ON THE EXTENSION OF HÖLDER MAPS WITH VALUES IN SPACES OF CONTINUOUS FUNCTIONS

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Abstract. We study the isometric extension problem for Hölder maps from subsets of any Banach space into c_0 or into a space of continuous functions. For a Banach space X, we prove that any α-Hölder map, with $0 < \alpha \leq 1$, from a subset of X into c_0 can be isometrically extended to X if and only if X is finite dimensional. For a finite dimensional normed space X and for a compact metric space K, we prove that the set of α's for which all α-Hölder maps from a subset of X into $C(K)$ can be extended isometrically is either $(0,1]$ or $(0,1)$ and we give examples of both occurrences. We also prove that for any metric space X, the described above set of α's does not depend on K, but only on finiteness of K.

1. Introduction - Notation

If (X,d) and (Y,ϱ) are metric spaces, $\alpha \in (0,1]$ and $K > 0$, we will say that a map $f : X \to Y$ is α-Hölder with constant K (or in short (K,α)-Hölder) if

$$\forall x, y \in X, \quad \varrho(f(x), f(y)) \leq K d(x,y)^\alpha.$$

Let us now recall and extend the notation introduced by Naor in [13]. For $C \geq 1$, $\mathcal{B}_C(X,Y)$ will denote the set of all $\alpha \in (0,1]$ such that any (K,α)-Hölder function f from a subset of X into Y can be extended to a (CK,α)-Hölder function from X into Y. If $C = 1$, such an extension is called an isometric extension. When $C > 1$, it is called an isomorphic extension. If a (CK,α)-Hölder extension exists for all $C > 1$, we will say that f can be almost isometrically extended. So, let us define:

$$\mathcal{A}(X,Y) = \mathcal{B}_1(X,Y), \quad \mathcal{B}(X,Y) = \bigcup_{C\geq 1} \mathcal{B}_C(X,Y), \text{ and } \widetilde{\mathcal{A}}(X,Y) = \bigcap_{C>1} \mathcal{B}_C(X,Y).$$

The study of these sets goes back to a classical result of Kirszbraun [10] asserting that if H is a Hilbert space, then $1 \in \mathcal{A}(H,H)$. This was extended by Grünbaum and Zarantonello [5] who showed that $\mathcal{A}(H,H) = (0,1]$. Then the complete description of $\mathcal{A}(L^p, L^q)$ for $1 < p, q < \infty$ relies on works by Minty [12] and Hayden, Wells and Williams [6] (see also the book of Wells and Williams [14] for a very nice exposition of the subject). More recently, K. Ball [1] introduced a very important notion of non linear type or cotype and used it to prove a general

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extension theorem for Lipschitz maps. Building on this work, Naor [13] improved the description of the sets $\mathcal{B}(L^p, L^q)$ for $1 < p, q < \infty$.

In this paper, we concentrate on the study of $\mathcal{A}(X,Y)$ and $\widetilde{\mathcal{A}}(X,Y)$, when X is a Banach space and Y is a space of converging sequences or, more generally, a space of continuous functions on a compact metric space. This can be viewed as an attempt to obtain a non linear version of the results of Lindenstrauss and Pelczyński [11] and later of Johnson and Zippin ([8] and [9]) on the extension of linear operators with values in $C(K)$ spaces.
So let us denote by c the space of all real converging sequences equipped with the supremum norm and by c_0 the subspace of c consisting of all sequences converging to 0. If K is a compact space, $C(K)$ denotes the space of all real valued continuous functions on K, equipped again with the supremum norm.
In section 2, we show that if X is infinite dimensional and Y is any separable Banach space containing an isomorphic copy of c_0, then $\widetilde{\mathcal{A}}(X,Y)$ is empty. On the other hand, we prove that $\mathcal{A}(X,c_0) = (0,1]$, whenever X is finite dimensional.
In section 3, we show that for any finite dimensional space X, $\widetilde{\mathcal{A}}(X,c) = (0,1]$ and $\mathcal{A}(X,c)$ contains $(0,1)$. Then the study of the isometric extension for Lipschitz maps turns out to be a bit more surprising. Indeed, we give an example of a 4-dimensional space X such that $\mathcal{A}(X,c) = (0,1)$. To our knowledge, this provides the first example of Banach spaces X and Y such that $\mathcal{A}(X,Y)$ is not closed in $(0,1]$ and also such that $\mathcal{A}(X,Y) \neq \widetilde{\mathcal{A}}(X,Y)$. On the other hand, we show that if the unit ball of a finite dimensional Banach space is a polytope, then $\mathcal{A}(X,c) = (0,1]$.
Finally, we prove in section 4, that c is the only $C(K)$ space that one needs to consider as the image space in the study of the isometric extension problem. More precisely, we show that for every infinite compact metric space K and every metric space X, $\mathcal{A}(X,c) = \mathcal{A}(X,C(K))$.

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2. Maps into c_0

It is well known that for any metric space (X,d), $\mathcal{A}(X,\mathbb{R}) = (0,1]$. Indeed, if M is a subset of X and $f\colon M \to \mathbb{R}$ is a (K,α)-Hölder function, then a (K,α)-Hölder extension g of f on X is given for instance by the inf-convolution formula:

$$\forall x \in X, \quad g(x) = \inf\{f(u) + K(d(u,x))^\alpha \colon u \in M\}.$$

It follows immediately that $\mathcal{A}(X,\ell_\infty) = (0,1]$, where ℓ_∞ is the space of all real bounded sequences equipped with the supremum norm. Now, since there is a 2-Lipschitz retraction from ℓ_∞ onto c_0 (see for instance [2, page 14]), it is clear that for any metric space X, $\mathcal{B}_2(X,c_0) = (0,1]$. Our first result shows that the difference between the isometric and isomorphic extension problems which is revealed in [13] is extreme when c_0 is the image space. More precisely:

Theorem 2.1. *Let X be an infinite dimensional normed vector space and Y be a separable Banach space containing an isomorphic copy of c_0. Then*

$$\widetilde{A}(X,Y)=\emptyset.$$

Proof. By a theorem of R.C. James [7], Y contains almost isometric copies of c_0. So, since we are studying the almost isometric extension problem, we may as well assume that there is a closed subspace Z of Y which is isometric to c_0. Let (e_n) be the isometric image in Z of the canonical basis of c_0 and (e_n^*) be the Hahn-Banach extensions to Y of the corresponding coordinate functionals (this sequence is included in the unit sphere of Y^*). Since Y is separable, there is a subsequence $(e_{n_k}^*)_{k\geq 1}$ which is weak*-converging to some y^* in the unit ball of Y^*.

On the other hand, by a theorem of Elton and Odell [4], there exists $\varepsilon>0$ and a sequence $(x_k)_{k\geq 1}$ in X such that:

$$\forall k\ \ \|x_k\|=1-\varepsilon \ \text{ and } \ \forall k\neq l\ \ \|x_k-x_l\|\geq 1.$$

Let now f be defined by $f(x_k)=(-1)^k e_{n_k}$. This is clearly a $(1,\alpha)$-Hölder function for any α in $(0,1]$. Let $\delta>0$ such that $(1+\delta)(1-\varepsilon)^\alpha<1$ and $\eta=1-(1+\delta)(1-\varepsilon)^\alpha>0$. Assume that f can be extended at 0 into a $(1+\delta,\alpha)$-Hölder function g with $g(0)=y$. Then, for any even k, $e_{n_k}^*(y)\geq\eta$ and for any odd k, $e_{n_k}^*(y)\leq-\eta$. This is in contradiction with the fact that $(e_{n_k}^*)$ is weak*-converging.

□

We will now solve the extension problem for Hölder maps from a finite dimensional space into c_0. Our result is.

Theorem 2.2. *If X is a finite dimensional normed vector space, then*

$$\mathcal{A}(X,c_0)=(0,1].$$

Proof. Let $\alpha\in(0,1]$, $M\subset X$ and $f\colon M\to c_0$ be a (K,α)-Hölder function. We may assume that $K=1$ and that M is closed. It is enough to show that for any $x_0\in X\setminus M$, f can be extended into a $(1,\alpha)$-Hölder function g on $M\cup\{x_0\}$ and we will assume that $x_0=0$.

Since X is finite dimensional, we can cover the unit sphere of X with $B_1,...,B_k$, balls of radius 1/4 and define

$$C_i=\{y\in X\setminus\{0\}\colon \frac{y}{\|y\|}\in B_i\}.$$

Let now $x,y\in C_i$ so that $\|x\|\geq\|y\|$. We have

$$\|x-y\|\leq\Big\|x-x\frac{\|y\|}{\|x\|}\Big\|+\|y\|\Big\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\Big\|\leq\|x\|-\frac{1}{2}\|y\|. \tag{2.1}$$

We denote $I = \{i\colon 1 \le i \le k \text{ and } C_i \cap M \neq \emptyset\}$. Since X is finite dimensional, for each i in I, we can pick x_i in $\overline{C_i} \cap M$ such that for any $x \in C_i \cap M$, $\|x\| \ge \|x_i\|$. Then, it follows from the inequality (2.1) that

$$\forall i \in I \ \ \forall x \in C_i \cap M,\ \|x - x_i\| \le \|x\| - \frac{1}{2}\|x_i\| \le \|x\|.$$

Let us now pick $\varepsilon > 0$ such that $\varepsilon < \frac{1}{2} dist(0,M)^\alpha$. Then

$$\exists N \in \mathbb{N}\ \forall n > N\ \forall i \in I :\ |f(x_i)(n)| < \varepsilon.$$

We will now choose $g(0) = (u(n))_{n\ge 1}$.
Since, in the metric space $(X, \|\ \|^\alpha)$, $\mathbb{R}$-valued contractions can be extended into contractions, we can pick $(\eta_n)_{n\ge1}$ in ℓ_∞ so that

$$\forall n \in \mathbb{N}\ \forall x \in M,\ |f(x)(n) - \eta_n| \le \|x\|^\alpha.$$

For $n \le N$, we set $u(n) = \eta_n$.
For $n > N$, let $\delta_n \in \{-1, 1\}$ be the sign of η_n. Now we set

$$u(n) = \delta_n \min\{|\eta_n|, \max_{i\in I} |f(x_i)(n)|\}.$$

Note that since I is finite and each $f(x_i) \in c_0$, we have that $g(0) = (u(n))_{n\ge1} \in c_0$.
Next we check that for all $x \in M$ and all $n > N$, $|f(x)(n) - u(n)| \le \|x\|^\alpha$. So let $x \in M$ and $i_0 \in I$ such that $x \in C_{i_0} \cap M$. We have four cases:

1) If $|f(x)(n)| \le |u(n)|$, then

$$|f(x)(n) - u(n)| \le 2\varepsilon \le \|x\|^\alpha.$$

2) If $|f(x)(n)| > |u(n)|$, $\operatorname{sgn}(f(x)(n)) = \delta_n$, and $|u(n)| = |\eta_n|$ then, by the definition of η_n:

$$|f(x)(n) - u(n)| \le \|x\|^\alpha.$$

3) If $|f(x)(n)| > |u(n)|$, $\operatorname{sgn}(f(x)(n)) = \delta_n$, and $|u(n)| = \max_{i\in I}|f(x_i)(n)| \ge |f(x_{i_0})(n)|$, then

$$\begin{aligned}|f(x)(n) - u(n)| &= |f(x)(n)| - |u(n)| \\ &\le |f(x)(n) - f(x_{i_0})(n)| + |f(x_{i_0})(n)| - |u(n)| \\ &\le \|x - x_{i_0}\|^\alpha \le \|x\|^\alpha.\end{aligned}$$

4) If $|f(x)(n)| > |u(n)|$ and $\operatorname{sgn}(f(x)(n)) \neq \delta_n$, then

$$|f(x)(n) - u(n)| = |f(x)(n)| + |u(n)| \le |f(x)(n)| + |\eta_n| = |f(x)(n) - \eta_n| \le \|x\|^\alpha.$$

□

Remark 2.3. The proof is much simpler in the case $\alpha = 1$. Indeed it is enough to set $u(n) = 0$ for $n > N$. Then, for $x \in M$, pick $i_0 \in I$ such that $x \in C_{i_0} \cap M$. Thus, the inequality (2.1) implies that for all $n > N$:

$$|f(x)(n) - u(n)| = |f(x)(n)| \leq |f(x)(n) - f(x_{i_0})(n)| + \varepsilon \leq \|x - x_{i_0}\| + \varepsilon$$
$$\leq \|x\| - \frac{1}{2}\|x_{i_0}\| + \varepsilon \leq \|x\|.$$

3. Maps into c

We now consider the isometric and almost isometric extension problems for Hölder maps from a normed vector space into c. If X is infinite dimensional, this question is settled by Theorem 2.1. Therefore, throughout this section, X will denote a finite dimensional normed vector space. The study of the almost isometric extensions is then rather simple. For this purpose, we recall that, for $\lambda > 1$, a Banach space Y is said to be a $\mathcal{L}^\infty_\lambda$ space if every finite dimensional subspace of Y is contained in a finite dimensional subspace F of Y which is λ-isomorphic to $\ell_\infty^{\dim F}$ (namely, there is an isomorphism T from F onto $\ell_\infty^{\dim F}$ such that $\|T\|\,\|T^{-1}\| \leq \lambda$).

Proposition 3.1. *Let X be a finite dimensional normed vector space and Y be a Banach space which is a $\mathcal{L}^\infty_\lambda$ space for any $\lambda > 1$. Then*

$$\widetilde{A}(X, Y) = (0, 1].$$

In particular, for every compact space K,

$$\widetilde{A}(X, C(K)) = (0, 1].$$

Proof. Let M be a closed subset of X and $f \colon M \to Y$ be a $(1, \alpha)$-Hölder map. We start with the following Lemma.

Lemma 3.2. *For any $x \in X \setminus M$ and any $\varepsilon > 0$, f admits a $(1 + \varepsilon, \alpha)$-Hölder extension to $M \cup \{x\}$.*

Proof. If M is compact and $\delta > 0$, we pick a δ-net $\{x_1, .., x_n\}$ of M and a finite dimensional subspace F of Y, containing $f(x_1), .., f(x_n)$ such that F is $(1 + \delta)$-isomorphic to some ℓ_∞^m. Then, it follows from the introductory remarks of section 2, that there exists $y \in F$ such that for all $1 \leq i \leq n$, $\|f(x_i) - y\| \leq (1 + \delta)\|x_i - x\|^\alpha$. If δ was chosen small enough, then for any $z \in M$, $\|f(z) - y\| \leq (1 + \varepsilon)\|z - x\|^\alpha$.
For a general M and a fixed $x \in X \setminus M$, we apply the compact case to the restriction of f to $M \cap KB_X$, for K big enough and where B_X denotes the closed unit ball of X.

□

We now finish the proof of Proposition 3.1. Let $(x_n)_{n\geq 1}$ be a dense sequence in $X \setminus M$. For a given $\varepsilon > 0$, we pick $(\varepsilon_n)_{n\geq 1}$ in $(0, 1)$ so that $\prod_{n\geq 1}(1+\varepsilon_n) < 1+\varepsilon$. It follows from the above Lemma and an easy induction that f can be extended to a $(1+\varepsilon, \alpha)$-Hölder function on $M \cup \{x_n,\ n \geq 1\}$, which in turn can be extended by density to X.

□

Remark 3.3. For $Y = C(K)$, there is a more concrete argument, which even allows to extend f isometrically when M is compact. We use the Inf-convolution formula and define

$$\forall t \in K \ f(x)(t) = \inf_{y \in M} [f(y)(t) + \|x - y\|^{\alpha}].$$

Clearly, $\|f(x) - f(y)\|_\infty \leq \|x - y\|^{\alpha}$. Since $f(M)$ is compact in $C(K)$, $f(x)$ is the infimum of an equicontinuous family of functions and therefore is continuous on K.

Let us now concentrate on the isometric extension problem. We will need the following characterization.

Lemma 3.4. *Let (X, d) be a metric space, M a subset of X, $f \colon M \to c$ a contraction and $x \in X \setminus M$. Then, the following statements are equivalent:*

(1) *f can be extended to a contraction $g \colon M \cup \{x\} \to c$.*
(2) $\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n, m > N \quad \forall y, z \in M$

$$|f(y)(n) - f(z)(m)| \leq d(y, x) + d(z, x) + \varepsilon.$$

Proof. Suppose that (1) holds. Then $(g(x)(n))_n$ is a Cauchy sequence. Thus

$$\forall \varepsilon > 0 \ \exists N \ \forall n, m > N \quad |g(x)(n) - g(x)(m)| < \varepsilon.$$

Since g is a contractive extension of f, we have that for all $n, m > N$ and all $y, z \in M$

$$\begin{aligned} |f(y)(n) - f(z)(m)| &\leq |f(y)(n) - g(x)(n)| + |g(x)(n) - g(x)(m)| + \\ &\quad + |g(x)(m) - f(z)(m)| \\ &\leq d(y, x) + d(z, x) + \varepsilon. \end{aligned}$$

Suppose now that (2) holds. Define

$$s(j) = \sup_{m \geq j} \sup_{z \in M} (f(z)(m) - d(z, x)).$$

Let us fix $z_0 \in M$. Then, it is easily seen that

$$\forall j \in \mathbb{N}, \quad |s(j)| \leq \|f(z_0)\|_\infty + d(x, z_0).$$

On the other hand $\{s(j)\}_{j \in \mathbb{N}}$ is a decreasing sequence and therefore converges. We will denote by $s(\infty)$ its limit.
In order to define $(g(x)(n))_{n \geq 1}$, we pick a sequence $(N_k)_{k \geq 1}$ of integers such that
(i) (2) holds with $\varepsilon = 2^{-k}$ and $N = N_k$;
(ii) $\forall j > N_k \quad s(j) \leq s(\infty) + 2^{-k}$;
(iii) $\forall k \in \mathbb{N} \quad N_{k+1} > N_k$.
Then we define $g(x)$ as follows:

(1) for $n \leq N_1$, let $g(x)(n)$ be any element of

$$\bigcap_{y\in M}[f(y)(n)-d(x,y),f(y)(n)+d(x,y)] = \\ [\sup_{y\in M}(f(y)(n)-d(x,y)),\ \inf_{y\in M}(f(y)(n)+d(x,y))].$$

(2) for $N_k < n \leq N_{k+1}$ we define

$$g(x)(n) = \max\{\sup_{y\in M}(f(y)(n)-d(x,y)), s(N_k)-2^{-k}\}.$$

It follows from (i) that

$$\forall n > N_k\ \forall y \in M \quad s(N_k)-2^{-k} \leq f(y)(n)+d(x,y).$$

So

$$\forall n \in \mathbb{N} \quad \sup_{y\in M}(f(y)(n)-d(x,y)) \leq g(x)(n) \leq \inf_{y\in M}(f(y)(n)+d(x,y)).$$

Thus $g(x) \in \ell_\infty$ and for all y in M, $\|g(x)-f(y)\|_\infty \leq d(x,y)$.
Finally, note that

$$\forall n > N_k \quad \sup_{y\in M}(f(y)(n)-d(x,y)) \leq s(N_k).$$

Thus

$$\forall n \in (N_k, N_{k+1}] \quad s(N_k)-2^{-k} \leq g(x)(n) \leq s(N_k).$$

It is now clear that $(g(x)(n))_{n\geq 1}$ converges to $s(\infty)$ and therefore belongs to c. □

As a first application we have

Theorem 3.5. *For any finite dimensional normed vector space X*

$$(0,1) \subset \mathcal{A}(X,c).$$

Proof. Let $0<\alpha<1$, M a closed subset of X such that $0 \notin M$ and $f\colon M \to c$ be a $(1,\alpha)$-Hölder function. It is enough to show that f admits a $(1,\alpha)$-Hölder extension to $M \cup \{0\}$. For this purpose, we will apply Lemma 3.4 on the metric space $(X, \|\ \|^\alpha)$.
We fix $\varepsilon > 0$ and pick $x_0 \in M$. Since $\alpha < 1$,

$$\lim_{\|x\|\to\infty}[(\|x\|+\|x_0\|)^\alpha - \|x\|^\alpha] = 0.$$

So, there is $K>0$ such that $\|x-x_0\|^\alpha \leq \|x\|^\alpha + \varepsilon/3$ for all x so that $\|x\| > K$. Let us also choose K such that $\|x_0\| \leq K$. Since $M_K = M \cap KB_X$ is compact,

$$\exists N \in \mathbb{N}\ \forall n,m > N\ \forall x \in M_K \quad |f(x)(n)-f(x)(m)| < \frac{\varepsilon}{3}.$$

Let now x and y in M.
If $x \in M_K$, then for all $n, m > N$:

$$|f(x)(n)-f(y)(m)| \leq \frac{\varepsilon}{3} + \|x-y\|^\alpha \leq \|x\|^\alpha + \|y\|^\alpha + \frac{\varepsilon}{3}.$$

If x and y belong to $M \setminus M_K$, then for all $n, m > N$:

$$|f(x)(n) - f(y)(m)| \leq \|x - x_0\|^\alpha + \|y - x_0\|^\alpha + \frac{\varepsilon}{3} \leq \|x\|^\alpha + \|y\|^\alpha + \varepsilon.$$

Then the conclusion follows directly from Lemma 3.4. □

We will now see that the possibility of extending isometrically all Lipschitz maps from a finite dimensional space into c may depend on the geometry of the space X. As a positive result, we have for instance

Theorem 3.6. *For any $n \in \mathbb{N}$*

$$\mathcal{A}(\ell_\infty^n, c) = (0, 1].$$

Proof. For $j \in \{1, \dots, n\}$, $\delta \in \{-1, 1\}$, we denote by $F_{j,\delta}$ the following $(n-1)-$face of the unit ball of ℓ_∞^n:

$$F_{j,\delta} = \{x = (x_1, \dots, x_n) \colon \|x\| = 1, x_j = \delta\}.$$

Let $C_{j,\delta}$ denote the cone supported by $F_{j,\delta}$:

$$C_{j,\delta} = \{x \in \ell_\infty^n \colon x_j = \delta\|x\|\}.$$

For $j, k \in \{1, .., n\}$, $j \neq k$, and $\delta, \eta \in \{-1, 1\}$ we denote by $F_{j,\delta,k,\eta}$ the $(n-2)-$face of $F_{j,\delta}$:

$$F_{j,\delta,k,\eta} = F_{j,\delta} \cap F_{k,\eta},$$

and by $C_{j,\delta,k,\eta}$ the corresponding cone:

$$C_{j,\delta,k,\eta} = C_{j,\delta} \cap C_{k,\eta}.$$

We also define a family of projections $P_{j,\delta,k,\eta} \colon C_{j,\delta} \longrightarrow C_{j,\delta,k,\eta}$ by

$$P_{j,\delta,k,\eta}(x) = y, \text{ where } \begin{cases} y_k = \eta|x_j| \\ y_i = x_i, \quad \text{if } i \neq k. \end{cases}$$

Note that for every $x \in C_{j,\delta}$, $\eta|x_j| = \eta\delta x_j$, so $P_{j,\delta,k,\eta}$ is linear on $C_{j,\delta}$ and

$$\forall x \in C_{j,\delta} \quad \|P_{j,\delta,k,\eta}(x)\| = \|x\|. \tag{3.1}$$

Further, since for all $x \in C_{j,\delta}$, $|x_j| \geq |x_k|$ we get

$$\operatorname{sgn}((P_{j,\delta,k,\eta}(x))_k - x_k) = \eta. \tag{3.2}$$

We also introduce the projection $Q_k \colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ defined by

$$Q_k(x_1, .., x_n) = (x_1, .., x_{k-1}, x_{k+1}, .., x_n).$$

The following Lemma will provide us with a convenient finite covering of the space ℓ_∞^n.

Lemma 3.7. *For any $M \subset X = \ell_\infty^n$, any $\varepsilon > 0$ and any $j \in \{1, \dots, n\}$, $\delta \in \{-1, 1\}$, such that $C_{j,\delta} \cap M \neq \emptyset$, there exist $A_1, .., A_\mu$ subsets of X such that*

$$(C_{j,\delta} \cap M) \subset \bigcup_{i=1}^{\mu} A_i$$

and $\forall i \in \{1,..,\mu\}$ $\exists x^i \in A_i \cap M$ *satisfying*

$$\forall x \in A_i \cap M \quad \|x - x^i\| \leq \|x\| - \|x^i\| + \varepsilon.$$

Proof of Lemma 3.7. We will give a proof by induction on the dimension of ℓ_∞^n. If $n = 1$, the statement is clear, so let us now assume that it is satisfied for $n-1$, where $n \geq 2$.

Let M, ε, j and δ be as in the statement of Lemma 3.7. We pick an element $x^{j,\delta} \in C_{j,\delta} \cap M$ and we denote

$$B_{j,\delta} = x^{j,\delta} + C_{j,\delta}.$$

Note that

$$\forall x \in B_{j,\delta} \quad \|x - x^{j,\delta}\| = \|x\| - \|x^{j,\delta}\|. \tag{3.3}$$

Denote $d_{j,\delta,k,\eta} = |(P_{j,\delta,k,\eta}(x^{j,\delta}))_k - (x^{j,\delta})_k|$. Let $x \in C_{j,\delta}$ such that for any $k \in \{1,..,n\} \setminus \{j\}$, and any $\eta \in \{-1,1\}$,

$$|(P_{j,\delta,k,\eta}(x))_k - x_k| \geq d_{j,\delta,k,\eta} \tag{3.4}$$

Then, we claim that $x \in B_{j,\delta}$.

Indeed, by (3.2)

$$|(P_{j,\delta,k,\eta}(x))_k - x_k| = \eta(P_{j,\delta,k,\eta}(x))_k - \eta x_k$$

and

$$|(P_{j,\delta,k,\eta}(x^{j,\delta}))_k - (x^{j,\delta})_k| = \eta(P_{j,\delta,k,\eta}(x^{j,\delta}))_k - \eta(x^{j,\delta})_k.$$

Thus (3.4) implies that

$$\eta(P_{j,\delta,k,\eta}(x))_k - \eta(P_{j,\delta,k,\eta}(x^{j,\delta}))_k \geq \eta x_k - \eta(x^{j,\delta})_k,$$

and hence

$$\eta\eta\delta x_j - \eta\eta\delta(x^{j,\delta})_j = \delta(x_j - (x^{j,\delta})_j) \geq \eta x_k - \eta(x^{j,\delta})_k.$$

Since this holds for all $\eta \in \{-1,1\}$, we get that for all $k \in \{1,..,n\} \setminus \{j\}$,

$$\delta(x - x^{j,\delta})_j \geq |(x - x^{j,\delta})_k|.$$

Thus $x - x^{j,\delta} \in C_{j,\delta}$ and $x \in B_{j,\delta}$.

Combining (3.2) and (3.4), we conclude that

$$C_{j,\delta} \setminus B_{j,\delta} \subset \bigcup_{\substack{k\in\{1,..,n\}\setminus\{j\},\\ \eta\in\{-1,1\}}} B_{j,\delta,k,\eta}$$

where

$$B_{j,\delta,k,\eta} = \{x \in C_{j,\delta} \colon ((P_{j,\delta,k,\eta}(x))_k - x_k) \in \eta[0, d_{j,\delta,k,\eta})\}.$$

Now, for each $k \in \{1,..,n\} \setminus \{j\}$, and $\eta \in \{-1,1\}$, we choose $N_{k,\eta} \in \mathbb{N}$ such that $\dfrac{d_{j,\delta,k,\eta}}{N_{k,\eta}} < \dfrac{\varepsilon}{3}$. Then we set

$\forall k \in \{1,..,n\} \setminus \{j\}$ $\forall \eta \in \{-1,1\}$ $\forall \nu \in \{1,\ldots,N_{k,\eta}\}$:

$$I^{\nu}_{j,\delta,k,\eta} = \Big[\frac{(\nu-1)d_{j,\delta,k,\eta}}{N_{k,\eta}}, \frac{\nu d_{j,\delta,k,\eta}}{N_{k,\eta}}\Big)$$

and

$$B^{\nu}_{j,\delta,k,\eta} = \{x \in C_{j,\delta}\colon ((P_{j,\delta,k,\eta}(x))_k - x_k) \in \eta I^{\nu}_{j,\delta,k,\eta}\}.$$

So we have

$$C_{j,\delta} \setminus B_{j,\delta} \subset \bigcup_{\substack{k\in\{1,..,n\}\setminus\{j\},\\ \eta\in\{-1,1\}}} \bigcup_{\nu=1}^{N_{k,\eta}} B^{\nu}_{j,\delta,k,\eta}. \tag{3.5}$$

We now fix $k \in \{1,..,n\}\setminus\{j\}, \eta \in \{-1,1\}$ and $\nu \le N_{k,\eta}$ such that $B^{\nu}_{j,\delta,k,\eta} \cap M \neq \emptyset$ and denote for simplicity:

$$B = B^{\nu}_{j,\delta,k,\eta},\ I = \eta I^{\nu}_{j,\delta,k,\eta},\ \widetilde{P} = P_{j,\delta,k,\eta},\ P = Q_k\widetilde{P},\ M' = P(M \cap B)$$

$$\text{and } C = P(C_{j,\delta}) = Q_k C_{j,\delta,k,\eta} = \{x \in \ell_\infty^{n-1}\colon x_{\phi(j)} = \delta\|x\|\},$$

where $\phi(j) = j$ if $k > j$ and $\phi(j) = j-1$ if $k < j$.
Since M' is a non empty subset of C, our induction hypothesis yields the existence of $A'_1,..,A'_L \subset C$ so that $M' \subset \bigcup_{l\le L} A'_l$ and $\forall l \in \{1,..,L\}\ \exists y^l \in A'_l \cap M'$ satisfying

$$\forall y \in A'_l \cap M' \quad \|y - y^l\| \le \|y\| - \|y^l\| + \frac{\varepsilon}{3}.$$

Now let $A_l = \{x = (x_i)_{i=1}^n \in C_{j,\delta}\colon P(x) \in A'_l, x_k \in \delta\eta x_j - I\}$. We have that

$$B \cap M \subset \bigcup_{l\le L} A_l.$$

Then, for any $l \le L$, we pick $x^l \in A_l \cap M$ such that $P(x^l) = y^l$. Note that

$$\forall x \in A_l \cap M \quad \|x\| = |x_j| = \|Px\| \ge \|y^l\| - \frac{\varepsilon}{3} = |x^l_j| - \frac{\varepsilon}{3} = \|x^l\| - \frac{\varepsilon}{3}.$$

Therefore

$$\forall x \in A_l \cap M \quad |x_j - x^l_j| \le |x_j| - |x^l_j| + \frac{2\varepsilon}{3}.$$

Now,

$$\|x - x^l\| = \max\{\|P(x) - P(x^l)\|, |x_k - x^l_k|\}.$$

We have

$$\|P(x) - P(x^l)\| \le \|P(x)\| - \|P(x^l)\| + \frac{\varepsilon}{3} = \|x\| - \|x^l\| + \frac{\varepsilon}{3}.$$

Since the diameter of I is less than $\frac{\varepsilon}{3}$, we get on the other hand that

$$\begin{aligned} |x_k - x^l_k| &= |(x_k - \eta\delta x_j) - (x^l_k - \eta\delta x^l_j) + \eta\delta x_j - \eta\delta x^l_j| \\ &\le \frac{\varepsilon}{3} + |x_j - x^l_j| \le \varepsilon + |x_j| - |x^l_j| = \varepsilon + \|x\| - \|x^l\|. \end{aligned} \tag{3.6}$$

So the conclusion of the lemma follows from (3.3) and (3.5). □

We now proceed with the proof of Theorem 3.6. As usual, we consider a contraction $f\colon M \to c$, where M is a closed subset of ℓ_∞^n with $0 \notin M$. We will only show, as we may, that f can be contractively extended to $M \cup \{0\}$.
Let $\varepsilon > 0$. It follows from Lemma 3.7 that there exist $A_1, .., A_\mu$ subsets of X such that $M \subset \bigcup_{i=1}^{\mu} A_i$ and

$$\forall 1 \le i \le \mu \ \exists x^i \in A_i \cap M \text{ such that } \forall x \in A_i \cap M \quad \|x - x^i\| \le \|x\| - \|x^i\| + \frac{\varepsilon}{2}.$$

There also exists $N \in \mathbb{N}$ such that

$$\forall n, m > N \ \forall i \in \{1, .., \mu\} \quad |f(x^i)(n) - f(x^i)(m)| < \frac{\varepsilon}{2}.$$

Let now x and y in M. Then we pick i such that $x \in A_i$. Thus, for all $n, m > N$

$$\begin{aligned} |f(x)(n) - f(y)(m)| &\le |f(x)(n) - f(x^i)(n)| + |f(x^i)(n) - f(x^i)(m)| \\ &\quad + |f(x^i)(m) - f(y)(m)| \\ &\le \|x - x^i\| + \frac{\varepsilon}{2} + \|x^i - y\| \\ &\le \|x\| - \|x^i\| + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \|y\| + \|x^i\| \\ &\le \|x\| + \|y\| + \varepsilon. \end{aligned}$$

Then we can apply Lemma 3.4 to conclude our proof.

□

Corollary 3.8. *Let X be any finite dimensional Banach space whose unit ball is a polytope. Then*

$$\mathcal{A}(X, c) = (0, 1].$$

Proof. If B_X is a polytope, we can find $f_1, ..., f_n$ in the unit sphere of the dual space of X such that

$$B_X = \bigcap_{i=1}^{n} \{x \in X \colon |f_i(x)| \le 1\}.$$

Then the map $T\colon X \to \ell_\infty^n$ defined by $Tx = (f(x_i))_{i=1}^n$ is clearly a linear isometry and the result follows immediately from Theorem 3.6. □

We will finish this section with a counterexample in dimension 4. We denote by $\ell_2^2 \oplus_1 \ell_2^2$ the space $\mathbb{R}^4$ equipped with the norm:

$$\forall (s, t, u, v) \in \mathbb{R}^4, \ \|(s, t, u, v)\| = (s^2 + t^2)^{1/2} + (u^2 + v^2)^{1/2}.$$

Then we have

Theorem 3.9.

$$\mathcal{A}(\ell_2^2 \oplus_1 \ell_2^2, c) = (0, 1).$$

Proof. For $n \in \mathbb{N}$, we define $x_n = (K^{2n}, K^n, 0, 0)$ and $y_n = (0, 0, K^{2n}, K^n)$, where $K > 1$ is to be chosen. Note that

$$\forall n \in \mathbb{N}, \quad \|x_n\| \leq K^{2n} + \frac{1}{2} \quad \text{and} \quad \|y_n\| \leq K^{2n} + \frac{1}{2}. \tag{3.7}$$

On the other hand,

$$\lim_{K \to +\infty} (\|x_n\| - K^{2n}) = \frac{1}{2}, \text{ uniformly for } n \in \mathbb{N}$$

and

$$\lim_{K \to +\infty} (\|x_n - x_m\| - (K^{2n} - K^{2m})) = \frac{1}{2}, \text{ uniformly for } n > m.$$

Thus, we can pick K large enough, so that

$$\forall n, m \in \mathbb{N} \quad \|x_n - y_m\| \geq K^{2n} + K^{2m} + \frac{7}{8}. \tag{3.8}$$

and

$$\forall n > m \in \mathbb{N} \quad \|x_n - x_m\| = \|y_n - y_m\| \geq K^{2n} - K^{2m} + \frac{3}{8}. \tag{3.9}$$

Then we denote $M = \{x_n \colon n \in \mathbb{N}\} \cup \{y_n,\ n \in \mathbb{N}\}$. We will now construct $u_n = f(x_n)$ and $v_n = f(y_n)$ in c so that $f\colon M \to c$ is 1-Lipschitz. So let $n \in \mathbb{N}$. For k odd and $k \leq n$, set $u_n(k) = K^{2n} + \frac{5}{8}$ and $u_n(k) = K^{2n} + \frac{1}{4}$ otherwise. For k even and $k \leq n$, set $v_n(k) = -(K^{2n} + \frac{5}{8})$ and $v_n(k) = -(K^{2n} + \frac{1}{4})$ otherwise.

We now check that f is 1-Lipschitz.

For all $n > m \in \mathbb{N}$, $\|u_n - u_m\|_\infty \leq K^{2n} + \frac{5}{8} - (K^{2m} + \frac{1}{4}) = K^{2n} - K^{2m} + \frac{3}{8}$.
Therefore, by (3.9), $\|u_n - u_m\|_\infty \leq \|x_n - x_m\|$.
We have, as well that $\|v_n - v_m\|_\infty \leq \|y_n - y_m\|$.
We also have that for all $n, m \in \mathbb{N}$, $\|u_n - v_m\|_\infty = K^{2n} + K^{2m} + \frac{7}{8}$.
Thus, (3.8) implies that $\|u_n - u_m\|_\infty \leq \|x_n - y_m\|$.
We have shown that f is 1-Lipschitz.

Assume now that f can be extented at 0 into a 1-Lipschitz function g and let $g(0) = w = (w(k))_{k \geq 1} \in c$. Then it follows from (3.7) that for all odd values of k, $w(k) \geq \frac{1}{8}$ and for all even values of k, $w(k) \leq -\frac{1}{8}$. This contradicts the fact that $w \in c$.

□

Remark 3.10. As we already mentioned in the introduction, this seems to be the first example of Banach spaces X and Y such that $\mathcal{A}(X, Y) \neq \widetilde{\mathcal{A}}(X, Y)$ and also such that $\mathcal{A}(X, Y)$ is not closed in $(0, 1]$.

4. Maps into $C(K)$ spaces

In this last section we show that if K is an infinite compact metric space, then the study of the isometric extension for Lipschitz maps with values in $C(K)$ reduces to the results of the previous section. More precisely, we prove the following.

Theorem 4.1. *Let (X,d) be a metric space and (K,ϱ) be an infinite compact metric space. Then*
$$\mathcal{A}(X,C(K))=\mathcal{A}(X,c).$$

The main step of the proof will be to establish the following generalization of Lemma 3.4.

Proposition 4.2. *Let M be a subset of X, $f\colon M\to C(K)$ a contraction and $x\in X\setminus M$. We denote by D the diameter of K for the distance ϱ. Then, the following statements are equivalent:*

1. *f can be extended to a contraction $g\colon M\cup\{x\}\to C(K)$.*
2. *$\forall\varepsilon>0\quad\exists\delta>0$ such that $\forall t,s\in K$ with $\varrho(t,s)<\delta\quad\forall y,z\in M$*
$$|f(y)(t)-f(z)(s)|\le d(y,x)+d(z,x)+\varepsilon.$$
3. *$\exists\varphi\colon[0,D]\longrightarrow[0,+\infty)$ such that φ is continuous, $\varphi(0)=0$ and*
$$\forall t,s\in K\ \forall y,z\in M\quad |f(y)(t)-f(z)(s)|\le d(y,x)+d(z,x)+\varphi(\varrho(t,s)).$$

Proof. Suppose that (1) holds. Then (2) follows from the triangle inequality and the fact that $g(x)$ is uniformly continuous on K.

Assume now that (2) holds. Let us define, for $\lambda\in(0,D]$:
$$\xi(\lambda)=\sup_{y,z\in M}\ \sup_{\varrho(t,s)\le\lambda}\left(|f(y)(t)-f(z)(s)|-d(x,y)-d(x,z)\right).$$
The function ξ is clearly non decreasing and bounded below by $-2\,dist\,(x,M)$. So we can set
$$\xi(0)=\lim_{\lambda\searrow 0}\xi(\lambda).$$
We have that
$$\forall t,s\in K\ \forall y,z\in M\quad |f(y)(t)-f(z)(s)|\le d(y,x)+d(z,x)+\xi(\varrho(t,s)).$$
It follows from (2) that $\xi(0)\le 0$. So, if we set $\psi=\xi-\xi(0)$, we get that ψ is non decreasing, $\psi(0)=0$ and ψ is continuous at 0. Since $\psi\ge\xi$, we still have
$$\forall t,s\in K\ \forall y,z\in M\quad |f(y)(t)-f(z)(s)|\le d(y,x)+d(z,x)+\psi(\varrho(t,s)).$$
We now define the function φ in the following way: $\varphi(0)=0$ and for $n\in\mathbb{N}$, $\varphi(\frac{D}{n+1})=\psi(\frac{D}{n})$. We also ask φ to be constant equal to $\psi(D)$ on $[\frac{D}{2},D]$, and affine on each $[\frac{D}{n+2},\frac{D}{n+1}]$ for $n\in\mathbb{N}$. It is now clear that φ is non decreasing, continuous on $[0,D]$ and that $\psi\le\varphi$ on $[0,D]$. So we have
$$\forall t,s\in K\ \forall y,z\in M\quad |f(y)(t)-f(z)(s)|\le d(y,x)+d(z,x)+\varphi(\varrho(t,s)).$$
This proves that (2) implies (3).

Suppose now that (3) holds and define, for $t \in K$,

$$g(x)(t) = \sup_{s\in K}\sup_{z\in M}\big(f(z)(s) - d(z,x) - \varphi(\varrho(t,s))\big).$$

Fix $y_0 \in M$. Then, for all $z \in M$ and for all $s \in K$,

$$\begin{aligned} f(z)(s) - d(z,x) - \varphi(\varrho(t,s)) &\le \|f(y_0)\|_{C(K)} + d(z,y_0) - d(z,x) \\ &\le \|f(y_0)\|_{C(K)} + d(x,y_0). \end{aligned}$$

So $g(x)(t)$ is well defined. Further, it follows from the uniform continuity of φ on $[0,D]$ that $g(x)$ is continuous on K.
Since $\varphi(0) = 0$, we have, by definition of $g(x)$, that for all $y \in M$ and all $t \in K$

$$f(y)(t) - g(x)(t) \le d(x,y). \tag{4.1}$$

By (3), we get that for all $y,z \in M$ and for all $t,s \in K$

$$|f(z)(s) - f(y)(t)| \le d(y,x) + d(z,x) + \varphi(\varrho(t,s)),$$

so

$$f(z)(s) - d(z,x) - \varphi(\varrho(t,s)) \le f(y)(t) + d(y,x),$$

and by taking the supremum over z and s we obtain

$$g(x)(t) - f(y)(t) \le d(x,y). \tag{4.2}$$

Combining (4.1) and (4.2), we get that for all $y \in M$ $\|g(x) - f(y)\|_{C(K)} \le d(x,y)$. Thus (3) implies (1) and this ends the proof of Proposition 4.2.

□

Proof of Theorem 4.1. Since K is an infinite compact metric space, it contains a closed subset F which is homeomorphic to the one point compactification of $\mathbb{N}$. Then, $C(F)$ is clearly isometric to c. On the other hand, by the linear version of Tietze extension theorem due to K. Borsuk [3], there is a linear isometry $T\colon C(F) \to C(K)$ such that for any f in $C(F)$, Tf is an extension of f to K. Let now R be the restriction operator from $C(K)$ onto $C(F)$. Then $P = TR$ is a projection of norm 1 from $C(K)$ onto an isometric copy of c. Therefore, it is clear that for any metric space X, $\mathcal{A}(X, C(K)) \subset \mathcal{A}(X, c)$.

For the other inclusion, it is enough to show that if $1 \notin \mathcal{A}(X, C(K))$, then $1 \notin \mathcal{A}(X, c)$. So let us assume that $1 \notin \mathcal{A}(X, C(K))$. Then there exist $M \subset X$, a contraction $f\colon M \to C(K)$ and $x \in X \setminus M$ such that f can not be contractively extended to $M \cup \{x\}$. Thus, by Proposition 4.2, there exists $\varepsilon > 0$ so that for all $n \in \mathbb{N}$ there exist $t_n, s_n \in K$ with $\varrho(t_n, s_n) < 1/n$ and $y_n, z_n \in M$ so that

$$|f(y_n)(t_n) - f(z_n)(s_n)| > d(y_n,x) + d(z_n,x) + \varepsilon. \tag{4.3}$$

Since K is compact, we may assume that the sequence $(t_n)_{n\in\mathbb{N}}$ is convergent. Define now a sequence $(w_n)_{n\in\mathbb{N}}$ in K by setting, for $n \in \mathbb{N}$, $w_{2n-1} = t_n$ and $w_{2n} = s_n$. Then the sequence $(w_n)_{n\in\mathbb{N}}$ is convergent. So we can define a 1-Lipschitz map $h : M \to c$ by

$$\forall y \in M \;\; h(y) = (h(y)(n))_{n\in\mathbb{N}} = (f(y)(w_n))_{n\in\mathbb{N}}.$$

It now clearly follows from (4.3) and Lemma 3.4 that h does not have any extension to a 1-Lipschitz map from $M \cup \{x\}$ into c. Therefore $1 \notin \mathcal{A}(X, c)$.

□

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