ON THE EXTENSIONS OF HÖLDER-LIPSCHITZ MAPS

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If (X,d) and (Y,ϱ) are metric spaces, $\alpha \in (0,1]$ and $K > 0$, we say that a map $f : X \to Y$ is α-Hölder with constant K (or in short (K,α)-Hölder) if

$$\forall x, y \in X, \quad \varrho(f(x), f(y)) \leq K d(x,y)^{\alpha}.$$

We refer to [2] for background and more information about Hölder maps.

In [12] and [9] the following notation was introduced: for $C \geq 1$, $\mathcal{B}_C(X,Y)$ denotes the set of all $\alpha \in (0,1]$ such that any (K,α)-Hölder function f from a subset of X into Y can be extended to a (CK,α)-Hölder function from X into Y. If $C = 1$, such an extension is called an isometric extension. When $C > 1$, it is called an isomorphic extension. If a (CK,α)-Hölder extension exists for all $C > 1$, we say that f can be almost isometrically extended. Thus the following sets are defined:

$$\mathcal{A}(X,Y) = \mathcal{B}_1(X,Y), \quad \mathcal{B}(X,Y) = \bigcup_{C \geq 1} \mathcal{B}_C(X,Y), \text{ and } \widetilde{\mathcal{A}}(X,Y) = \bigcap_{C>1} \mathcal{B}_C(X,Y).$$

The study of these sets goes back to a classical result of Kirszbraun [8] asserting that if H is a Hilbert space, then $1 \in \mathcal{A}(H,H)$. This was extended by Grünbaum and Zarantonello [4] who showed that $\mathcal{A}(H,H) = (0,1]$. Then the complete description of $\mathcal{A}(L^p, L^q)$ for $1 < p, q < \infty$ relies on works by Minty [11] and Hayden, Wells and Williams [5] (see also the book of Wells and Williams [13] for a very nice exposition of the subject). More recently, K. Ball [1] introduced a very important notion of non linear type or cotype and used it to prove a general extension theorem for Lipschitz maps. Building on this work, Naor ([12] and a forthcoming preprint) described completely the sets $\mathcal{B}(L^p, L^q)$ for $1 < p, q < \infty$.

In [9] we studied $\mathcal{A}(X,Y)$ and $\widetilde{\mathcal{A}}(X,Y)$, when X is a Banach space and Y is a space of continuous functions on a compact space equipped with the supremum norm. (This can also be viewed as a non linear generalization of the results of Lindenstrauss and Pełczyński [10] and of Johnson and Zippin [6, 7] on the extension of linear operators with values in $C(K)$ spaces.) We showed that for any finite dimensional space X, $\widetilde{\mathcal{A}}(X, C(K)) = (0,1]$ and $\mathcal{A}(X, C(K))$ is either $(0,1]$ or $(0,1)$ and we gave examples of both occurrences. To our knowledge, this is the first example of Banach spaces X and Y such that $\mathcal{A}(X,Y)$ is not closed in $(0,1]$ and also such that $\mathcal{A}(X,Y) \neq \widetilde{\mathcal{A}}(X,Y)$.

This leads us to a number of questions concerning the above defined sets:

Question 1. *Is $\widetilde{\mathcal{A}}(X,Y)$ always closed? If yes, is $\widetilde{\mathcal{A}}(X,Y) = \overline{\mathcal{A}(X,Y)}$?*

Question 2. *Is $\mathcal{B}(X,Y)$ always closed? Is $\widetilde{\mathcal{B}}_C(X,Y) \stackrel{def}{=} \bigcap_{\varepsilon>0} \mathcal{B}_{C+\varepsilon}(X,Y)$ always closed? If yes, is $\widetilde{\mathcal{B}}_C(X,Y) = \overline{\mathcal{B}_C(X,Y)}$? Or, more generally, is $\widetilde{\mathcal{B}}_C(X,Y) \subseteq \overline{\mathcal{B}_C(X,Y)}$?*

Question 3. *Is the collection of sets $\mathcal{B}_C(X,Y)$ continuous with respect to C?*

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Question 4. *Does there always exist $C > 0$ so that $\mathcal{B}(X,Y) = \mathcal{B}_C(X,Y)$? (It is so in the examples that we know.)*

Brudnyi and Shvartsman [3] proved that if Y is a Banach space then the set $\mathcal{B}(X,Y)$ is always a subinterval of $(0,1]$ with the left endpoint equal to 0 (see also Naor [12]). Naor asked whether the same is true for the set $\mathcal{A}(X,Y)$. It is also natural to ask

Question 5. *Do the sets $\mathcal{B}_C(X,Y)$ or $\widetilde{\mathcal{B}}_C(X,Y)$ have to be intervals? If yes, does the left endpoint have to be 0?*

We note that all the above questions make sense in the setting when X and Y are assumed to be either metric spaces or Banach spaces, and the answers may differ in these two settings.

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