ON TSFASMAN–VLĂDUŢ INVARIANTS OF INFINITE GLOBAL FIELDS

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Abstract. In this article we study certain asymptotic properties of global fields. We consider the set of Tsfasman–Vlăduţ invariants of infinite global fields and answer some natural questions arising from their work. In particular, we prove the existence of infinite global fields having finitely many strictly positive invariants at given places, and the existence of infinite number fields with certain prescribed invariants being zero. We also give precisions on the deficiency of infinite global fields and on the primes decomposition in those fields.

1. Introduction

In the 80', Ihara [8] initiated the asymptotic theory of global fields in the particular case of unramified infinite Galois number fields. More recently, motivated by its connection to asymptotic problems arising from codes and sphere packings, Tsfasman and Vlăduţ [21] generalized his work to arbitrary infinite global fields, and defined a set of invariants of such fields. The aim of this paper is to investigate their properties further.

1.1. **Definitions.** Following [21], let us first recall some definitions and some basic facts about the theory of infinite global fields. A global field is a finite algebraic extension either of the field $\mathbb{Q}$ of rational numbers, or of the field $\mathbb{F}_r(t)$ of rational functions in one variable over a finite field of constants. Unless otherwise stated, the function fields we consider are extensions of $\mathbb{F}_r(t)$ and have $\mathbb{F}_r$ as field of constants, where r is a fixed power of a prime p. In this case, all the extensions we consider are separable and without constants extension. We write (NF) (respectively (FF)) in order to signify that an assertion concerns the number field case (resp. the function field case). For any global field K, put n_K its degree over $\mathbb{Q}$ or $\mathbb{F}_r(t)$, g_K its genus ($\frac{1}{2}\log|D_K|$ in the number field case, where D_K denotes its discriminant). For a number field (resp. a function field) K, let g_K^* denote g_K (resp. $g_K - 1$), so that the Hurwitz formula is exactly the same in the case of number fields and of function fields without constants extensions. Denote by $P(K)$ (resp. $P_f(K)$) its set of places (resp. of non-archimedean places). For any place $\mathfrak{p} \in P_f(K)$, let $\mathrm{N}\mathfrak{p}$ be its norm. For any prime power q, let $\Phi_q(K)$ denote the number of its places of norm q, and let $\Phi_{\mathbb{R}}(K)$, $\Phi_{\mathbb{C}}(K)$ be the number of its real and complex places respectively. Let A denote the set of parameters $\{p^k \mid p \text{ prime number}, k \in \mathbb{N}^*\} \cup \{\mathbb{R}, \mathbb{C}\}$ in the number field case, $\{r^k \mid k \in \mathbb{N}^*\}$ in the function field case, and let A_f be the subset of the parameters which are prime powers. We also define relative Φ-numbers as follows: given a place $\mathfrak{p}$ of a global field E, and K/E a finite extension, let $\Phi_{\mathfrak{p},q}(K)$ denote the number of places of K above $\mathfrak{p}$ with the norm q (or which are real or complex in the case $q = \mathbb{R}$ or $\mathbb{C}$). We will omit the field K in our notation if there is no possible confusion.

2000 *Mathematics Subject Classification.* 11R34, 11R37, 11R45, 11R58.
Key words and phrases. Global fields, infinite global fields, class field towers, decomposition of primes, Grunwald–Wang theorem.

Given a set of primes T of a global field K, put $s(T)=\sum_{\mathfrak{p}\in T}\mathrm{N}\mathfrak{p}^{-1}\in\mathbb{R}\cup\{+\infty\}$ and denote by $\delta(T)$ its Dirichlet density if it exists:

$$\delta(T)=\lim_{s\to 1^+}\frac{\sum_{\mathfrak{p}\in T}\mathrm{N}\mathfrak{p}^{-s}}{\sum_{\mathfrak{p}\in P_f(K)}\mathrm{N}\mathfrak{p}^{-s}}.$$

If it does not exist, we put

$$\bar{\delta}(T)=\limsup_{s\to 1^+}\frac{\sum_{\mathfrak{p}\in T}\mathrm{N}\mathfrak{p}^{-s}}{\sum_{\mathfrak{p}\in P_f(K)}\mathrm{N}\mathfrak{p}^{-s}}.$$

For any set of places T of a global field K and any extension of global fields L/K (resp. K/L), let $T(L)$ be the set of places of L lying above (resp. under) T.

Recall that a sequence $\{K_i\}_{i\in\mathbb{N}}$ of global fields (meaning that they are either all number fields, or all function fields over $\mathbb{F}_r$) is said to be a family if K_i is not isomorphic to K_j for $i\neq j$, and if the constants field of all the K_i equals one and the same $\mathbb{F}_r$ in the function field case (see [21]). In any family of global fields, we have $g_i\to\infty$, because for any real number g_0, there are only finitely many number fields (respectively function fields over a given constant field up to isomorphism) whose genus does not exceed g_0.

We say that a family of global fields $\{K_i\}_{i\in\mathbb{N}}$ is asymptotically exact if the limit

$$\phi_q(\{K_i\})=\lim_{i\to\infty}\frac{\Phi_q(K_i)}{g_{K_i}}$$

exists for every $q\in A$. A family $\{K_i\}_{i\in\mathbb{N}}$ of global fields is called a tower if K_i is strictly included in K_{i+1} for every integer i. An infinite global field $\mathcal{K}$ is an infinite separable algebraic extension of $\mathbb{Q}$ or $\mathbb{F}_r(t)$ which is the limit of such a tower of global fields (so that $\mathcal{K}\cap\overline{\mathbb{F}_r}=\mathbb{F}_r$ in the function field case). Remark that this condition is always satisfied in the number field case.

Tsfasman and Vlăduţ (see [21]) proved that any tower of global fields $\{K_i\}$ is an asymptotically exact family, and the ϕ_q's depend only on $\cup_{i\in\mathbb{N}}K_i$. Thus one can define Tsfasman-Vlăduţ invariants $\phi_q(\mathcal{K})$ of an infinite global field $\mathcal{K}$ as being the ϕ_q's corresponding to any tower $\{K_i\}_{i\in\mathbb{N}}$ such that $\cup_{i\in\mathbb{N}}K_i=\mathcal{K}$. Define the support of an infinite global field as

$$\mathrm{Supp}(\mathcal{K})=\{q\in A,\ \phi_q(\mathcal{K})>0\}.$$

We also define the quantity

$$\phi_\infty(\mathcal{K})=\lim_{i\to\infty}\frac{n_{K_i}}{g_{K_i}}.$$

We say that an infinite global field is asymptotically good if its support is not empty, and asymptotically bad if it is empty. In the same manner we can define the relative invariants $\phi_{\mathfrak{p},q}(\mathcal{K})$, for a global field E, a place $\mathfrak{p}$ of E and an infinite global field $\mathcal{K}$ containing E, as the limit of the ratio $\Phi_{\mathfrak{p},q}(K_i)/g_{K_i}$ for any tower $\{K_i\}$ such that $E\subset K_i$ for any i and $\cup_{i\in\mathbb{N}}K_i=\mathcal{K}$. As there are only finitely many places of E whose norm is not greater than q and the number of archimedean places is also finite, we have

$$\phi_q(\mathcal{K})=\sum_{\mathfrak{p}\in P(E)}\phi_{\mathfrak{p},q}(\mathcal{K}).$$

Define also the prime support of $\mathcal{K}/E$ as the set

$$\mathrm{PSupp}(\mathcal{K}/E)=\{\mathfrak{p}\in P_f(E)\mid \exists\, q\in A_f\ \phi_{\mathfrak{p},q}>0\}.$$

Finally, for any (possibly infinite) global field $\mathcal{K}$, any global field $K\subset\mathcal{K}$, let $Ram(\mathcal{K}/K)$ (respectively $WRam(\mathcal{K}/K)$, resp. $Dec(\mathcal{K}/K)$) denote the ramification locus (resp. the wild ramification locus, resp. the decomposition locus) of $\mathcal{K}/K$, that is the set of places of K that are ramified in some (resp. that are wildly ramified in some, resp. that split completely in any) finite subextension L/K of $\mathcal{K}/K$. In the case $K = \mathbb{Q}$ or $\mathbb{F}_r(t)$, we omit it in the notation.

1.2. **The set of invariants.** Tsfasman and Vlăduţ proved that an infinite number field is asymptotically good if and only if $\phi_\infty > 0$. In the function field case that is necessary but not sufficient. More precisely, for any prime number p,

$$(NF) \quad \sum_{m \geq 1} m\phi_{p^m} \leq \phi_{\mathbb{R}} + 2\phi_{\mathbb{C}}.$$

Except for the case of asymptotically bad infinite global fields or of those constructed from the optimal ones (see [2]), we do not know any example where the set of the invariants of an infinite global field or even its support is completely known. But Tsfasman and Vlăduţ also gave the following inequalities for the invariants which generalize the Drinfeld-Vlăduţ bound (see [21], Theorems 3.1, 3.2 and Proposition 3.2):

Theorem (Tsfasman-Vlăduţ Basic Inequalities)**.** *For any infinite global field, we have the following inequalities:*

$$(NF-GRH) \ \sum_q \frac{\phi_q \log q}{\sqrt{q}-1} + (\log\sqrt{8\pi} + \frac{\pi}{4} + \gamma/2)\phi_{\mathbb{R}} + (\log 8\pi + \gamma)\phi_{\mathbb{C}} \leq 1,$$

$$(NF) \quad \sum_q \frac{\phi_q \log q}{q-1} + (\gamma/2 + \log 2\sqrt{\pi})\phi_{\mathbb{R}} + (\gamma + \log 2\pi)\phi_{\mathbb{C}} \leq 1,$$

$$(FF) \quad \sum_{m=1}^{\infty} \frac{m\phi_{r^m}}{r^{\frac{m}{2}}-1} \leq 1,$$

where the (GRH) designation indicates, here and in all that follows, that an assertion is true assuming the Generalised Riemann Hypothesis.

Several questions arise naturally. For any family $(a_q)_{q\in A}$ of real numbers satisfying the basic inequalities, does there exist an infinite global field such that $\phi_q = a_q$ for all $q \in A$? We will see that the answer to this question is negative in the number field as well as function field cases (Corollary 6.7). Some other natural weaker questions are far from being within reach at the moment. For example, we do not know if there exists an infinite global field with an infinity of ϕ_q being positive, neither if there are infinite number fields with all but one invariants equal to zero (the function field case being known in the case where r is a square, using optimal towers). Even if we are not able to give answers to these two questions, we can prove the following result:

Theorem A. *Let n be an integer and $t_1, ..., t_n \in A_f$. There exists an infinite global field (in both the number field and function field cases) such that $\phi_{t_1}, ..., \phi_{t_n}$ are all > 0, and such that, in the number field case, for any other q with $\gcd(q, \prod t_i) > 1$, $\phi_q = 0$.*

Note that an analogous result holds for function fields, if we consider the $\phi_{\mathfrak{p},r^m}$ numbers instead of ϕ_{r^m}.

Theorem B. *Let P be a finite set of prime numbers. There exists an asymptotically good infinite Galois number field $\mathcal{K}$ such that, for any positive integer m and any $p \in P$, $\phi_{p^m}(\mathcal{K}) = 0$. Moreover,* $\mathrm{PSupp}(\mathcal{K}/\mathbb{Q})$ *equals $Dec(\mathcal{K}/\mathbb{Q})$ (and therefore has a zero Dirichlet density, see Proposition E).*

1.3. **The deficiency.** Tsfasman and Vlăduţ also defined the deficiency of an infinite global field as the difference between the two terms of the basic inequalities:

$$\begin{aligned}
(NF-GRH)\quad \delta^{(1)} &= 1-\sum_q \frac{\phi_q \log q}{\sqrt{q}-1} \\
&\quad -(\log\sqrt{8\pi}+\frac{\pi}{4}+\gamma/2)\phi_{\mathbb{R}}-(\log 8\pi+\gamma)\phi_{\mathbb{C}}, \\
(NF)\quad \delta^{(2)} &= 1-\sum_q \frac{\phi_q \log q}{q-1} \\
&\quad -(\log 2\sqrt{\pi}+\gamma/2)\phi_{\mathbb{R}}-(\log 2\pi+\gamma)\phi_{\mathbb{C}}, \\
(FF)\quad \delta^{(3)} &= 1-\sum_{m=1}^{\infty} \frac{m\phi_{r^m}}{r^{\frac{m}{2}}-1}.
\end{aligned}$$

In the case of function fields, when r is a square, there are infinite function fields which reach the deficiency zero, and they are called optimal (see [9] or [3] for example). But in the case of number fields, we do not know any instances of $\delta^{(1)}=0$ and of course we do not expect to find any examples with $\delta^{(2)}=0$.

Theorem C. *For any $i\in\{1,2,3\}$, the map $\mathcal{K}\mapsto\delta^{(i)}(\mathcal{K})$ is an increasing map for the inclusion of infinite global fields.*

The deficiency is an increasing function, therefore optimal fields, if they exist, have to satisfy some similar properties to those satisfied by just-infinite global fields (meaning that they have no proper infinite subextension). One can refine in some sense Theorem A, looking for the best possible field having n distinct positive invariants, in term of deficiency.

Theorem D. *There is an infinite number field having at least n positive non archimedean invariants, such that its deficiency δ_n satisfies $\delta_n\leq 1-\varepsilon_n$, with:*

$$\begin{aligned}
(NF-GRH)\quad \varepsilon_n &\sim \frac{8}{3\sqrt{n\log n}} \text{ and,} \\
(NF)\quad \varepsilon_n &\sim \frac{4}{3n}.
\end{aligned}$$

One can obtain a corresponding result for function fields, but the class field theory would likely give a very bad estimation comparing to what we could obtain starting with an optimal tower. This should be done in a further work.

1.4. **On the decomposition of primes.** As one can see from the definition of the $\phi_q{}'s$, these invariants are closely related to the decomposition of primes in infinite global fields. The most general result concerning it is an easy corollary of the Cebotarev density theorem.

Proposition E. *Let $\mathcal{K}$ be an infinite number field (resp. function field), and let T be the set of places of $\mathbb{Q}$ (resp. $\mathbb{F}_r(t)$) that split completely in $\mathcal{K}$. Then $\delta(T)$ exists and is equal to* 0.

This result implies in particular that, for an infinite Galois global field $\mathcal{K}$ over a global field K, the set

$$\{\mathfrak{p}\in P(K)\mid \phi_{\mathfrak{p},\mathrm{N}\mathfrak{p}}>0\}$$

has a zero Dirichlet density, and in the particular case of Galois number fields, the set $\{p\mid p \text{ is a prime number and } \phi_p>0\}$ has to be very small. It seems to be hard to make this statement more precise, but in his paper [8], Ihara considered the primes decomposition in infinite unramified Galois extensions of global fields, and proved that, for any number field k and any unramified Galois extension $\mathcal{K}$ of k

$$\sum_{\mathfrak{p}\in P_f(k)} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{f(\mathfrak{p})}-1} \leq C(k),$$

where $f(\mathfrak{p})$ is the inertia index of a place of $\mathcal{K}$ above $\mathfrak{p}$, and $C(k)$ is a real number depending only on k. Using the Tsfasman-Vlăduţ-basic inequalities, one deduces the following:

Proposition F. *Let $\mathcal{K}$ be an infinite number field (resp. function field) and let T denote the set of places of $\mathbb{Q}$ (resp. $\mathbb{F}_r(t)$) that split completely in $\mathcal{K}$. Then*

$$(NF) \quad \sum_{\mathfrak{p}\in T} \frac{\log \mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})-1} \leq \frac{1}{\phi_\infty},$$

$$(FF) \quad \sum_{\mathfrak{p}\in T} \frac{\log_r \mathrm{N}(\mathfrak{p})}{\sqrt{\mathrm{N}(\mathfrak{p})}-1} \leq \frac{1}{\phi_\infty},$$

where $1/\phi_\infty \in \mathbb{R}\cup\{+\infty\}$.

However in the case of asymptotically bad infinite global field, the sum $s(T) = \sum_{\mathfrak{p}\in T} \mathrm{N}\mathfrak{p}^{-1}$ can be infinite, even if the set of ramification is relatively small:

Theorem G. *There exists an infinite global field $\mathcal{K}$ (in both cases of number and function fields) without wild ramification such that $s(Dec(\mathcal{K})) = \infty$ and $\delta(Ram(\mathcal{K})) = 0$.*

One can improve this result a bit, giving a more precise information on the ramification locus: for any $\varepsilon > 0$ there is an infinite global field $\mathcal{K}$ satisfying the above properties, and such that $s(Ram(\mathcal{K})) \leq \varepsilon$.

The structure of the paper is as follows. First we recall basic facts concerning infinite global fields, which will be useful for the comprehension of the remainder parts. In §3 we prove Theorem A using class field towers. The following paragraph is devoted to the study of the deficiency and the proof of Theorem C. In §5, we estimate the defect of towers involved in Theorem A. After that, we consider the problem of prime decomposition in infinite global fields, which is central in our study and prove Theorem G. Finally, we prove Theorem B in the last section.

2. Basic facts on infinite global fields

In this paragraph we recall briefly some basic properties of asymptotically good infinite global fields which are essentially known but that we need to keep in mind for the following. For more details and complete proofs see [11], or [3],[4] for the the function field Galois case. In practice, if we want to construct an asymptotically good infinite global field, we ask for three conditions. First, the tower should be tamely ramified, or we should be able to control the wild ramification which is often the most difficult part. Second, the tower should be unramified outside of a finite set of primes. There are examples in the function field case, where $\phi_\infty > 0$ and where the set of ramification is infinite (in fact everywhere ramified, see [1]), but we do not know any example where the field is asymptotically good. And third, there should be at least one split prime (in the function fields case).

Controlling the wild ramification leads to specific calculations, as you can find in [2]. In the following, we will always avoid wild ramification. If there is no deep wild ramification, and if the field is Galois, then an asymptotically good tower has to be finitely ramified. We say that an infinite global field $\mathcal{K}$ is stepwise-Galois (almost-Galois in [21]) if there is a tower $\{K_i\}_{i\in\mathbb{N}}$ of global fields such that $\cup K_i = \mathcal{K}$ and K_{i+1}/K_i is Galois for every $i \in \mathbb{N}$.

Proposition 2.1. *Let $\mathcal{K}$ be an infinite number field (resp. function field) such that*

$$\inf_{K' finite} \# WRam(\mathcal{K}/K') = 0.$$

Then $\mathcal{K}$ is asymptotically good (resp. $\phi_\infty(\mathcal{K}) > 0$) if its ramification locus S is finite. Moreover, if $\mathcal{K}/K$ (for a global field K) is Galois, the converse is true. If $\mathcal{K} = \cup K_i$ is stepwise-Galois (for a tower $\{K_i\}_{i\in\mathbb{N}}$ of global fields), then $\mathcal{K}$ is asymptotically good (resp. $\phi_\infty(\mathcal{K}) > 0$) if and only if

$$\sum_{i=0}^{\infty} \frac{1}{n_{K_i}} \sum_{\mathfrak{p}\in Ram(K_{i+1}/K_i)} \log \mathrm{N}\mathfrak{p} < \infty,$$

where log *denote the base e (resp. base r) logarithm.*

Proof. This is a straightforward application of the Riemann-Hurwitz formula that can be essentially found in [4]. Let us prove the assertion concerning the stepwise-Galois case. Let S_i denote the ramification locus of the Galois extension K_{i+1}/K_i. Applying the Riemann-Hürwitz formula, we get:

$$\frac{g_i^*}{n_i} + \frac{1}{4\, n_i} \sum_{\mathfrak{p}\in S_i} \log \mathrm{N}\mathfrak{p} \le \frac{g_{i+1}^*}{n_{i+1}} \le \frac{g_i^*}{n_i} + \frac{1}{2\, n_i} \sum_{\mathfrak{p}\in S_i} \log \mathrm{N}\mathfrak{p}.$$

We obtain by induction

$$\frac{g_0^*}{n_0} + \frac{1}{4} \sum_{j=0}^{i} \sum_{\mathfrak{p}\in S_j} \frac{\log \mathrm{N}\mathfrak{p}}{n_j} \le \frac{g_{i+1}^*}{n_{i+1}} \le \frac{g_0^*}{n_0} + \frac{1}{2} \sum_{j=0}^{i} \sum_{\mathfrak{p}\in S_j} \frac{\log \mathrm{N}\mathfrak{p}}{n_j},$$

which concludes the proof. □

In particular, any infinite global field $\mathcal{K}$ containing a global field K such that $\mathcal{K}/K$ is unramified, satisfies $\phi_\infty > 0$. The following result explains why the split places are of particular interest in the study of the ϕ_q's.

Proposition 2.2. *Let $\mathcal{K}$ be an infinite global field, and let K be a global field contained in $\mathcal{K}$. Suppose that $\phi_\infty > 0$ and that a non-archimedean place $\mathfrak{p}$ of K splits completely in $\mathcal{K}/K$. Then $\phi_{\mathrm{N}\mathfrak{p}} > 0$. Moreover, if $\mathcal{K}$ is Galois, and if there is a $q \in A_f$ such that $\phi_q > 0$, then $\phi_\infty > 0$ and there is a global field $L \subset \mathcal{K}$ containing K and a non-archimedean place $\mathfrak{p}$ of K such that any place of L above $\mathfrak{p}$ is of norm q and splits completely in $\mathcal{K}/L$.*

Proof. For the sake of notation, let us prove it in the number fields case. If $\mathfrak{p}$ splits completely in $\mathcal{K}/K$ then, for any tower $\{K_i\}$ representing $\mathcal{K}$ such that $K_0 = K$, $\Phi_{\mathrm{N}\mathfrak{p}}(K_i) \ge n_{K_i}/[K_0 : \mathbb{Q}]$, which implies the first assertion. Suppose now that $\mathcal{K}/K$ is Galois, and that $\phi_q > 0$, for a prime power q. There is a place $\mathfrak{p}$ of K such that $\phi_{\mathfrak{p},q} > 0$, because $\sum_{\mathfrak{p}\in P(K)} \phi_{\mathfrak{p},q} = \phi_q$. Then $\phi_\infty \ge \phi_{\mathfrak{p},q}$ has to be non zero. All the places above $\mathfrak{p}$ in $\mathcal{K}$ have the same ramification index and inertia degree, which have to be both finite: indeed, if K' is a global field contained in $\mathcal{K}$, the ramification index e, the inertia degree f, and the number d of places above $\mathfrak{p}$ satisfy $e\, f = [K' : K]/d$, and this ratio is bounded over the subfields K' of $\mathcal{K}$ because of our assumption. Therefore all the places above $\mathfrak{p}$ have to split completely in $\mathcal{K}/L$ for some global field L. $\square$

3. On class field towers

3.1. **Unramified class field towers.** This section is devoted to the construction of infinite global fields using unramified class field towers. For definition and results concerning S-ramified T-split class field towers, see [12]. Let us recall first the construction: given a global field K, a prime number ℓ and a set T of places of K, let $H_\ell^T(K)$ denote the maximal unramified abelian ℓ-extension of K where all the places of T split completely. Construct a tower of field as follows: let K_0 be our global field K and T_0 a set of non archimedean places of K_0. For $i \geq 1$ put $K_i = H_\ell^{T_{i-1}}(K_{i-1})$, and let T_i be the set of places of K_i above T_{i-1}. The tower $\{K_i\}$ is called the ℓ-T_0-class field tower of K_0. The question of the finiteness of this tower remained open for years until Golod and Schafarevitch gave in the 1960's a criterion to prove that such a tower is infinite. This result in particular implies the following theorem:

Theorem 3.1. *[Tsfasman–Vlăduţ* [21] *(NF), Serre* [19] *, Niederreiter–Xing* [15] *(FF)] Let K/k be a cyclic extension of global fields of degree ℓ. Let $T(k)$ be a finite set of non archimedean places of k and $T(K)$ be the set of places above $T(k)$ in K. Suppose in the function field case that* $\gcd\{\ell, \deg \mathfrak{p}, \mathfrak{p} \in T(K)\} = 1$*. Let Q be the ramification locus of K/k. Let*

$$\begin{aligned}(FF) \quad C(T, K/k) =& \#T(k) + 2 + \delta_\ell + 2\sqrt{\#T(K) + \delta_\ell},\\ (NF) \quad C(T, K/k) =& \#T(K) - t_0 + r_1 + r_2 + \delta_\ell + 2 - \rho + \\ & 2\sqrt{\#T(K) + \ell(r_1 + r_2 - \rho/2) + \delta_\ell},\end{aligned}$$

where $\delta_\ell = 1$ if K contains the ℓ-root of unity, and 0 otherwise, t_0 is the number of principal ideals in $T(k)$, $r_1 = \Phi_{\mathbb{R}}(K)$, $r_2 = \Phi_{\mathbb{C}}(K)$ and ρ is the number of real places of k which become complex in K. Suppose that $\#Q \geq C(T, K/k)$. Then K admits an infinite unramified ℓ-$T(K)$-class field tower.

Remark that the assumption on the degree of the places in T in the function field case guaranties that there is no constant field extension in the tower. These unramified towers give us infinite global fields such that $\phi_{\mathrm{N}\mathfrak{p}} > 0$ for every $\mathfrak{p} \in T$, provided we can construct sufficiently ramified cyclic extensions. Even though this point can be made explicitly (see §5), we will use the Grunwald–Wang theorem.

3.2. **The Grunwald–Wang theorem.** Let S be a set of primes of a global field k, containing archimedean places in the number field case. Let k_S denote the maximal extension of k unramified outside of S, which is Galois over k. Put $G_S = Gal(k_S/k)$. We set

$$(NF) \qquad \mathbb{N}(S) = \{n \in \mathbb{N} \mid n \in \mathcal{O}_{k,S}^\times\}$$

in the number fields case, where $\mathcal{O}_{k,S} = \{a \in k \mid v_\mathfrak{p}(a) \geq 0 \text{ for any } \mathfrak{p} \notin S\}$, and let $\mathbb{N}(S)$ denote the set of numbers prime to the characteristic of k in the function fields case.

Let us recall that the exponent of a finite group $\#A$ is the smallest integer a such that $x^a = 1_A$ for every x in A.

Let us now state the Grunwald–Wang theorem, as we can find it in [14] in the case where S contains all the places of k.

Theorem 3.2 (Grunwald–Wang)**.** *Let S be a set of places of a global field k, containing the archimedean primes in the number field case, such that $\delta(S) = 1$, let $T \subset S$ be a finite set of primes of k and A be a finite abelian group of exponent a such that $\#A \in \mathbb{N}(S)$. Let $K_\mathfrak{p}/k_\mathfrak{p}$ be, for any $\mathfrak{p} \in T$, local abelian extensions such that $G(K_\mathfrak{p}/k_\mathfrak{p})$ can be imbedded in A. Then there is a global abelian extension K/k with Galois group A, unramified outside of S such that K has the given completions $K_\mathfrak{p}$ for any $\mathfrak{p} \in T$, with the exception of the special case (k, a, T) when the following four conditions hold:*

(i) $a = 2^r a', \quad r > 2$
(ii) *k is a number field*
(iii) *$k(\mu_{2^r})/k$ is not cyclic*
(iv) $\{\mathfrak{p} \in S \ \mid \mathfrak{p}$ *divides* 2 *and* $\mathfrak{p}$ *is not split in* $k(\mu_{2^r})/k\} \subset T$.

Let us remark that as T is finite, there is no special case for function fields (see [14]).

Proof. It is nothing but the proof of 9.2.3. of [14], where k is replaced by k_S. For the sake of completeness, let us recall it. Consider A as a trivial G_S-module. First recall the following. For any $\mathfrak{p} \in S$, we choose a k-embedding $k_S \to \bar{k}_\mathfrak{p}$ by chosing a place $\bar{\mathfrak{p}}$ of k_S above $\mathfrak{p}$. This induces the restriction map from $G(\bar{k}_\mathfrak{p}/k_\mathfrak{p})$ to $G_S = G(k_S/k)$ (whose image is the decomposition group of $\bar{\mathfrak{p}}$ over k). It induces the map

$$H^1(G_S, A) \to H^1(G(\bar{k}_\mathfrak{p}/k_\mathfrak{p}), A)$$

independent from the choice of the embedding.

$$H^1(k_S|k, A) \xrightarrow{res} \prod\nolimits_{\mathfrak{p} \in T} H^1(\bar{k}_\mathfrak{p}|k_\mathfrak{p}, A).$$

Let us prove the theorem now. We want to show that the map

$$\mathrm{Epi}(G_S, A) \to \prod_T \mathrm{Hom}(G(\bar{k}_\mathfrak{p}/k_\mathfrak{p}), A)$$

is onto, where $\mathrm{Epi}(G_S, A)$ denotes the onto morphisms from G_S to A. Let $\mathfrak{q}_1, ..., \mathfrak{q}_r$ be finite places of $S - T$ not dividing 2 in the number field case, and let

$$\psi_{\mathfrak{q}_i} : G(\bar{k}_{\mathfrak{q}_i}/k_{\mathfrak{q}_i}) \to A$$

be morphisms such that their images generate A. For any $\mathfrak{p} \in T$, $\psi_\mathfrak{p} : G(\bar{k}_\mathfrak{p}/k_\mathfrak{p}) \to A$ denotes the canonical map induced by the chosen embedding $G(K_\mathfrak{p}/k_\mathfrak{p}) \to A$. Let T' denote the set $T \cup \{\mathfrak{q}_1, ..., \mathfrak{q}_r\}$. The map

$$H^1(k_S|k, A) \xrightarrow{res} \prod\nolimits_{T'} H^1(\bar{k}_\mathfrak{p}|k_\mathfrak{p}, A)$$

is known to be onto in the case where $\delta(S) = 1$ and A is a trivial $G_S(k)$-module, with exception of the special case $(k, exp(A), T') = (k, exp(A), T)$ (see [14, 9.2.2]). An inverse image $\psi : G_S \to A$ of $(\psi_\mathfrak{p})_{\mathfrak{p} \in T'}$ in $H^1(G_S, A) = \mathrm{Hom}(G_S, A)$ realises the local extensions and is onto because of the choice of the $\psi_{\mathfrak{q}_i}$. □

3.3. **Application to infinite global fields.** Using this theorem, we can now prove the following important result:

Corollary 3.3. *Let P be a finite set of places of $\mathbb{Q}$ (resp. of $\mathbb{F}_r(t)$, containing at least one rational place). Then there is an asymptotically good infinite number field $\mathcal{K}$ (resp. infinite function field) such that all the places in P are totally split, and such that there exists a global field K of prime degree 2 such that $\mathcal{K}/K$ is unramified.*

Proof. Let Q be a finite set of places of non archimedean places of $\mathbb{Q}$ (resp. of $\mathbb{F}_r(t)$) such that $P \cap Q = \emptyset$, and $\#Q$ is big enough to satisfy the condition of theorem 3.1 for any quadratic extension of $\mathbb{Q}$ (resp. $\mathbb{F}_r(t)$).

Applying the theorem with $A = \mathbb{Z}/2\mathbb{Z}$, $T = P \cup Q$, we obtain a cyclic extension of degree 2 of $\mathbb{Q}$ (resp. de $\mathbb{F}_r(t)$) where all the places of Q, at least, are ramified, and such that P is totally split. Note that we can obtain the totally ramified local extensions of degree 2 by adjoining a root of Eisenstein separable polynomials. Then we use Theorem 3.1 to obtain an unramified 2-class field tower where all the places in P split completely. This tower does not have constants extension because P contains a rational place, and is asymptotically good because of the results of §2. □

We remark that we can choose a tower with $\phi_{\mathbb{R}}$ or $\phi_{\mathbb{C}}$ positive, choosing in the first step a real quadratic field or an imaginary one. In the corollary, we can also take towers of degree p different from the characteristic of our fields, but in that case, the norm of the places constituting Q must equal 1 modulo p so that the totally ramified local p-extensions exist.

3.4. **Proof of Theorem A.** In the function field case, we apply the corollary 3.3 with a set of places P containing at least one place of degree $\log_r t_i$ for any $i = 1, ..., n$, which gives directly the desired result. If we want to obtain such additional properties as in the number field case, we just follow the number field case proof.

In the number field case, we consider the set $P = \{p^{(1)}, \ldots, p^{(k)}\}$ of prime numbers $p^{(1)}, \ldots, p^{(k)}$ which divide one of the t_i.

Lemma 3.4. *There is a finite extension L of $\mathbb{Q}$ having the following properties:*

(i) *for any $i = 1 \ldots n$ there exists $\mathfrak{p} \in P(L)$ such that $\mathrm{N}\mathfrak{p} = t_i$,*
(ii) *for any $\mathfrak{p} \in P(L)$, if there exists $p \in P$ such that $p|\mathrm{N}\mathfrak{p}$ then there is $i \in \{1, ..., n\}$ such that $\mathrm{N}\mathfrak{p} = t_i$.*

Assuming this lemma, let us apply the corollary 3.3 to the set of places P. It gives us an infinite number field $\mathcal{K} = \cup K_i$ represented by the tower of number fields $\{K_i\}$, $K_0 = \mathbb{Q}$, unramified over K_1, where all the places of P split completely. Then the tower $\{L.K_i\}_{i \in \mathbb{N}}$ is unramified over $L.K_1$, therefore asymptotically good, and all the places of L above P are totally split in $L.K_i$. So we obtain the field $L.\mathcal{K}$ having the desired properties.

Proof. We construct the tower $L_0 = \mathbb{Q} \subset L_1, ... \subset L_k = L$ using the Grunwald–Wang theorem. Let us begin by ordering the set T. Write

$$T = \left\{ t_1^{(1)}, ..., t_{i_1}^{(1)}; t_1^{(2)}, ..., t_{i_2}^{(2)}; t_1^{(k)}, ..., t_{i_k}^{(k)} \right\}$$

such that $p^{(r)}$ divide all $t_1^{(r)}, ..., t_{i_r}^{(r)}$ but no others. Put $d_j^{(r)} = \log_{p^{(r)}} t_j^{(r)}$.

Recall first the following properties of local fields guarantees that we can use the Grunwald–Wang theorem (see [18, §III.5, Theorem 2]):

Lemma 3.5. *Let K be a local field with a finite residue field k, and let k'/k be a cyclic extension of k. Then there exists an unramified cyclic extension K'/K such that k' is the residue field of K'.*

Given a global field K and a place $\mathfrak{p}$, the completion $K_{\mathfrak{p}}$ of K at $\mathfrak{p}$, an integer $n > 0$, there exists an unramified cyclic extension K' of $K_{\mathfrak{p}}$, of degree n (meaning that $\mathfrak{p}$ is totally inert in it).

Using the Grunwald–Wang theorem, let us construct by induction a tower $L_0 \subset \cdots \subset L_k$ having the following properties:

(i) in L_{r+1}/L_r, all the places above $p^{(j)}$ are totally split for $j \neq r+1$,
(ii) there exists an extension L_r^1 of L_r of degree i_{r+1}, contained in L_{r+1} such that all the places above $p^{(r+1)}$ are totally split.
(iii) in L_{r+1}/L_r^1 there are $[L_{r+1} : L_0]/(i_{r+1} d_j^{(r+1)})$ places of norm $t_j^{(r+1)}$ for any j.

L_0 is given. Let us suppose that L_r/L_0 has been constructed. Put $m_r = [L_r : L_0]$. Let L_r^1/L_r be an extension of degree i_{r+1} where all the places above any $p^{(i)}$ are totally split. In L_r^1 one has got $i_{r+1}m_r$ places

$$p_{1,1}^{(r+1)}, \dots, p_{1,m_r}^{(r+1)}, \dots, p_{i_{r+1},1}^{(r+1)}, \dots, p_{i_{r+1},m_r}^{(r+1)}$$

above $p^{(r+1)}$, put into i_{r+1} packets containing m_r places each.

Consider then successive cyclic extensions of prime degree (so we do not fall in the special case). We obtain an extension L_{r+1}/L_r^1 such that:

(i) all the places above $p^{(i)}$ are totally split for any $i \neq r+1$,
(ii) above $p_{j,l}^{(r+1)}$, for any j and any l, there are $[L_{r+1}/L_r^1]/d_j^{(1)}$ places of norm $t_j^{(r+1)}$.

In order to deal with j^{th} packet, we ask for the existence of a cyclic extension of prime degree dividing $t_j^{(r+1)}$, in which all the places above the $p_{j,l}^{(r+1)}$ are totally inert for any l, and in which all the other pointed places are totally split, until we obtain places of norm $t_j^{(r+1)}$.

By induction, we obtain therefore $L_k = L$, satisfying by the construction the two given properties. This ends the proof of the lemma and that of Theorem A. $\square$

4. Deficiency and optimal fields

The deficiency of an infinite global field is defined as the difference between the two sides of the basic inequality. Tsfasman and Vlăduţ proved that it is related to the limit distribution of the zeroes of the zeta function. This distribution admits a continuous density, and the deficiency is its value at 0 (see [21]). The study of the deficiency is therefore not only simpler than the study of the invariants itself, but also gives us some very interesting knowledge about the field.

4.1. **Proof of Theorem C.** The first but not the least result concerning the deficiency is Theorem C. Let us prove it in the case of $\delta^{(1)}$, the proof being exactly the same in the two other cases (we replace log by $\log_r$ in the function field case). We begin by treating the non-archimedean terms:

Lemma 4.1. *For any prime number p, any $m \in \mathbb{N}$, and any infinite global fields $\mathcal{K} \subset \mathcal{L}$,*

$$(NF) \qquad \sum_{k=1}^{m} \frac{k\phi_{p^k}(\mathcal{L}) \log p}{\sqrt{p^k} - 1} \leq \sum_{k=1}^{m} \frac{k\phi_{p^k}(\mathcal{K}) \log p}{\sqrt{p^k} - 1},$$

$$(FF) \qquad \sum_{k=1}^{m} \frac{k\phi_{r^k}(\mathcal{L})}{\sqrt{r^k} - 1} \leq \sum_{k=1}^{m} \frac{k\phi_{r^k}(\mathcal{K})}{\sqrt{r^k} - 1}.$$

Remark that we could replace $\log p/(p^{k/2} - 1)$ by any decreasing function. We can also prove the same inequality for the $\phi_{\mathfrak{p},q}$ along the same lines.

Proof. Let $\mathcal{K} \subset \mathcal{L}$ be two infinite number fields and p a prime number. Recall that, for any m, we have:

$$A_m(\mathcal{K}) := \sum_{k=1}^{m} k\phi_{p^k}(\mathcal{K}) \geq A_m(\mathcal{L}).$$

Using Abel transform on $\sum_{k=1}^{m} \frac{k\phi_{p^k}(\mathcal{K}) \log p}{\sqrt{p^k}-1}$, we get:

$$\begin{aligned}\sum_{k=1}^{m} \frac{k\phi_{p^k}(\mathcal{K}) \log p}{\sqrt{p^k}-1} =& A_m(\mathcal{K}) \frac{\log p}{\sqrt{p^m}-1} \\ &+ \sum_{k=1}^{m-1} A_k(\mathcal{K}) \left(\frac{\log p}{\sqrt{p^k}-1} - \frac{\log p}{\sqrt{p^{k+1}}-1} \right),\end{aligned}$$

from which we deduce the lemma. □

Taking the limit, we obtain, for two infinite global fields $\mathcal{K} \subset \mathcal{L}$:

$$(NF) \quad \sum_{k=1}^{\infty} \frac{k\phi_{p^k}(\mathcal{L}) \log p}{\sqrt{p^k}-1} \leq \sum_{k=1}^{\infty} \frac{k\phi_{p^k}(\mathcal{K}) \log p}{\sqrt{p^k}-1},$$

$$(FF) \qquad \sum_{k=1}^{\infty} \frac{k\phi_{r^k}(\mathcal{L})}{\sqrt{r^k}-1} \leq \sum_{k=1}^{\infty} \frac{k\phi_{r^k}(\mathcal{K})}{\sqrt{r^k}-1}.$$

Let us now consider the archimedean terms:

Lemma 4.2. *For two infinite number fields $\mathcal{K} \subset \mathcal{L}$ and two positive real numbers α_1, α_2 such that $2\alpha_1 \geq \alpha_2$, one has:*

$$\alpha_1 \phi_{\mathbb{R}}(\mathcal{L}) + \alpha_2 \phi_{\mathbb{C}}(\mathcal{L}) \leq \alpha_1 \phi_{\mathbb{R}}(\mathcal{K}) + \alpha_2 \phi_{\mathbb{C}}(\mathcal{K}).$$

Proof. Let us consider two towers of number fields $\mathcal{K} = \cup K_i$, $\mathcal{L} = \cup L_i$, with $K_i \subset L_i$, and let us write the relations between archimedean places of K_i and L_i.

We forget the indices and denote by K and L these fields. Let $P_{\mathbb{R}}(K)$ denote the set of real places of K. Recall that the complex places are always split in L/K, giving birth to $[L:K]$ complex places of L and each real place v of K decomposes into $\Phi_{v,\mathbb{R}}(L/K)$ real places and $\Phi_{v,\mathbb{C}}(L/K)$ complex places of L, so that $\Phi_{v,\mathbb{R}}(L/K) + 2\Phi_{v,\mathbb{C}}(L/K) = [L:K]$.

Therefore one has

$$\Phi_{\mathbb{R}}(L) = \sum_{v \in P_{\mathbb{R}}(K)} \Phi_{v,\mathbb{R}}(L/K),$$

and

$$\Phi_{\mathbb{C}}(L) = [L:K]\Phi_{\mathbb{C}}(K) + \sum_{v \in P_{\mathbb{R}}(K)} \Phi_{v,\mathbb{C}}(L/K).$$

Let α_1 and α_2 be real numbers such that $2\alpha_1 \geq \alpha_2$. Then

$$\begin{aligned}\alpha_1\Phi_{\mathbb{R}}(L)+\alpha_2\Phi_{\mathbb{C}}(L) &= \alpha_1\sum_{v\in P_{\mathbb{R}}(K)}\Phi_{v,\mathbb{R}}(L/K)+\alpha_2[L:K]\Phi_{\mathbb{C}}(K)\\ &+\alpha_2\sum_{v\in P_{\mathbb{R}}(K)}\Phi_{v,\mathbb{C}}(L/K)\\ &\leq \alpha_1\sum_{v\in P_{\mathbb{R}}(K)}\left(\Phi_{v,\mathbb{R}}(L/K)+2\,\Phi_{v,\mathbb{C}}(L/K)\right)\\ &+\alpha_2[L:K]\Phi_{\mathbb{C}}(K)\\ &\leq [L:K]\big(\alpha_1\Phi_{\mathbb{R}}(K)+\alpha_2\Phi_{\mathbb{C}}(K)\big).\end{aligned}$$

As $g_L \geq [L:K]g_K$, one obtains, for $g_K > 0$:

$$\alpha_1\frac{\Phi_{\mathbb{R}}(L)}{g_L}+\alpha_2\frac{\Phi_{\mathbb{C}}(L)}{g_L}\leq \alpha_1\frac{\Phi_{\mathbb{R}}(K)}{g_K}+\alpha_2\frac{\Phi_{\mathbb{C}}(K)}{g_K}.$$

Taking the limit we get the lemma. □

These two lemmas (the first in the function field case) give us directly, taking for α_1 and α_2 corresponding to the archimedean terms in the deficiency, the decreasing property of the map $\mathcal{K} \mapsto 1-\delta^{(i)}$, for any i, which ends the proof.

4.2. **Optimal Fields.** We say that an infinite global field is optimal for $\delta^{(i)}$ if its deficiency $\delta^{(i)}$ is equal to 0. In the function field case, when r is a square, there are examples of optimal fields. Different constructions can be used, such as tower of modular curves or recursive towers. In the number field case, or even in the function field case where r is not a square, the question whether optimal towers exist or not remains open. Let us give first some properties that should be satisfied by optimal fields. We will prove then that most of the infinite number fields constructed using the class field theory cannot be optimal.

Proposition 4.3. *Let $\mathcal{K}$ be an optimal infinite number field (resp. function field) for $\delta^{(i)}$. If $\mathfrak{p}$ is a non archimedean place of $\mathbb{Q}$ (resp. $\mathbb{F}_r(t)$) such that $\phi_{\mathfrak{p},q}=0$ for any q, there is no infinite global field $\mathcal{L}$ contained in $\mathcal{K}$ such that $\phi_{\mathfrak{p},q}(\mathcal{L})>0$.*

Proof. Let us prove it for $i=1$. Suppose there exists such an extension $\mathcal{L}$. The proof of Theorem C shows that, for any prime number p,

$$\delta_p(\mathcal{L}) := \sum_{k=1}^{\infty}\frac{k\phi_{p^k}(\mathcal{L})\log p}{\sqrt{p^k}-1}\geq \sum_{k=1}^{\infty}\frac{k\phi_{p^k}(\mathcal{K})\log p}{\sqrt{p^k}-1}=\delta_p(\mathcal{K}).$$

Because of Lemma 4.2, one has

$$1-\Big(\sum_{p\neq \ell}\delta_p(\mathcal{L})+\delta_\ell(\mathcal{L})+\alpha_1\phi_{\mathbb{R}}(\mathcal{L})+\alpha_2\phi_{\mathbb{C}}(\mathcal{L})\Big)\leq 1-(1+\delta_\ell(\mathcal{L})),$$

where α_1 and α_2 are the archimedean coefficients involved in the deficiency $\delta^{(1)}$. Therefore $\delta^{(1)}(\mathcal{L})\leq -r\phi_{\ell^r}\log\ell/(\ell^{r/2}-1)<0$, which leads to a contradiction. □

Proposition 4.4. *Let $\mathcal{K}$ be an optimal infinite Galois number field (resp. function field) for $\delta^{(i)}$. Let $\mathfrak{p}$ be a place of $\mathbb{Q}$ (resp. $\mathbb{F}_r(t)$). Suppose there is a prime number p and an integer d such that $\phi_{\mathfrak{p},p^d}(\mathcal{K})>0$. Then $\mathcal{K}$ does not contain any infinite global subfield $\mathcal{L}$ such that $\phi_{\mathfrak{p},p^m}(\mathcal{L})>0$, $m\neq d$.*

Proof. We show the result for $\delta^{(1)}$. Let $\mathcal{L} \subset \mathcal{K}$ such that one $\phi_{\mathfrak{p},p^m} > 0$, and $m < d$ (the other case cannot happen because $\mathcal{K}$ is Galois). We prove that $\mathcal{L}$ is a subfield of $\mathcal{K}$ whose deficiency is strictly smaller than the deficiency of $\mathcal{K}$. Indeed, for any prime $q \neq p$, we have $\delta_q(\mathcal{L}) \geq \delta_q(\mathcal{K})$. As for $\mathfrak{p}$, one has

$$\sum_{k \leq d} k\phi_{p^k}(\mathcal{L}) \geq d\phi_{p^d}(\mathcal{K}),$$

since $\mathcal{L} \subset \mathcal{K}$, we deduce

$$\sum_{k \leq d} \frac{k\phi_{p^k}(\mathcal{L})}{\sqrt{p^k} - 1} > \sum_{k \leq d} \frac{k\phi_{p^k}(\mathcal{L})}{\sqrt{p^d} - 1} \geq \frac{d\phi_{p^d}(\mathcal{K})}{\sqrt{p^d} - 1}.$$

But this is impossible, because $\mathcal{K}$ is optimal. □

Let us prove now that fields constructed with class field theory are mainly not optimal. Consider the deficiency without GRH in the number field case (the GRH-deficiency result can easily be deduced), and call it δ.

Let $\mathcal{Q}$ denote $\mathbb{Q}$ in the number field case (resp. $\mathbb{F}_r(t)$ in the function field case). Let $K/\mathcal{Q}$ be a finite Galois extension. Let $S(K)$ be a finite set of non-archimedean places of K. Let P be a minimal set of non-archimedean places of K such that:

(i) P is disjoint from $S(K)$,
(ii) P is stable under the action of the Galois group of $K/\mathcal{Q}$,
(iii) For any non-archimedean places $\mathfrak{p}$ and $\mathfrak{q}$ of K outside of $S(K)$, such that $\mathrm{N}\mathfrak{p} < \mathrm{N}\mathfrak{q}$, if $\mathfrak{q}$ belongs to P, $\mathfrak{p}$ belongs to P.
(iv) In the number fields case,

$$(NF) \quad \sum_{\mathfrak{p} \in P} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p} - 1} > g^*(K) - \frac{\ell\alpha_2}{2},$$

where $\alpha_2 = \gamma + \log 2\pi$, and in the function fields case,

$$(FF) \quad \sum_{\mathfrak{p} \in P} \frac{\log_r \mathrm{N}\mathfrak{p}}{\sqrt{\mathrm{N}\mathfrak{p}} - 1} > g^*(K).$$

The sum $\sum_{\mathfrak{p} \in P} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}-1}$ (resp. $\sum_{\mathfrak{p} \in P} \frac{\log_r \mathrm{N}\mathfrak{p}}{\sqrt{\mathrm{N}\mathfrak{p}}-1}$) taken over all non archimedean places of K, is divergent; therefore such P exists. One constructs it taking consecutively all the non archimedean places of K outside of $S(K)$ (and those obtained by applying the Galois action), until the sum becomes bigger than the right hand side term. Let $\mathfrak{p}_0$ be a prime of maximal norm in P, and p_0 the finite place of $\mathcal{Q}$ under it. Put

$$(NF) \quad \alpha(K,S) := \frac{\ell \log p_0}{g^*(K)} \left(\frac{1}{\mathrm{N}\mathfrak{p}_0 - 1} - \frac{1}{\mathrm{N}\mathfrak{p}_0^{\ell} - 1} \right)$$

$$(FF) \quad \alpha(K,S) := \frac{\ell \log_r \mathrm{N}p_0}{g^*(K)} \left(\frac{1}{\mathrm{N}\mathfrak{p}_0^{\frac{1}{2}} - 1} - \frac{1}{\mathrm{N}\mathfrak{p}_0^{\frac{\ell}{2}} - 1} \right)$$

Let us now state the theorem.

Theorem 4.5. *Let $K/\mathcal{Q}$ be a cyclic extension of prime degree ℓ ramified exactly at S a finite set of non archimedean places of $\mathcal{Q}$. Let $\mathfrak{q}, \mathfrak{q}' \notin S$ be two places of K (of relatively prime degrees in the function field case). Let $\mathcal{K} = K_{\emptyset}(\ell)$ be the maximal unramified ℓ-extension of K (resp. let $\mathcal{K} = K_{\emptyset}^{\{\mathfrak{q},\mathfrak{q}'\}}(\ell)$ be the maximal unramified ℓ-extension of*

K where $\mathfrak{q}$ and $\mathfrak{q}'$ split completely). Suppose that $\#S \geq 2 + 2\ell + 2\sqrt{\ell^2+\ell+1}$ (resp. $\#S \geq 5 + \varepsilon + 2\sqrt{3\,\ell+\varepsilon}$— see Theorem 3.1 for notation).

Then $\mathcal{K}$ is not optimal. Moreover its deficiency satisfies $\delta(\mathcal{K}) \geq \alpha(K,S)$.

Remark that it is also the case of any field containing $\mathcal{K}$, in particular it is the case of $\mathbb{Q}_S$ (the maximal extension of $\mathbb{Q}$ unramified outside of S) in the number field case. The condition on S guarantees that $g(K) > 1$ and that $\mathcal{K}$ is infinite because of Theorem 3.1.

Proof. Suppose $\delta(\mathcal{K}) < \alpha$. Let log denote as usual the logarithm with base e in the number field case, and with base r in the function field case. Prove first that there is a finite place with no contribution in δ.

Lemma 4.6. *There is a place $\mathfrak{p}_1 \in P$ such that, for any $m > 0$, $\phi_{p_1^m} = 0$, where $p_1 = \mathfrak{p}_1 \cap \mathbb{Q}$.*

Proof. Suppose that for every place $\mathfrak{p}$ in P, there is an $m_\mathfrak{p} > 0$ such that $\phi_{\mathfrak{p},\mathrm{N}\mathfrak{p}^{m_\mathfrak{p}}} > 0$. As $\mathcal{K}$ is Galois, it is also the case for all the places above $\mathfrak{p} \cap \mathcal{Q}$. The set S is by our hypothesis sufficiently large so that, for any place $\mathfrak{r}$ not in S, the maximal extension $\mathcal{K}^\mathfrak{r}$ of K, contained in $\mathcal{K}$, with all the places above $\mathfrak{r}$ in K totally split, is not finite. Let us prove first that, for any place $\mathfrak{p}$ of P, $m_\mathfrak{p} = 1$. If we only want to prove the non optimality, it is sufficient to apply the proposition 4.4.

Let us suppose that $m_\mathfrak{r} > 1$ for a place $\mathfrak{r}$. All the places above $\mathfrak{r}$ in K have the same norm, they are unramified in any tower contained in $\mathcal{K}$, therefore the difference of their contributions to the deficiency of $\mathcal{K}^\mathfrak{r}$ and their contributions to that of $\mathcal{K}$ is given by

$$(NF) \quad \frac{\ell \log \mathrm{N}\mathfrak{r}}{g^*(K)(\mathrm{N}\mathfrak{r}-1)} - \frac{\ell \log \mathrm{N}\mathfrak{r}}{g^*(K)(\mathrm{N}\mathfrak{r}^{m_\mathfrak{r}}-1)}.$$

$$(FF) \quad \frac{\ell \log_r \mathrm{N}\mathfrak{r}}{g^*(K)(\sqrt{\mathrm{N}\mathfrak{r}}-1)} - \frac{\ell \log_r \mathrm{N}\mathfrak{r}}{g^*(K)(\sqrt{\mathrm{N}\mathfrak{r}^{m_\mathfrak{r}}}-1)}.$$

It cannot exceed α, because the deficiency belongs to $[0,1]$. Moreover, this quantity is decreasing in p, increasing in m, so it is sufficient to verify it for the biggest prime in our set, and for the smallest m (meaning $m = \ell$).

Because of the definition of α, this is not satisfied and therefore we have a contradition. Consequently $m_\mathfrak{p} = 1$ for any place $\mathfrak{p}$ of P.

Then for any $\mathfrak{p} \in P$,

$$\phi_{\mathrm{N}\mathfrak{p}} = \frac{\Phi_{\mathrm{N}\mathfrak{p}}(K)}{g^*(K)}.$$

Indeed, all the places above $\mathfrak{p} \cap \mathcal{Q}$ in K are totally split and the equality follows.

We have

$$(NF) \quad \sum_q \phi_q \frac{\log q}{q-1} + \delta_\infty \geq \frac{1}{g^*(K)} \sum_{\mathfrak{p}\in P} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}-1} + \frac{\ell\alpha_2}{2\,g^*(K)} > 1,$$

$$(FF) \quad \sum_q \phi_q \frac{\log_r q}{\sqrt{q}-1} \geq \frac{1}{g^*(K)} \sum_{\mathfrak{p}\in P} \frac{\log_r \mathrm{N}\mathfrak{p}}{\sqrt{\mathrm{N}\mathfrak{p}}-1} > 1,$$

where $\delta_\infty = \alpha_1\phi_\mathbb{R}(\mathcal{L}) + \alpha_2\phi_\mathbb{C}(\mathcal{L})$ is the contribution to the deficiency of the archimedean places. Indeed $\delta_\infty \geq \frac{1}{2}\alpha_2\phi_\infty$ and $\phi_\infty = \ell\, g^*(K)^{-1}$. This contradicts Tsfasman–Vlăduţ Basic Inequalities (see §1). □

Consider now the maximal unramified extension of K such that $\mathfrak{p}_1$ splits completely. The contribution of $\mathfrak{r}_1 = \mathfrak{p}_1 \cap \mathcal{Q}$ to the deficiency of this extension is $\frac{\ell \log p_1}{g^*(K)(\mathrm{N}\mathfrak{p}_1 - 1)} >$

$\alpha(K,S)$, whereas it is zero in $\mathcal{K}$. This leads to a contradiction since this infinite global fields would have a strictly negative deficiency. □

5. An Effective Version of Theorem A

The aim of this paragraph is to produce an example of infinite number field with at least n positive invariants having a deficiency δ_n as small as we can make it.

In order to achieve our goal, we will apply Theorem 3.1. Take for the set P the first prime n numbers greater than 2: $P = \{3, 5, 7, ..., p_n\}$. We will take for K a quadratic extension, $\mathbb{Q}(\sqrt{q_1 ... q_{r_n} r})$, where r_n is the smallest integer satisfying the conditions of Theorem 3.1 with $s = 2n$:

$$r_n = 1 + \lfloor n + 5 + 2\sqrt{2n + 4} + 1\rfloor.$$

Then $r_n = n + 2\sqrt{2n} + c_n$ where c_n is a bounded sequence. Remark that for imaginary quadratic extensions, we could take a smaller r_n, but that does not change anything asymptotically.

Now let us choose r. We want the p_i to be split (otherwise they are inert and we lose a $\sqrt{n}$ factor). Take for q_i, $i \leq r_n$, the r_n primes following p_n in the prime progression, and choose r in the following way so that the p_i are split:

Lemma 5.1. *There exists $r \in \{0, ..., 2\prod p_i \prod q_j - 1\}$ prime to $2, p_i, q_j$ for any i, j such that the p_i are split in $K/\mathbb{Q}$.*

Proof. Recall that the p_i split in $K/\mathbb{Q}$ if $r\prod q_j$ is a square mod p_i for any i, i.e. if for any $i \leq n$

$$\left(\frac{r\prod q_j}{p_i}\right) = \left(\frac{r}{p_i}\right)\prod_j \left(\frac{q_j}{p_i}\right) = 1.$$

When the p_i and the q_j, $j \leq r_n$ have been chosen, r has to satisfy

$$\forall i \in \{1, ..., n\} \qquad \left(\frac{r}{p_i}\right) = \prod_j \left(\frac{q_j}{p_i}\right).$$

We choose a non zero solution modulo p_i for any i. We lift it to an odd integer of $\{0, ..., 2\prod p_i \prod q_j - 1\}$ equal to 1 modulo q_j for any j. If this integer is different from 1, it splits into a product of prime numbers, which are distinct from the p_i, q_i and from 2. We choose it squarefree, which is possible because if r has a square factor p^2, r/p^2 suits us too. Note that r can be equal to 1, and that its factors ramify in $K/\mathbb{Q}$ but these factors do not influence the validity of Theorem 3.1. □

We remark that we could probably improve the upper bound on r. But it would only improve the involved constants in the following proposition. For instance we took r in the most trivial way. We could have computed the number of integers smaller than $2.\prod p_i$ satisfying the conditions on the Legendre's symbols and compared it to the number of integers prime to all the q_i and 2.

From now on we denote by K_n the field K corresponding to n and an r obtained from Lemma 5.1.

Proposition 5.2. *The genus g_{K_n} of K_n satisfies*

$$g_{K_n} \leq g_n, \text{ where } \quad g_n = \frac{3}{2} n \log n + o(n \log n),$$

$$g_{K_n} \geq g'_n, \text{ where } \quad g'_n = \frac{1}{2} n \log n + o(n \log n).$$

Proof.

$$g_{K_n} = \frac{1}{2}\sum_j \log q_j + \frac{1}{2}\log r + \frac{1}{2}\varepsilon \log 2,$$

where $\varepsilon = 2$ if 2 is ramified and 0 otherwise. One has $\log r \le \sum_i \log p_i + \sum_j \log q_j + \log 2$. Therefore

$$\log r \le \sum_{p \le q_{r_n}} \log p.$$

Thus

$$g_{K_n} \le \sum_{p \le q_{r_n}} \log p - \frac{1}{2}\sum_{p \le p_n} \log p + \log 2 = g_n.$$

From the analytic number theory (see [7]), we know that the function $\vartheta(x) = \sum_{p \le x} \log p$ is asymptotically equal to $x + o(x)$, as $x \to \infty$, and that the n^{th} odd prime number p_n satisfies $p_n = n \log n + o(n \log n)$. Thus

$$\sum_{p \le p_n} \log p = n \log n + o(n \log n).$$

As $r_n = n + 2\sqrt{2n} + \mathcal{O}(1)$, one has $q_{r_n} = (2n + 2\sqrt{2n}) \log n + o(n \log n)$, q_{r_n} being the $(n + r_n)^{\text{th}}$ odd prime number. Therefore:

$$\sum_{p \le q_{r_n}} \log p = (2n + 2\sqrt{2n}) \log n + o(n \log n),$$

and we get

$$g_n = \frac{3}{2} n \log n + o(n \log n).$$

The second inequality can be obtained in the same way, using $r \ge 1$. □

Apply now Theorem 3.1 to $K_n/\mathbb{Q}$. The field K_n admits an infinite unramified class field tower $\mathcal{K}_n = (K_n^i)_{i \ge 0}$, with $K_n^0 = K_n$, where the p_i split completely in $\mathcal{K}_n/\mathbb{Q}$ for $i \le n$.

Proposition 5.3. *We have $\phi_\infty(\mathcal{K}_n) = O((n \log n)^{-1})$.*

Proof. For any i, we have $n_{K_n^i} = 2^{i+1}$ and $g_{K_n^i} = 2^i g_{K_n}$ as the tower is unramified, so that

$$\frac{n_{K_n^i}}{g_{K_n^i}} = \frac{2}{g_{K_n}}$$

and, taking the limit, we get

$$\phi_\infty(\mathcal{K}_n) = \frac{2}{g_{K_n}}.$$

Therefore $\phi_\infty(\mathcal{K}_n) \le \frac{2}{g_n'}$ and the result follows from Proposition 5.2. □

Let us evaluate now the sum involved in the deficiency. Because the p_i are totally split and because of Proposition 5.2, we have $\phi_{p_i} = \frac{2}{g_{K_n}}$. Therefore we need to study the following sums:

Proposition 5.4. *Let $S_n = \sum_{i=1}^n \frac{\log p_i}{\sqrt{p_i} - 1}$, $S_n' = \sum_{i=1}^n \frac{\log p_i}{p_i - 1}$. Then*

$$S_n \sim 2\sqrt{n \log n}$$
$$S_n' \sim \log (n \log n).$$

Proof. Let us introduce the function $\Lambda : \mathbb{N} \to \mathbb{R}$ defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p \text{ is a prime number,} \\ 0 & \text{otherwise.} \end{cases}$$

As $\vartheta(x) = \sum_{n \leq x} \Lambda(n)$, we have

$$\sum_{n \leq x} \frac{\Lambda(n)}{\sqrt{n}} = \frac{\vartheta(x)}{\sqrt{x}} + \int_2^x \frac{\vartheta(t)dt}{2t\sqrt{t}}.$$

Because $\vartheta(x) = x + o(x)$, the first term in the sum is equal to $\sqrt{x} + o(\sqrt{x})$ as $x \to \infty$. Concerning the second, it is a divergent integral. As $\frac{\vartheta(x)}{2x\sqrt{x}} \sim \frac{1}{2\sqrt{x}}$, we find:

$$\int_2^x \frac{\vartheta(t)dt}{2t\sqrt{t}} \sim \int_2^x \frac{dt}{2\sqrt{t}} \sim \sqrt{x}$$

As $p_n \sim n \log n$ and as S_n is divergent, we get

$$S_n \sim \sum_{i \leq n} \frac{\Lambda(i)}{\sqrt{i}} \sim 2\sqrt{n \log n}.$$

The same computation, using

$$\sum_{p \leq x} \frac{\log p}{p-1} \sim \log x, \text{ as } x \to \infty,$$

leads to the estimate of S'_n. □

Corollary 5.5. *For $i = 1, 2$, $\delta_n^{(i)} \leq 1 - \varepsilon_n^{(i)}$, where*

$$\varepsilon_n^{(1)} \sim \frac{8}{3\sqrt{n \log n}} \text{ and,}$$
$$\varepsilon_n^{(2)} \sim \frac{4}{3n}.$$

Proof. We will prove it in the case of $\delta^{(1)}$ the second case being the same. As $\sum_q \phi_q \frac{\log q}{\sqrt{q}-1} \geq \frac{2}{g_{K_n}} S_n$, we get

$$\delta_n \leq 1 - \frac{2}{g_{K_n}} S_n - (\gamma + \log 8\pi)\phi_{\mathbb{C}}(\mathcal{K}_n) - (\gamma/2 + \pi/4 + \log \sqrt{8\pi})\phi_{\mathbb{R}}(\mathcal{K}_n).$$

Then the result come from our estimates of the genus and of S_n, and from the fact that archimedean factor are negligible. □

6. On Prime Decomposition in Infinite Global Fields

The class field theory produces examples of infinite global fields with finitely many split places. But as we can see from the last paragraph, it seems impossible to use it directly to obtain asymptotically good infinite global fields having infinitely many split places, and thus infinitely many positive invariants. The question of prime decomposition in infinite global field is central, and the analytic theory (in particular the Cebotarev density theorem) allows us to understand a bit what happens in such field. Let us recall a corollary of this theorem:

Proposition 6.1. *Let K be a global field and let T be the set of places of K which split completely in a separable extension L/K of degree n such that, in the function field case, K and L have the same field of constants. Then $\bar{\delta}(T) \leq \frac{1}{n}$.*

Corollary 6.2. *Let K be a global field, $\mathcal{K}/K$ an infinite global field and T the set of places of K which split completely in $\mathcal{K}$. Then $\delta(T)$ exists and is equal to 0.*

6.1. **Proof of Proposition 6.1.** Although this proposition is well known, we recall the way it can be proven because of Corollary 6.7. Let us state the Cebotarev density theorem. Let K/L be a Galois extension, $G = Gal(L/K)$ and $\sigma \in G$. Put

$$P_{L/K}(\sigma) = \left\{ \mathfrak{p} \in P(K) \mid \quad \exists \mathfrak{P} | \mathfrak{p} \in P(L) \quad \sigma = \left(\frac{L/K}{\mathfrak{P}} \right) \right\},$$

and $C(\sigma) = \left\{ \tau^{-1} \sigma \tau \mid \ \tau \in G \right\}$.

Theorem 6.3. *(Cebotarev) The set $P_{L/K}$ has a Dirichlet density. It is given by*

$$\delta(P_{L/K}(\sigma)) = \frac{\# C(\sigma)}{\# G}.$$

Proof. See [13, 7.13.4] □

Recall the following lemma:

Lemma 6.4. *Let L/K be a separable extension of global fields and N be the Galois closure of L/K. Then $\mathfrak{p}$ splits completely in L/K if and only if it splits completely in N/K.*

Proof. See [13]. □

For any separable extension of global fields L/K, define $P_{L/K}$ as the set of unramified places $\mathfrak{p}$ of K admitting a place $\mathfrak{P}$ of L above it, whose inertia degree over $\mathfrak{p}$ is 1.

Lemma 6.5. *Using notation and assumptions of the last lemma, put $G = Gal(N/K)$ and $H = Gal(N/L)$, One has :*

$$P_{L/K} = \bigsqcup_{C(\sigma) \cap H \neq \emptyset} P_{N/K}(\sigma).$$

Proof. [13, 7.13.5] □

Lemma 6.6. *Under this hypothesis, we have $\delta(P_{L/K}) \geq \frac{1}{n}$ with the equality if and only if L/K is Galois. In addition, we have $\delta(P_{L/K}) \leq 1 - \dfrac{n-1}{\# G}$.*

Proof. For the first point see [13, 7.13.6]. Concerning the second inequality, we start from:

$$\delta(P_{L/K}) = \delta \left(\bigsqcup_{C(\sigma) \cap H \neq \emptyset} P_{N/K}(\sigma) \right) \leq \sum_{C(\sigma) \cap H \neq \emptyset} \frac{\# C(\sigma)}{\# G}.$$

Saying that $C(\sigma) \cap H$ is not empty is the same as saying that σ belongs to the set

$$C(H) = \bigcup_{\tau \in G} \tau H \tau^{-1},$$

and in this case $C(\sigma)$ lies in $C(H)$. Therefore

$$\delta(P_{L/K}) \leq \frac{1}{\# G} \# C(H).$$

An easy basic exercise in group theory (making G act on $C(H)$ by conjugation) shows that

$$\# C(H) \leq 1 + \# G - [G : H],$$

from which we deduce the second inequality. □

In particular the set of places of K having no degree 1 places above them in L is of positive Dirichlet density. Therefore we have the following result:

Corollary 6.7. *Let $\mathcal{K}$ be an infinite number field (resp. function field). Let U be the set of primes of $\mathbb{Q}$ (resp. $\mathbb{F}_r(t)$) such that $\phi_{\mathfrak{p},\mathrm{N}\mathfrak{p}} = 0$. Then U contains a set of positive Dirichlet density.*

Proof of Proposition 6.1: Consider the Galois closure N/K of L/K, and let T be the set of places of K totally split in L/K, or equivalently in N/K. The last lemma implies that

$$\bar{\delta}(T) \leq \delta(P_{N/K}) = \frac{1}{[N:K]} \leq \frac{1}{n}.$$

To obtain the corollary, we just let $n \to +\infty$. □

6.2. **Proof of Proposition F.** We will prove in fact the following result:

Proposition 6.8. *Let $\mathcal{K}$ be an infinite global field and $\{K_i\}_{i\in\mathbb{N}}$ be a tower representing $\mathcal{K}$. Let T be the set of primes of K_0 totally split in $\mathcal{K}/K_0$. Then*

$$(NF) \quad \sum_{\mathfrak{p}\in T} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}-1} \leq \frac{n_{K_0}^2}{\phi_\infty}(1-\frac{1}{2}(\log 2\pi+\gamma)\phi_\infty).$$

$$(NF-GRH) \quad \sum_{\mathfrak{p}\in T} \frac{\log \mathrm{N}\mathfrak{p}}{\sqrt{\mathrm{N}\mathfrak{p}}-1} \leq \frac{n_{K_0}^2}{\phi_\infty}(1-\frac{1}{2}(\log 8\pi+\gamma)\phi_\infty)$$

$$(FF) \quad \sum_{\mathfrak{p}\in T} \frac{\log_r \mathrm{N}\mathfrak{p}}{\sqrt{\mathrm{N}\mathfrak{p}}-1} \leq \frac{n_{K_0}^2}{\phi_\infty},$$

where the right hand terms can be infinite.

Proof. Let us make the proof in the number field case, the function field one and the GRH one being deduced using the appropriate inequalities. Suppose that $\mathcal{K}$ is asymptotically good. Because of Tsfasman–Vlăduţ Basic Inequalities (see §1), one has

$$\sum_q \frac{\phi_q \log q}{q-1} + (\gamma/2 + \log 2\sqrt{\pi})\phi_\mathbb{R} + (\gamma + \log 2\pi)\phi_\mathbb{C} \leq 1.$$

As $\phi_\mathbb{R} + 2\phi_\mathbb{C} = \phi_\infty$, we obtain the inequality

$$\sum_q \frac{\phi_q \log q}{q-1} \leq 1 - \frac{1}{2}(\log 2\pi + \gamma)\phi_\infty.$$

Let T be the set $Dec(\mathcal{K}/K_0)$ of places of K_0 totally split in $\mathcal{K}$. For any q in A, let T_q be the set of places of T of norm q, and let $\#T_q = N_q(T)$. Remark that, because of the Riemann-Hurwitz formula, the ratio n_{K_i}/g_{K_i} is decreasing to ϕ_∞. Therefore, for q such that T_q is not empty,

$$\frac{\Phi_q(K_n)}{g(K_n)} \geq \frac{[K_n:K_0]}{g_{K_n}} = \frac{1}{n_{K_0}}\frac{n_{K_n}}{g_{K_n}} \geq \frac{1}{n_{K_0}}\phi_\infty.$$

Therefore

$$\phi_q \geq \frac{1}{n_{K_0}}\phi_\infty.$$

Then

$$\sum_{\mathfrak{p}\in T}\frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}-1}=\sum_q N_q(T)\frac{\log q}{q-1}\leq \sum_{T_q\neq\emptyset} n_{K_0}\frac{\log q}{q-1}\leq \frac{n_{K_0}^2}{\phi_\infty}\sum_{T_q\neq\emptyset}\phi_q\frac{\log q}{q-1},$$

$$\text{Thus}\qquad \sum_{\mathfrak{p}\in T}\frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}-1}\leq \frac{n_{K_0}^2}{\phi_\infty}\sum_q \phi_q\frac{\log q}{q-1}\leq \frac{n_{K_0}^2}{\phi_\infty}(1-\frac{1}{2}(\log 2\pi+\gamma)\phi_\infty).$$

□

6.3. **Asymptotically bad Case.** One question arises naturally. We know that the Dirichlet density of the set of the totally split places is zero. When the field is asymptotically good, we have also proved that $s(T)$ has to be bounded. Can $s(T)$ be infinite in asymptotically bad towers? Can the set T be infinite? A negative answer would be a disaster, because of the link between the positive ϕ_q and the split places. We do not know at the moment how to construct asymptotically good fields having infinitely many positive ϕ_q or split places, and this seems to be a very difficult problem because of the restricted ramification condition. However, we can construct such examples in the asymptotically bad case:

Theorem 6.9. *For any global field K, there is an infinite Galois global field $\mathcal{K}$ containing K such that infinitely many places split completely in $\mathcal{K}$, $\sum_{\mathfrak{p}\in Dec(\mathcal{K})}\mathrm{N}\mathfrak{p}^{-1}$ is infinite and such that it has no wild ramification over the Galois closure of K (in some given separable closure of K) and $\delta(Ram(\mathcal{K}))$ is zero.*

In the number field case (and also likely in the function field case) we could ask in addition that $s(Ram(\mathcal{K}))\leq\epsilon$ (meaning: for any given $\epsilon>0$, there is an infinite number field such that...) using a result of Gras (see [6, Corollary V.2.4.7]) and doing the same as we will do in our proof. But he only proved it for the number fields case, that is why we will use once again Grunwald–Wang Theorem, which allows us to prove the theorem in both cases of number and function fields.

Proof. Recall that, for a set E of places of a global field K, we put $s(E):=\sum_{\mathfrak{p}\in E}\mathrm{N}\mathfrak{p}^{-1}$, eventually infinite. Let K_0 be the Galois closure of K. Consider, for the simplicity of notation, the number field case. Let S_0 (resp. D_0) be the ramification locus of $K_0/\mathbb{Q}$ (resp. the splitting locus of $K_0/\mathbb{Q}$). We will extend this notation to other indices than 0 by putting $S_n=Ram(K_n/\mathbb{Q})$ and $D_n=Dec(K_n/\mathbb{Q})$. Let T_0 be a finite set of places of $\mathbb{Q}$ totally split in $K_0/\mathbb{Q}$ such that $s(T_0)\geq 1$. Such a set exists since $\delta(D_0)>0$ by the Cebotarev density theorem. Let ℓ be a prime in T_0 (or a prime number different from the characteristic of K_0 in the function field case).

We construct by induction the tower of fields K_n, Galois over $\mathbb{Q}$ and the set $T_n\subset D_n$ with $s(T_n)\geq n+1$ having the desired properties in the following way. Suppose that we have constructed a field K_{n-1}, $n\geq 1$, Galois over Q, and a set T_{n-1} as above. Let K_n^* be a ℓ-extension of K_{n-1} unramfied outside of $D_{n-1}(K_{n-1})$, such that all the places above T_{n-1} in K_{n-1} split completely. Let S_n^* denotes the ramified places of K_n^*/K_{n-1}. Then we consider the maximal ℓ-extension K_n of K_{n-1}, unramified outside of the places of S_n^* and their conjugates by the Galois action, where the places of $T_{n-1}(K_{n-1})$ split completely in K_n. This extension is non trivial, and moreover K_n is Galois over $\mathbb{Q}$. To see that, let us take a morphism σ from K_n to a separable closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. $\sigma(K_n)$ is a ℓ-extension of $\sigma(K_{n-1})=K_{n-1}$ unramified outside of the conjugates of places of S_n^*, where the places above T_{n-1} split completely, therefore it is contained in K_n. For the set T_n, let us take a subset of D_n containing T_{n-1} and such that $s(T_n)\geq n+1$.

If we denote by T the splitting locus of $\mathcal{K}/\mathbb{Q}$ and put $S = \cup S_n$ for the ramification locus of $\mathcal{K}/\mathbb{Q}$, then we have $s(T) \geq s(T_n)$ for any n and therefore $s(T) = \infty$. In addition, S_{n+1} is contained in D_n for any $n \geq 0$. Therefore $\cup_{m>n} S_m \subset D_n$ for any $n \geq 0$. So we have $\delta(S) = 0$. Indeed, suppose that $\bar{\delta}(S) > 1/\ell^{n_0}$ for a given $n_0 > 0$. Then $\bar{\delta}(S - \cup_{i \leq n_0} S_i) > 1/\ell^{n_0}$, but this is impossible because $S - \cup_{i \leq n_0} S_i \subset D_{n_0}$ and $\delta(D_{n_0}) \leq 1/\ell^{n_0}$. ℓ is not ramified in $\mathcal{K}/\mathbb{Q}$, thus there is no wild ramification over K_0. □

To conclude, we remark that there are infinite global field having no split prime: this is for example the case for the maximal abelian extension of $\mathbb{Q}$.

6.4. **Abelian Case.** Let us give now the negative answer to a natural conjecture of Michel Balazard formulated at Poncelet Laboratory seminar (Moscow). In the case of $\mathbb{Q}$ and in that of cyclotomic extensions, we have the following beautiful result of Norton [16], which can be obtained using Brün–Titchmarch and Siegel–Walfisz Theorems:

Proposition 6.10 (Norton)**.** *Let q be a prime number and a be an integer non divisible by q. Let*

$$I_x = \{p \text{ prime number} \mid p = a \mod q, p \leq x\}.$$

Then there exists a constant M independent from q, a such that

$$\left| \sum_{p \in I_x} \frac{1}{p} - \frac{1}{q-1} \log\log x \right| \leq M.$$

It seems to be hard to obtain good generalisations of these two theorems to general global fields (see [5] for the number fields case). However Michel Balazard put forward the following conjecture:

Conjecture 6.11. *Let K be a global field. Let L/K be an abelian extension of degree n. Then there exists a constant M depending only on K such that*

$$\left| \sum_{\mathfrak{p} \in Dec(L/K),\ \mathrm{N}\mathfrak{p} \leq x} \frac{1}{\mathrm{N}\mathfrak{p}} - \frac{1}{n} \log\log x \right| \leq M_K.$$

Unfortunately, this conjecture is not true, at least this formulation, as we will see it producing an example of pro-cyclic infinite global field which violates it.

Proposition 6.12. *Both in the case of number fields and in the case of function fields, there exists an infinite Galois global field $\mathcal{K}/K$ of pro-cyclic Galois group such that $\sum_{\mathfrak{p} \in Dec(\mathcal{K})} \frac{1}{\mathrm{N}\mathfrak{p}} = \infty$.*

Proof. Grunwald-Wang Theorem will allow us to produce such an example. We will give the construction in the number field case for simplicity of notation. Let us take a prime number $p_0 = 3$ and consider a cyclic extension K_0 of the field K (K being a finite separable extension of $\mathbb{Q}$) of degree p_0.

Let us take then a finite set T_0 of primes $\mathfrak{p}_1^0, \ldots, \mathfrak{p}_n^0$ of the ground field $\mathbb{Q}$ totally split in K_0 such that $\sum_{\mathfrak{p} \in T_0} \frac{1}{\mathrm{N}\mathfrak{p}} \geq 1$.

Take for the field K_1 a cyclic extension of $\mathbb{Q}$ of prime degree $p_1 > p_0$ which does not divide the degree of K, such that T_0 is totally split in K_1. Thus $K_1.K_0/K$ is a cyclic extension of degree $p_1 p_0$, where T_0 is totally split. Take for T_1 a set of places of $\mathbb{Q}$ totally split in $K_1.K_0$, containing T_0 and satisfying $\sum_{\mathfrak{p} \in T_1} \frac{1}{\mathrm{N}\mathfrak{p}} \geq 1$.

Suppose that we have constructed in this way a field $K_n.K_{n-1}\ldots K_0/K$ of degree $p_n \ldots p_0$ together with a set T_n of split places, satisfying $\sum_{\mathfrak{p} \in T_n} \frac{1}{\mathrm{N}\mathfrak{p}} \geq n$. We construct K_{n+1}, T_{n+1} from $K_n.K_{n-1}\ldots K_0, T_n$ in the same way that we have constructed K_1, T_1

from K_0, T_0. Then we obtain our tower by induction. This tower satisfies the hypotheses of the proposition. □

Corollary 6.13. *Conjcture 6.11 is false, even for cyclic extensions.*

Proof. Suppose it is true for a given K. Consider the tower of the last proposition starting from K, and let T be its splitting locus. Taking the limit as $n \to +\infty$, for any x we have:

$$\sum_{\mathfrak{p}\in T,\ \mathrm{N}\mathfrak{p}\leq x} \frac{1}{\mathrm{N}\mathfrak{p}} \leq M_K.$$

Taking the limit as $x \to +\infty$ we get a contradiction. □

7. Proof of Theorem B

First let us recall the following definitions and notation from Galois cohomology. Let G be a profinite group. The cohomological dimension $cd\,G$ of G is the smallest integer n, if it exists, such that

$$H^q(G, A) = 0 \text{ for every } q > n$$

and every torsion G-module A. If no such integer n exists, then $cd\,G = \infty$. For a pro-p-group, $cd\,G \leq n$ if and only if $H^{n+1}(G, \mathbb{Z}/p\mathbb{Z}) = 0$ (see [14, §III.3]). For example, any non trivial finite group has an infinite cohomological dimension.

We will prove now the following result which implies Theorem B.

Theorem 7.1. *Let p be an odd prime number and let S_0 be a set of primes congruent to* 1 *modulo p, such that $\mathbb{Q}_{S_0}(p)$ — the maximal p-extension of $\mathbb{Q}$ unramified outside of S_0— is of cohomological dimension* 2*. Let P be a finite set of prime numbers. Then there exists a finite set of primes S containing S_0 and not containing p such that, for any $m > 1$ and any prime number ℓ, $\phi_{\ell^m}(Q_S(p)) = 0$, and that, for any $\ell \in S \cup P$, $\phi_\ell(\mathbb{Q}_S(p)) = 0$.*

Remark that the only primes which can be ramified in a p-extension are congruent to $0, 1$ modulo p, thus the condition on S_0 is natural.

As a corollary, we deduce directly Theorem B. Indeed, Labute (see [10]) gave some examples having the cohomological dimension 2 property, for example

$$cd\,Gal(\mathbb{Q}_{\{7,19,61,163\}}(3)/\mathbb{Q}) = 2,$$

so one can apply Theorem 7.1 to the set P. The resulting infinite number field $\mathcal{K} = \mathbb{Q}_S(3)$ is asymptotically good because it is tamely ramified and unramified outside of a finite set of primes. In addition, the splitting prime numbers ℓ in $\mathcal{K}$ satisfy $\phi_\ell > 0$ because of Proposition 2.2, and no other prime number belongs to $\mathrm{PSupp}(\mathcal{K}/\mathbb{Q})$ because of Theorem 7.1.

Let us now prove Theorem 7.1. In order to do that, recall a result due to Schmidt ([17, Theorems 2.2 and 2.3]). Denote by $G_S(p)$ the Galois group of the maximal p-extension of $\mathbb{Q}$ unramified outside of S.

Theorem 7.2 (Schmidt)**.** *Let p be an odd prime number and let S be a finite set of prime numbers congruent to* 1 *modulo p.*

(i) *Suppose that $G_S(p) \neq 1$ and that $cd\,G_S(p) \leq 2$. Then $cd\,G_S(p) = 2$, and for any $\ell \in S$, $\mathbb{Q}_S(p)$ realises the maximal p-extension of $\mathbb{Q}_\ell$.*

(ii) *Suppose $cd\,G_S(p) = 2$. Then, if $\ell \notin S$ is an other prime number congruent to* 1 *modulo p, which is not totally split in $\mathbb{Q}_S(p)/\mathbb{Q}$. Then $cd\,G_{S\cup\{\ell\}}(p) = 2$.*

Consider now an odd prime number p, S_0 a set of primes congruent to 1 modulo p, such that $cd\, G_{S_0} = 2$ and a finite set $P = \{p_1, ..., p_r\}$ of prime numbers. Let us begin by proving two lemmas:

Lemma 7.3. *If $cd\, G_S(p) = 2$, then for any $m > 1$, for any prime number ℓ not belonging to S, $\phi_{\ell^m}(\mathbb{Q}_S(p)) = 0$.*

Proof. If $\mathrm{cd} G_S(p) = 2$, then for any closed subgroup H of $G_S(p)$, $H^n(H) = 0$ for any $n > 2$ ([20, §I prop. 21′]). Suppose that $\phi_{\ell^m} > 0$ for $\ell \notin S$ and $m > 1$. Then the Frobenius of $\mathbb{F}_{\ell^m}$ which is of finite order, can be lifted in the decomposition subgroup in $G_S(p)$ of any place over ℓ to an element of finite order. The subgroup generated by it is finite therefore of infinite cohomological dimension, so it does not satisfy the above property. This leads to a contradiction. □

Lemma 7.4. *If $cd\, G_S(p) = 2$, then for any m, for any prime ℓ in S, $\phi_{\ell^m}(\mathbb{Q}_S(p)) = 0$.*

Proof. We apply Theorem 7.2, which tells us that the inertia index of ℓ in $\mathbb{Q}_S(p)$ is infinite (indeed, there exists an unramified infinite algebraic extension of $\mathbb{Q}_\ell$). □

Let us prove now Theorem 7.1.

Proof. Consider $\mathbb{Q}_{S_0}(p)$ and its splitting locus T. We will complete the set S_0 by a finite number of places in order to make the ϕ_p vanish one after the other. We will construct a set S of primes having the following properties:

(i) any $q \in S$ is congruent to 1 modulo p,
(ii) $\mathbb{Q}_S(p)$ is asymptotically good and $cd\, G_S(p) = 2$,
(iii) p_i, $i \leq r$, is not totally split in $\mathbb{Q}_S(p)$.

The key ingredient of the proof is the existence of a tamely ramified p-extension, unramified at T, such that the p_i are inert. Such an extension exists because of the precise version of Grunwald–Wang Theorem.

Indeed, because of the Grunwald–Wang theorem, there is a cyclic extension K of degree p, unramfied outside of $(P(\mathbb{Q}) - T)$, such that p and the p_i are inert. Put $S = S_0 \cup \mathrm{Ram}(K)$. Thus no p_i is totally split in $\mathbb{Q}_S(p)$.

As the only primes which can be ramified in a p-extension are congruent to $0, 1$ modulo p, S satisfies the first condition. Moreover $\mathbb{Q}_S(p)$ is asymptotically good, because it is tamely and finitely ramified. In order to prove (ii), we have to prove that one can apply Theorem 7.2 (ii) to all the places in $\mathrm{Ram}(K) = \{s_1, ..., s_m\}$. We have to verify that s_i is not totally split in $\mathbb{Q}_{S_0 \cup \{s_0\} \cdots \cup \{s_{i-1}\}}(p)$. As this field contains $\mathbb{Q}_{S_0}$, and as s_i is not totally split in it since T is unramified, s_i is not totally split in $\mathbb{Q}_{S_0 \cup \{s_0\} \cdots \cup \{s_{i-1}\}}(p)$. Thus we can apply Theorem 7.2 (ii) to all the set $\mathrm{Ram}(K)$ and obtain (ii).

From the first lemma, we deduce that the $\phi_{p_i^m}(\mathbb{Q}_S)$ are all zero and we have proven the theorem. □

Let us conclude this section by the following remark: in order to prove Theorem B we could have first considered a ℓ-extension $K/\mathbb{Q}$ where all the primes in P are inert and its ramification locus $\mathrm{Ram}(K)$, and then extended this set to a set S such that $cd\, G_S(p) = 2$ (see [17, 7.3]). Using this, the ramification locus and thus its deficiency may be much bigger.

Acknowledgment

The author would like to thank Michael Tsfasman for all he has done as his advisor, Christian Maire for the time he spent answering his questions and for his advice concerning §7, Alexey Zykin for all of his comments on the first version of this paper and the referee for his numerous suggestions.

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