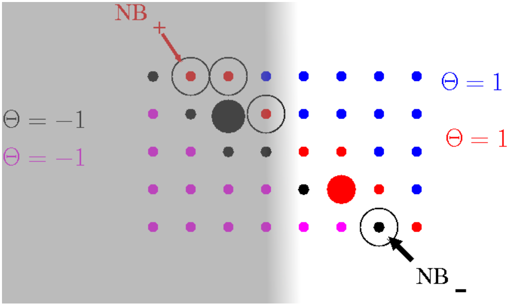
1 Compute the time $\tilde{u}_{i,j}^{n-1}$ in the NB_{+}^{n-1} and NB_{-}^{n-1}

GFMM



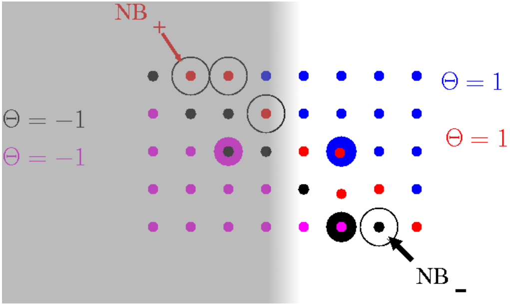
1. Compute the time $\tilde{u}_{i,j}^{n-1}$ in the NB_+^{n-1} and NB_-^{n-1}
2. Call t_n the minimal time $\tilde{u}_{i,j}^{n-1}$ on the $NB^{n-1} = NB_+^{n-1} \cup NB_-^{n-1}$

GFMM



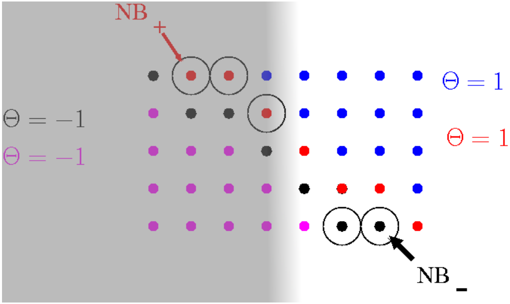
1. Compute the time $\tilde{u}_{i,j}^{n-1}$ in the NB_+^{n-1} and NB_-^{n-1}
2. Call t_n the minimal time $\tilde{u}_{i,j}^{n-1}$ on the $NB^{n-1} = NB_+^{n-1} \cup NB_-^{n-1}$ and accept at the time t_n the minimizing points (i, j)

GFMM



1. Compute the time $\tilde{u}^{n-1}_{i,j}$ in the NB^{n-1}_{+} and NB^{n-1}_{-}
2. Compute the minimal time $\tilde{u}^{n-1}_{i,j}$ on the NB^{n-1} and accept (i,j)
3. The new $F^n_{\pm}$ is defined as the boundary of the new regions $\Theta^n \equiv 1$ and $\Theta^n \equiv -1$

GFMM



1. Compute the time $\tilde{u}_{i,j}$ in the NB_+ and NB_-
2. Compute the minimal $\tilde{u}_{i,j}$ on the NB^{n-1} and accept (i,j)
3. The new $F^n_\pm$ is defined as the boundary of the new regions $\Theta^n \equiv 1$ and $\Theta^n \equiv -1$
4. Update the NB^n and return to **Step 1**

Convergence result

Theorem (Carlini, F., Forcadel, Monneau)

Let $c(x,y,t)$ be globally Lipschitz continuous in space and time, the initial set Ω_0 be with piece wise smooth boundary and $\theta^\Delta(x,y,t)$ be an appropriate extension of the discrete function $\theta^n_{i,j}$ over all the continuous space,then

$$\overline{\theta}^0(x,y,t) = \limsup_{\Delta\to 0,(z,w)\to(x,y),s\to t} \theta^\Delta(z,w,s)$$

(resp. $\underline{\theta}^0(x,y,t) = \liminf_{\Delta\to 0,(z,w)\to(x,y),s\to t} \theta^\Delta(z,w,s)$) is a viscosity sub-solution (resp. super-solution) of the problem

$$\begin{cases} \theta_t = c(x,y,t)|\nabla\theta| & \mathbb{R}^2\times(0,T) \\ \theta = 1_{\Omega_0} - 1_{\Omega_0^c} & \mathbb{R}^2. \end{cases}$$

Non constant time step!

The time step $\Delta t_n = t_{n+1} - t_n$ is not constant and we can actually have:

1. $\Delta t_n >> 1$ too large time step
2. $\Delta t_n < 0$ not increasing time

In order to avoid Case 1, we choose

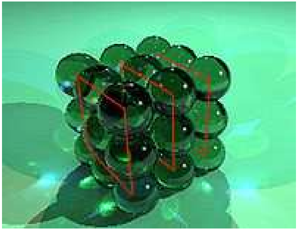
$$\widehat{t}_n \equiv t_n + \Delta t$$

and to avoid Case 2 we set $t_n = t_{n-1}$. Then one always gets

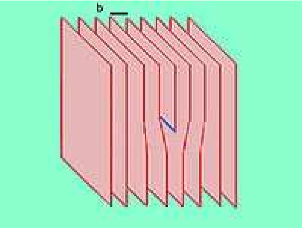
$$0 \leq \Delta t_n < \Delta t$$

WARNING: If Case 1 occurs we do not advance the front!

Dislocations



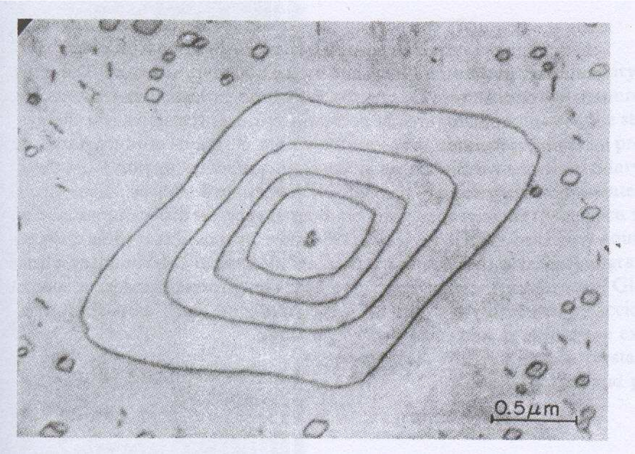
Crystal lattice showing atoms and lattice planes[1]



An edge dislocation [2]

[1] Picture from: http://en.wikipedia.org
[2] Picture from: http://en.wikipedia.org

A picture of dislocations



A model for dislocations

We study the phase field model of dislocation dynamics which has been proposed by Rodney, Le Bouar and Finel.
To simplify the model, let us assume that:

- the thickness of the dislocation is zero
- there is only one dislocation in the domain
- the dislocation is planar (it is contained in the slip plane)

A model for dislocations

We assume that the dislocation line is represented by the boundary Γ_t of a smooth bounded domain $\Omega_t \subset \mathbb{R}^2$.
Let us define

$$u(x,t) = \begin{cases} > 0 & \text{if} \quad x \in \Omega_t, \\ < 0 & \text{if} \quad x \notin \Omega_t \\ = 0 & \text{if} \quad x \in \partial\Omega_t. \end{cases}$$

$$\begin{cases} u_t = c(1_{u>0}, x, t)|Du| & \text{in} \quad \mathbb{R}^2 \times (0, +\infty), \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^2. \end{cases} \tag{1}$$

where c(x,t) is our non local velocity.

Dislocations dynamics, Peierls-Nabarro model

The resolved Peach-Koehler force acting on the dislocation is

$$c(x,t) = c_0 * 1_{u>0}(x,t)$$

The Fourier transform of c_0 is given by:

$$\widehat{c_\delta^0}(\xi_{x_1},\xi_{x_2}) = -\frac{1}{2}\left(\frac{\xi_{x_1}^2 + (\frac{1}{1-\nu})\xi_{x_2}^2}{\sqrt{\xi_{x_1}^2 + \xi_{x_2}^2}}\right) e^{-\delta\sqrt{\xi_{x_1}^2+\xi_{x_2}^2}}, \qquad (2)$$

$\delta \simeq$ size of the core of the dislocation
ν influences the anisotropy of the evolution

Short time existence and uniqueness

Theorem (Alvarez, Carlini, Monneau, Rouy)
Let $c^0 \in L^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$. *If* u^0 *satisfies*

$$|\nabla u^0(x,y)| < B \quad in\ \mathbb{R}^2$$

and

$$\frac{\partial u^0}{\partial y}(x,y) > b > 0 \quad in\ \mathbb{R}^2,$$

then there exists T^* *such that a unique viscosity solution of the problem in* $\mathbb{R}^2 \times [0, T^*)$ *exists .*

A finite difference scheme for the continuous model

$$\begin{cases} v_{i,j}^{n+1} = S(c_{i,j}(1_{\{v^n>0\}}), v^n) & \text{for } n = 0, ..., N \\ v_{i,j}^n = u^0(x_i, y_j) \end{cases}$$

$$S(c_{i,j}(1_{\{v^n>0\}}), v^n) = v_{i,j}^n + \Delta t H_d(c_{i,j}(1_{\{v^n>0\}}), D_x^{\pm} v_{i,j}^n, D_y^{\pm} v_{i,j}^n)$$

where the discrete numerical Hamiltonian is

$$H_d(c_{i,j}([v^n]), D_x^{\pm} v_{i,j}^n, D_y^{\pm} v_{i,j}^n) = \begin{cases} c_{i,j}([v^n])H^+ & c_{i,j}[v^n] \geq 0 \\ c_{i,j}([v^n])H^- & c_{i,j}[v^n] < 0. \end{cases}$$

H^+, H^- are the standard numerical Hamiltonian:

$$H^+ = \left\{ \max(D_x^+ v_{i,j}^n, D_y^+ v_{i,j}^n, 0)^2 + \min(D_x^- v_{i,j}^n, D_y^- v_{i,j}^n, 0)^2 \right\}^{\frac{1}{2}}$$

Properties of the FD scheme

The FD scheme is

- consistent
- NOT monotone
- convergent under the CFL condition

$$0 < \frac{\Delta t}{\Delta x} \leq \frac{1}{2\sqrt{2}|c_\delta^0(\cdot,\cdot)|_1}$$

for small time

Convergence result

Theorem (Alvarez, Carlini, Monneau, Rouy)
If u^0 *satisfies*

$$|\nabla u^0(x,y)| < B \quad in\ \mathbb{R}^2$$

and

$$\frac{\partial u^0}{\partial y}(x,y) > b > 0 \quad in\ \mathbb{R}^2,$$

then there exists a positive constant C *such that*

$$\sup_{i,j\in\mathbb{Z}} |u(x_i, y_j, n\Delta t) - v^n_{i,j}| \le C\sqrt{\Delta t} \quad n = 1, ..., N_{T^*}$$

with $\Delta t \simeq \Delta x$.

Computation of the discrete convolution

Under periodic assumption on w one have

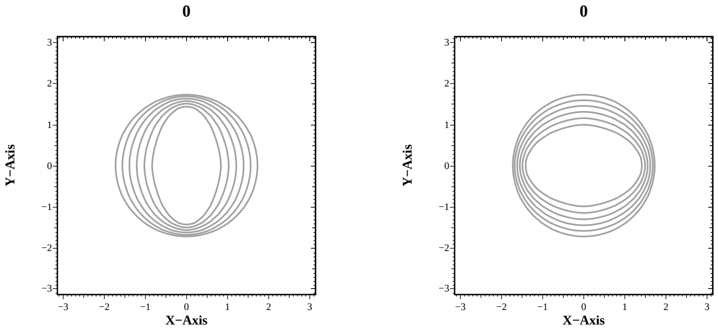
$$\left(\widehat{c^{\Delta}}\right)_{(p,m)} = \widehat{w}_{(p,m)} \cdot (\widehat{\widetilde{c}^{0}})_{(p,m)} \quad \text{for every} \quad (p,m) \in \mathbb{Z}^2$$

where $\widehat{w}$ is the Fourier transform of $w = [v^n]$ and $\widetilde{c}^0$ is well approximated by:

$$\widehat{\widetilde{c}^0}_{(p,m)} \simeq \widehat{c^0}(\pi p/L, \pi m/L)$$

where $\widehat{c}^0$ is the Fourier transform of the kernel c^0.

Numerical tests: Anisotropic Shrinking of a circle



Convergence result for non-local dislocation dynamics

$$\begin{cases} \theta_t(x,t) = c[\theta](x,t)|D\theta(x,t)| & \text{on} \quad \mathbb{R}^N \times (0,T) \\ \theta(\cdot,0) = 1_{\Omega_0} - 1_{\Omega_0^c}. \end{cases} \tag{2}$$

$c[\theta](x,t) = c_1(x,t) + (c_0 \star \theta(\cdot,t))(x).$

Main assumptions

(A1) Existence and uniqueness for problem (2)

(A2) Existence and uniqueness for the perturbed problem with $c^e(x,t) = c[\theta](x,t) + e$

(A3) Stability of the perturbed problem

$$|\theta^e - \theta|_{L^\infty((0,T);^1(\mathbb{R}^N))} \leq CeT$$

Convergence result for non-local dislocation dynamics

Theorem (Carlini, Forcadel, Monneau)
Under assumptions (A1)-(A2)-(A3).
Let $\theta^{\Delta}(x,t)$ be the solution of GFMM algorithm applied to problem (2) with discrete speed c^{Δ} defined by

$$c^{\Delta} = c[\theta^{\Delta}]$$

Then

$$|\theta^{\Delta} - \theta|_{L^{\infty}((0,T);L^1(\mathbb{R}^N))} \leq \omega_T(\Delta).$$

with $\omega_T(\Delta)$ modulus of continuity with respect to Δ and T small enough.

Checking assumption

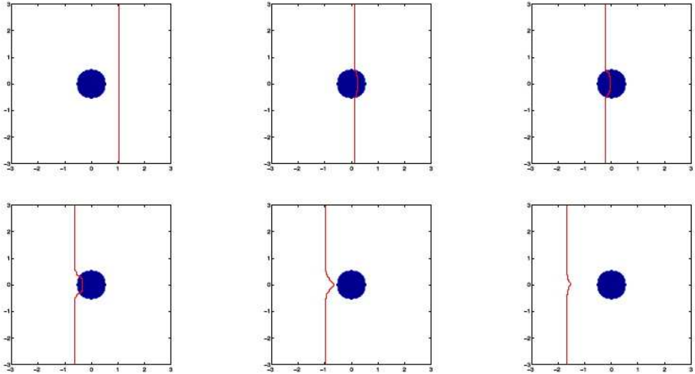
- If a closed dislocation loop is a smooth curve $\partial\Omega_0$ in $\mathbb{R}^2$ at the initial time,
  if Ω_0 bounded and $\partial\Omega_0$ smooth enough and
  if $c_1 \in W^{1,\infty}$, $c_0 \in W^{1,1} \cap L^\infty$ then (A1)-(A3) are verified for short time
  (see Alvarez ,Hoch, LeBouar, Monneau '04).
- If dislocation dynamics has a non-negative velocity and the initial curve satisfyes an *interior ball condition*,
  if $c_1 \in W^{2,\infty}$, $c_0 \in W^{1,1} \cap L^1$ then (A1)-(A3) are verified for large time
  (see Alvarez, Cardialiaguet, Monneau '05).

Local dynamics a rotating line

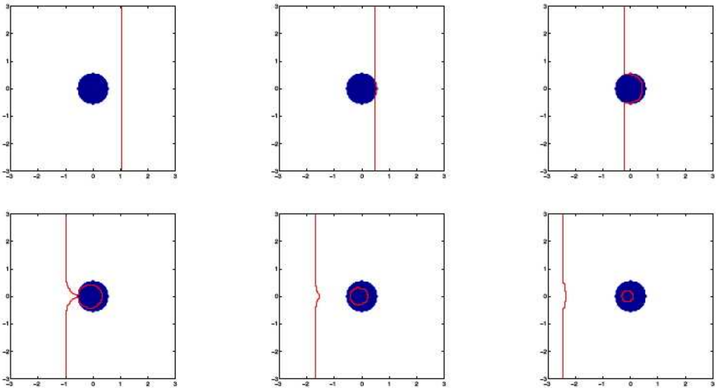
Speed: $c(x,t) = \sin(2\pi t)x_1$

	GFMM		**FD**	
Δx	$\mathcal{H}(\mathcal{C},\tilde{\mathcal{C}})$	CPU	$\mathcal{H}(\mathcal{C},\tilde{\mathcal{C}})$	CPU
0.04	$5.21 \cdot 10^{-2}$	$0.52s$	$4.82 \cdot 10^{-2}$	$1.82s$
0.02	$3.07 \cdot 10^{-2}$	$1.71s$	$2.46 \cdot 10^{-2}$	$13.3s$
0.01	$1.54 \cdot 10^{-2}$	$10.5s$	$1.35 \cdot 10^{-2}$	$102s$
0.005	$9.00 \cdot 10^{-3}$	$130s$	$7.00 \cdot 10^{-3}$	$842s$

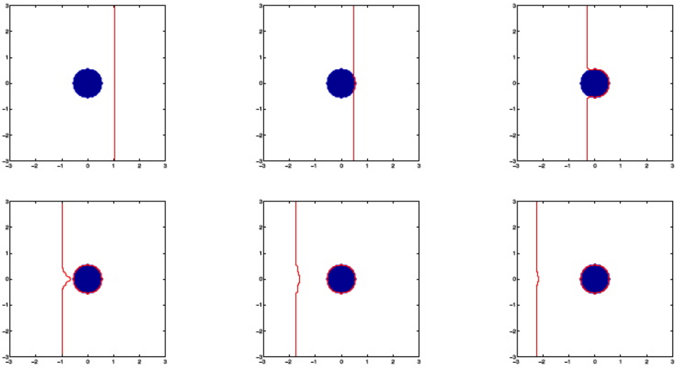
The dislocation line passes the obstacle



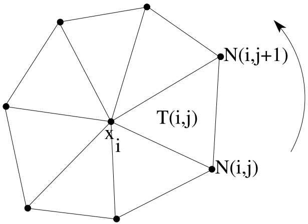
The obstacle breaks the line



The obstacle captures the line



GFMM on UNSTRUCTURED grids: local solver

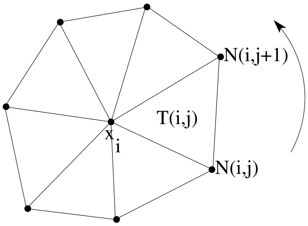


The neighborhood of the node i, is the set of nodes defined

$$V(i) = \{N(i,l), l \in \mathcal{V}(i)\}$$

where $\mathcal{V}(i) = \{1, \ldots, \mathcal{N}_v(i)\}$.

GFMM on UNSTRUCTURED grids: local solver



We suppose there exists a $\gamma_0 > 0$ s.t. for any mesh

$$\gamma_0 \leq \frac{h}{\Delta} \leq 1$$

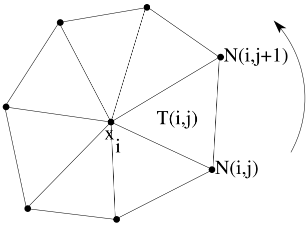
where $\Delta := \max\{l_{ij},\ i,j \in \{1,\ldots,\mathcal{N}_v\}\}$,
and $h := \min\{l_{ij},\ i,j \in \{1,\ldots,\mathcal{N}_v\}\}$

GFMM on UNSTRUCTURED grids

Local problem

$$|Du(x)| = \frac{1}{|c(x_i, t_n)|} \quad \text{in} \quad D_i \times [t_n, t_{n+1}[$$

where D_i is:



General local solver

$$Q\left(x_i, u_i, \{u_{N(i,j)}, u_{N(i,j+1)}\}_{j\in\mathcal{V}(i)}\right) = \frac{1}{|c(x_i, t_n)|} \quad i \in \{1, \dots, \mathcal{N}_v\}.$$

Properties Local Solver: Consistency

(H1)
For any $\psi \in C^2(\mathbb{R}^2)$, let us denote by $\psi_i := \psi(x_i)$ for any $i \in \{1 \ldots \mathcal{N}_v\}$ and consider true the following assumptions:

$$\lim_{m\to\infty} Q\left(x_{i_m}, \psi_{i_m}, \{\psi_{N(i_m,j_m)}, \psi_{N(i_m,j_m+1)}\}_{j_m\in\mathcal{V}(i_m)}\right) = |D\psi(x)|$$

where m is an index of refinement for a family of grids $\{\mathcal{M}_m^T\}_{m\geq 0}$ and $(x_{i_m}) \in \mathcal{M}_m^T$ is a sequence of nodes such that for $m \to \infty$

$$\Delta_m \to 0 \quad \text{and} \quad x_{i_m} \to x.$$

Properties Local Solver: Monotonicity

(H2)
Let us suppose $u_i \leq t$ and define

$$\mathcal{C}(i) := \{j \in \mathcal{V}(i), \text{ s. t. } u_{N(i,j)} \geq \psi_{N(i,j)}, \ \ u_{N(i,j+1)} \geq \psi_{N(i,j+1)}\}$$

then

$$Q(x_i, u_i, \{u_{N(i,j)}, u_{N(i,j+1)}\}_{j\in\mathcal{C}(i)}) \leq Q(x_i, t, \{\psi_{N(i,j)}, \psi_{N(i,j+1)}\}_{j\in\mathcal{C}(i)}).$$

Properties Local Solver

(H3)

$$\frac{K}{\Delta} \leq Q(x_i, t, \{t, t-K\}) \leq \frac{K}{h}$$

for any positive constant K

Example of Local Solver

1. Local problem

$$\begin{cases} |Du(x)| = \frac{1}{|c(x_i,t_n)|} & x \in D_i \\ u(x) = u_h(x) & x \in \partial D_i \end{cases}$$

2. the Hopf-Lax formula :

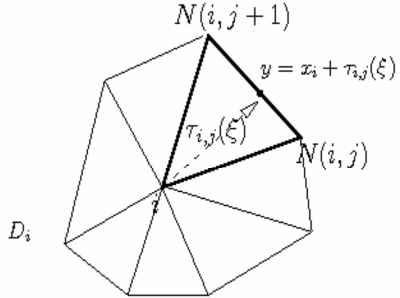
$$u_h^*(x_i) = \min_{y \in \partial D_i} (u_h(y) + \frac{|x_i - y|}{|c(x_i, t_n)|})$$

3. Local Solver:

$$Q = \max_{y \in \partial D_i} \left(\frac{u_h^*(x_i) - u_h(y)}{|x_i - y|} \right)$$

Example of Local Solver: Semi-Lagrangian

$$Q = \max_{j \in \mathcal{V}(i)} \max_{0 \le \xi \le 1} \left(\frac{u_i - (1-\xi) u_{N(i,j+1)} - \xi u_{N(i,j)}}{|\tau_{i,j}(\xi)|} \right)$$



Refs: Tsitsiklis (95), Cristiani-Falcone (2005), Sethian Vladimirsky(2006), Bornemann Rash(2005))

GFMM on UNSTRUCTURED grids

We introduce an auxiliary discrete function

$$\theta_i^n = \begin{cases} 1 & \text{if } x_i \in \Omega_n \\ -1 & \text{otherwise.} \end{cases}$$

We define the two phases

$$\Theta_\pm^n \equiv \{i : \theta_i^n = \pm 1\},$$

and the fronts

$$F_+^n \equiv V(\Theta_-^n) \backslash \Theta_-^n, \quad F_-^n \equiv V(\Theta_+^n) \backslash \Theta_+^n$$

GFMM algorithm on Unstructured grids

Initialization

- *Initialization of the matrix* θ^0
  $$\theta_i^0 = \begin{cases} 1 & x_i \in \Omega_0 \\ -1 & x_i \notin \Omega_0 \end{cases}$$
- *Initialization of the time on the front*
  $u_i^0 = 0$ for all $i \in F^0$

GFMM algorithm on Unstructured grids

Main Cycle

1. Compute the time $\tilde{u}_i^{n-1}$ in the NB_+^{n-1} and NB_-^{n-1} using a local solver

$$Q(\tilde{u}_i^{n-1}, \{u_{N(i,j)}^{n-1}, u_{N(i,j)}^{n-1}\}_{j\in V(i)}) = \frac{1}{|c(x_i, t_n)|}$$

   using respectively the values u^{n-1} defined on F_-^{n-1} or F_+^{n-1}.
2. Compute the minimal time $\tilde{u}^{n-1}$ on the $NB_\pm^{n-1}$
3. Initialize the new accepted points $NA_\pm^n = \{i \in NB_\pm^{n-1}\ u_i^n = t_n\}$,
4. Update $\Theta_\pm^n$ on the $NA_\pm^n$
5. Update $F_\pm^n$ and $NB_\pm^n$ and return to **1**

GFMM on UNSTRUCTURED grids: Definition of $\theta^\epsilon(x,t)$

$\{t_{k_n}, n \in \mathbb{N}\}$ is a strictly increasing subsequence of $(t_n)_n$ such that

$$t_{k_n} = t_{k_n+1} = ... = t_{k_{n+1}-1} < t_{k_{n+1}}.$$

Extension of $(\theta_i^n)_{n,i}$ on the continuous time interval $[0,T]$

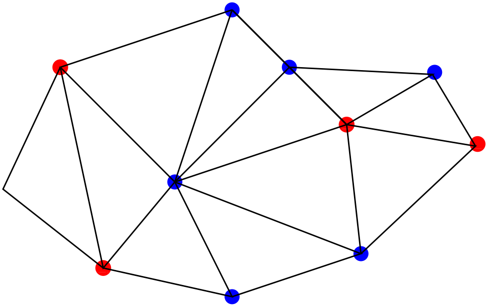
$$\theta(x_i,t) = \theta_i^{k_{n+1}-1} \quad \text{if } (x_i,t) \in \{x_i\} \times [t_{k_n}, t_{k_{n+1}}[$$

(Same extension on structured grids.)

GFMM on UNSTRUCTURED grids: Definition of $\theta^\epsilon(x,t)$

Let $\epsilon = (\Delta, \Delta t)$ and $\theta^\epsilon(x,t)$ be an extension of $(\theta(x_i, t_n))_i$ on a continuous domain Ω of $\mathbb{R}^2$

• $\theta = 1$, • $\theta = -1$



(Different than structured grids!)

GFMM on UNSTRUCTURED grids: Definition of $\theta^\epsilon(x,t)$

Let $\epsilon = (\Delta, \Delta t)$ and $\theta^\epsilon(x,t)$ be an extension of $(\theta(x_i,t_n)_i)$ on a continuous domain Ω of $\mathbb{R}^2$

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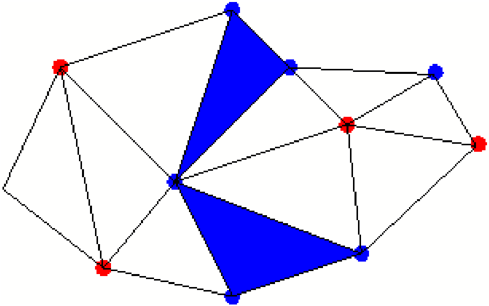

GFMM on UNSTRUCTURED grids: Definition of $\theta^\epsilon(x,t)$

Let $\epsilon = (\Delta, \Delta t)$ and $\theta^\epsilon(x,t)$ be an extension of $(\theta(x_i, t_n)_i)$ on a continuous domain Ω of $\mathbb{R}^2$

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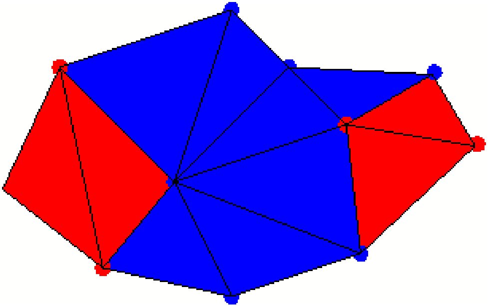


Partial Convergence result

Theorem (Carlini, F., Hoch)

Let $c(x,t) \neq 0$ be globally Lipschitz continuous in space and time, the initial set Ω_0 be with piecewise smooth boundary then

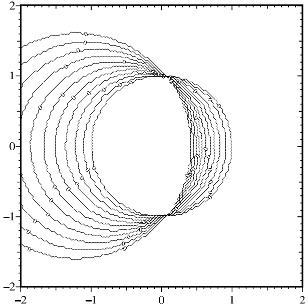
$$\overline{\theta}^0(x,t) = \limsup_{\epsilon \to 0, z \to x, s \to t} \theta^\epsilon(z,s)$$

(resp. $\underline{\theta}^0(x,t) = \liminf_{\epsilon \to 0, z \to x, s \to t} \theta^\epsilon(z,s)$)
is a viscosity sub-solution (resp. super-solution) of the problem

$$\theta_t = c(x,t)|\nabla\theta| \quad \mathbb{R}^2 \times (0,T)$$

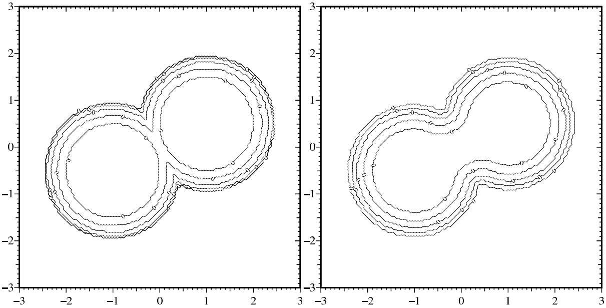
Numerical tests: evolution of one circle

Speed $c(x, y, t) = 0.1t - x$



Numerical tests: evolution of two circles

Speed $c(x, y, t) = 1 - t$



Increasing (left) and decreasing (right) evolution of two circles

Conclusions and perspectives

- The GFMM can deal with unsigned front propagation
- It can work on structured and unstructured grids
- The GFMM can deal with non local velocities
- A general result of convergence on unstructured grids is still missing (on going)
- To have a complete convergence result we need
  1. to prove that $\overline{\theta}^0$ is sub-solution in the case $c = 0$
  2. to prove a comparison principle

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