Introduction to Riemann surfaces, moduli spaces and its applications

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M2, Mathematical Physics

Outline

1. Introduction to Moduli Spaces
    1.1 Moduli of Lines
    1.2 Line Bundles
2. Moduli space of curves
    2.1 Riemann Surfaces
    2.2 $g = 0$
    2.3 $g = 1$
    2.4 Higher genus

Motivation

Problem:

Describe the collection of all lines in the real plane $\mathbb{R}^2$ that pass through the origin.

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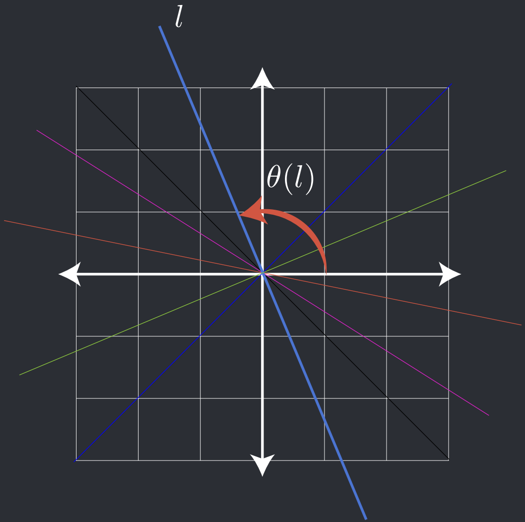
A way to solve this is to assign to each line l a parameter

Example

Define a function $\theta(l)$ between l and the x-axis, so $0 \leq \theta < \pi$.



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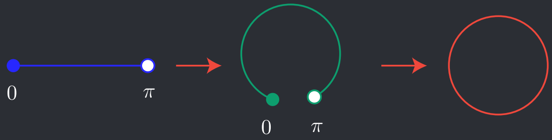
When we do this, we get a circle



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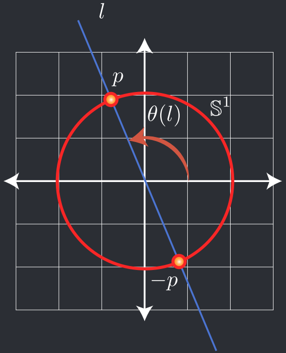

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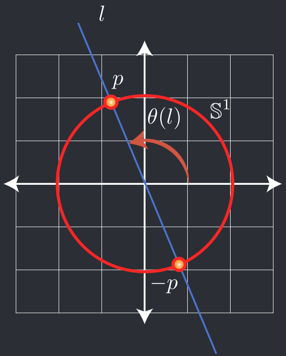


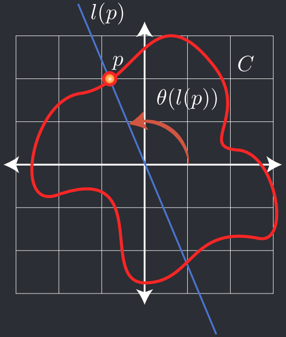
The advantage of this procedure is that we can endow $\mathbb{RP}^1$ with a topology inherited of $\mathbb{S}^1$.

Now we are allowed to study continuous functions $f : C \longrightarrow \mathbb{RP}^1$ with C a topological space.

Example

Let $C \subset \mathbb{R}^2 \setminus \{0\}$ be a curve given by

$$\gamma : I \subset \mathbb{R} \longrightarrow C$$



Because γ is continuous in $\mathbb{R}^2$, $l(c)$ is also a continuous function from

$$\begin{aligned} l : C &\to \mathbb{RP}^1 \\ c &\mapsto l(c) \end{aligned}$$

In the same way we can extend the notion of continuity for maps from $\mathbb{R}^n$ to $\mathbb{RP}^1$ by assigning to each $x \in \mathbb{R}^n$, $\theta(l(x)) \in \mathbb{RP}^1$, in the case $\theta(l) = 0$ or π we can use the angle φ from the vertical axis.

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We say $\mathbb{RP}^1$ is the **moduli space of lines** in $\mathbb{R}^2$

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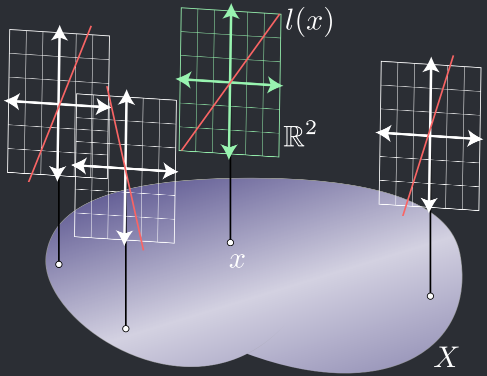

If $X = \mathbb{RP}^1$, we call this the **tautological line bundle**.

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- Family of lines in $\mathbb{R}^2$ parametrized by a topological space X
- Continuous functions from X to $\mathbb{RP}^1$
- Line bundle on X contained in the trivial bundle $X \times \mathbb{R}^2$

Outline

Moduli space of curves

Problem Classify compact Riemann surfaces

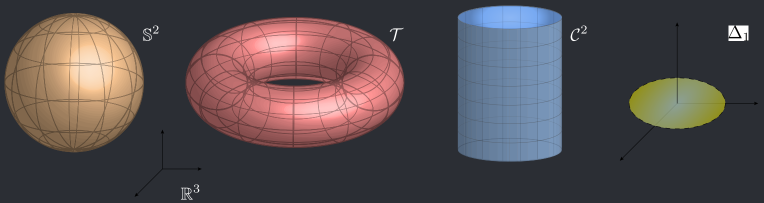
A geometric solution is Moduli spaces $\mathscr{M}_g$

- Each point of $\mathscr{M}_g$ is a Riemann surface
- Studying $\mathscr{M}_g$ can tell us things of the geometry of the Riemann surfaces.

Riemann Surfaces

A Riemann surface is a $1_{\mathbb{C}}$-dimensional manifold where the transitions functions are holomorphic.

Examples



- Classification of compact R.S. up to topological equivalence is given by the integer number g (genus), where $g = 0$ corresponds to $\mathbb{S}^2 \cong \mathbb{CP}^1$.
- Such Classification ignores the complex structure.
- Contrary to the genus (discrete), there are inequivalent R.S. that can be parametrized by continuous parameters.

$g = 0$

Riemann Uniformization Theorem

Any symply connected R.S. is biholomorphic to either $\mathbb{CP}^1$, $\mathbb{C}$ or $\mathbb{H}$.

- Any compact R.S. with $g = 0$ is simply connected, then by RUT, it can only be $\mathbb{CP}^1$.
- For this case the topological and holomorphic classifications agree.

$g = 1$

- Torus with one handle can be obtained by taking the quotient $\mathbb{C}/(\mathbb{Z} \times \mathbb{Z})$
- Replace $\mathbb{Z} \times \mathbb{Z}$ by a lattice L, and $z \sim w$ if $z - w \in L$.
- A basis for L is a pair of numbers $w_1, w_2 \in \mathbb{C}$, such that $\forall z \in L$, $z = aw_1 + bw_2$ with $a, b \in \mathbb{Z}$.
- Any compact R.S with $g = 1$ can be obtained as the quotient $\mathbb{C}/L$.
- The zero of $\mathbb{C}$ is preserved by the quotient, such point is a marked point in the torus.

Definition

An elliptic curve over C is a Riemann surface of genus one with a marked point.

- There is a one -to -one correspondence between elliptic curves and lattices.
- IfL is a lattice λL is also a lattice, also $\mathbb{C}/L \cong \mathbb{C}/\lambda L$
- Given a lattice L and its oriented basis $(w_1, w_2) = (1, \tau = \dfrac{w_2}{w_1})$, where $\tau \notin \mathbb{R}$, this suggest $\mathbb{H}$ is the parameter space.

Given two basis on $\mathbb{H}$, the change of basis is given by the matrix:

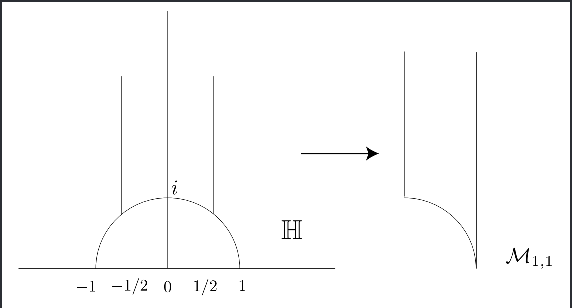
$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{where } a, b, c, d \in \mathbb{Z}$$

This is fact must be an element of $PSL(2, \mathbb{R})$ as well, so:

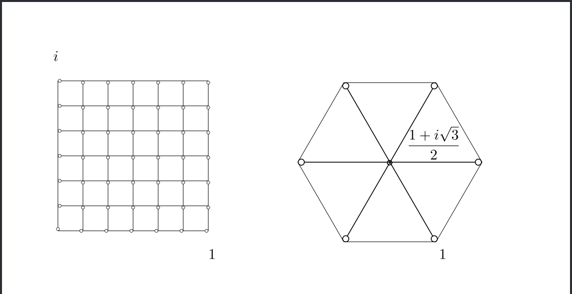
- Two points in $\mathbb{H}$ correspond to the same elliptic curve if there exists an element $T \in PSL(2, \mathbb{Z})$ that relates them.

Elliptic curves are in one-to-one correspondence with orbits of $PSL(2, \mathbb{Z})$ in $\mathbb{H}$, i.e. elements of $\mathbb{H}/PSL(2, \mathbb{Z})$

- However, NOT every continuous family of elliptic curves over a topological space X can be identified with a map from X to $\mathscr{M}_{1,1}$
- There are two points in $\mathbb{H}/PSL(2,\mathbb{Z})$ with additional symmetry.
- These two points have symmetries corresponding to the groups $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$



To see this we note that one of the points correspond to the lattice $(1, i)$, this lattice is the same if we rotate it by $\pi/2$ and $\pi/6$ for the lattice $(1, \frac{1+i\sqrt{3}}{2})$, this means the automorphism group is discontinuous at these points.



Because of this "hidden" symmetry, we can no longer have the correspondence of maps from X to $\mathscr{M}_{(1,1)}$, because there will be more elliptic curves than maps in these points.

The structure that actually deals with this situations is called: **orbifold**.

Orbifolds can handle points with internal symmetries and therefore play an important role on the description of moduli spaces.

Higher genus

- Any compact Riemann surface X of genus $g > 1$ can be obtained as a quotient $X = \mathbb{H}/\Gamma$.
- Γ is a representation of the fundamental group $\pi_1(X)$ in $PSL(2, \mathbb{R})$ with 2g generators.

There is a one-to-one correspondence between holomorphic maps from a complex manifold Y to T_g and biholomorphism classes of marked Riemann surfaces of genus g

- Unfortunately the Teichmüller space (space of parameters for the complex structure) associated to X might not have a complex structure.
- This is caused by the fact that $\pi_1(X)$ is not abelian, therefore cannot be identified to the homology group $H_1(X, \mathbb{Z})$
- Fortunately there is an "abelian" moduli, called: **Abelian varieties**, but such descriptions are quite complicated.

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