Bounded Palais-Smale Mountain-Pass Sequences

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Abstract: Let $I(\lambda, .)$, $\lambda \in \mathbb{R}$, be a family of C^1-functionals having mountain-pass geometry. Under hypotheses which do not ensure that the mountain-pass level $c(\lambda)$ is a monotone function of λ, it is shown that $I(\lambda, .)$ has a bounded Palais-Smale sequence at level $c(\lambda)$, for almost every λ.

Suites de Palais-Smale bornées dans le lemme du col

Résumé: Soit $I(\lambda, .)$, $\lambda \in \mathbb{R}$ une famille de fonctionnelles de classe C^1 ayant une géométrie de col. Sous des hypothèses qui n'impliquent pas que le niveau du col $c(\lambda)$ soit une fonction monotone de λ, on montre que $I(\lambda, .)$ possède une suite de Palais-Smale bornée au niveau $c(\lambda)$, pour presque tout λ.

Version française abrégée

Soient $(X, \|\cdot\|)$ un espace de Banach, $J \subset \mathbb{R}$ un intervalle compact et $\mathfrak{I} = \{I(\lambda, \cdot)) : \lambda \in J\}$ une famille de fonctionnelles de classe C^1 sur X. On ne suppose pas que $I : \mathbb{R} \times X \to \mathbb{R}$ soit continue.

Définition 0.1. *On dit que $\mathfrak{I}$ a une géométrie de col s'il existe deux points v_1, v_2 dans X tel que pour tout $\lambda \in J$*

$$c(\lambda) := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\lambda, \gamma(t)) > \max\{I(\lambda, v_1), I(\lambda, v_2)\},$$

où $\Gamma := \{\gamma \in C([0, 1], X), \gamma(0) = v_1, \gamma(1) = v_2\}$ est l'ensemble des chemins continus joignant v_1 et v_2.

Le but de cette note est de montrer, sous des hypothèses très générales, que $I(\lambda, .)$ possède, pour presque tout $\lambda \in J$, une suite de Palais-Smale bornée au niveau $c(\lambda)$ (une SPSBλ). Des résultats similaires ont été obtenu par Struwe (voir [4, Chapter II, Section 9], [1], [5]) dans des cas particuliers et une version abstraite de son approche est due à Jeanjean [2]. Dans ces travaux il est nécessaire que la fonction $\lambda \to c(\lambda)$ soit monotone. Elle est alors dérivable presque partout et l'on montre que si c est dérivable en $\lambda_0 \in J$ alors $I(\lambda_0, .)$ possède une SPSBλ_0. Nous prouvons ici que ces hypothèses de monotonie et de dérivabilité sont superflues. Notre approche repose un résultat classique de Denjoy [3, Theorem (4.4), page 270] qui implique que l'ensemble D des points $\lambda_0 \in J$ pour lesquels il existe une suite strictement croissante $\{\lambda_n\} \subset J$ telle que

$$\lambda_n \to \lambda_0 \text{ et } \frac{c(\lambda_n) - c(\lambda_0)}{\lambda_0 - \lambda_n} \le M(\lambda_0) \tag{0.1}$$

pour un $M(\lambda_0) < \infty$, est de mesure pleine dans J. Notre résultat principal est le suivant.

Théorème 0.1. *On suppose que $\mathfrak{I}$ a une géométrie de col et que l'hypothèse (**H**) est vérifiée*

(**H**) *Lorsque $\{(\lambda_n, u_n)\} \subset J \times X$, avec $\{\lambda_n\}$ strictement croissante, est telle que $\lambda_n \nearrow \lambda_0 \in J$ et les suites*

$$-I(\lambda_0, u_n), \quad I(\lambda_n, u_n) \text{ et } \frac{I(\lambda_n, u_n) - I(\lambda_0, u_n)}{\lambda_0 - \lambda_n} \text{ sont toutes bornées supérieurement ,}$$

alors $\{\|u_n\|\}$ *est bornée et, pour tout* $\epsilon > 0$*, il existe* $N > 0$ *tel que*

$$I(\lambda_0, u_n) \leq I(\lambda_n, u_n) + \epsilon \text{ pour tout } n \geq N.$$

Alors pour tout $\lambda_0 \in D$*,* $I(\lambda_0, \cdot)$ *possède une suite de Palais-Smale bornée au niveau* $c(\lambda_0)$*. On rappelle que* D *est de mesure pleine dans* J*.*

Idée de la Preuve: Soient $\lambda_0 \in D$ et $\{\lambda_n\}$ une suite strictement croissante telle que (0.1) soit vérifiée. On montre qu'il existe une suite de chemins $\{\gamma_n\} \subset \Gamma$ et $K = K(\lambda_0) > 0$ tels que

(i) $\|\gamma_n(t)\| \leq K$ si $I(\lambda_0, \gamma_n(t)) \geq c(\lambda_0) - (\lambda_0 - \lambda_n)$.

(ii) Pour tout $\epsilon > 0$, $\max_{t\in[0,1]} I(\lambda_0, \gamma_n(t)) \leq c(\lambda_0) + \epsilon$ lorsque $n \in \mathbb{N}$ est suffisamment grand.

La suite $\{\gamma_n\} \subset \Gamma$ vérifie $\max_{t\in[0,1]} I(\lambda_0, \gamma_n(t)) \to c(\lambda_0)$ et pour chaque $n \in \mathbb{N}$ toute la partie supérieure du chemin, à partir d'un niveau strictement inférieur à $c(\lambda_0)$, est contenue dans une même boule de rayon $K > 0$ centrée à l'origine. Par un argument de déformation on en déduit que pour tout $a > 0$

$$\inf\{\|\partial_u I(\lambda_0, u)\| : u \in X, \ \|u\| \leq K + 1 \text{ et } |I(\lambda_0, u) - c(\lambda_0)| \leq a\} = 0.$$

Par suite $I(\lambda_0, .)$ possède bien une suite de Palais-Smale bornée au niveau $c(\lambda_0)$ car contenue dans la boule de rayon $K + 1$ centrée à l'origine.

Exemple: Lorsque $I(\lambda, \cdot) \in C^1(X, \mathbb{R})$ est de la forme

$$I(\lambda, u) = A(\lambda, u) - \lambda B(u), \ \lambda \in J,$$

(**H**) est vérifiée si pour toute suite $\{(\lambda_n, u_n)\} \subset J \times X$ avec $\lambda_n \nearrow \lambda_0 \in J$ strictement croissante, $\{I(\lambda_n, u_n)\}$ bornée supérieurement et $\{I(\lambda_0, u_n)\}$ bornée inférieurement on a:

(B1) si $\|u_n\| \to \infty$ alors $B(u_n) \to +\infty$;

(B2) si $\{u_n\}$ est bornée il existe $M > 0$ telle que $B(u_n) \geq -M$ pour tout $n \in \mathbb{N}$;

(B3) $A(\lambda_0, u_n) - A(\lambda_n, u_n) \leq C(\lambda_0 - \lambda_n)$ uniformément en $n \in \mathbb{N}$ pour un $C > 0$.

1 Introduction

When $(X, \|\cdot\|)$ is a Banach space and $J \subset \mathbb{R}$ is a compact interval, let $\mathfrak{I} = \{I(\lambda, \cdot)) : \lambda \in J\}$ denote a family of C^1- functionals on X. It is not assumed that $I : \mathbb{R} \times X \to \mathbb{R}$ is continuous.

Definition 1.1. $\mathfrak{I}$ *is said to have mountain-pass geometry if there exist two points* v_1, v_2 *in* X *such that for all* $\lambda \in J$

$$c(\lambda) := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\lambda, \gamma(t)) > \max\{I(\lambda, v_1), I(\lambda, v_2)\}, \tag{1.2}$$

where $\Gamma := \{\gamma \in C([0,1], X), \gamma(0) = v_1, \gamma(1) = v_2\}$ *is the set of continuous paths joining* v_1 *and* v_2*.*

Definition 1.2. *For* $\lambda \in J$*,* $\{u_n\} \subset X$ *is a Palais-Smale sequence at level* a *for* $I(\lambda, \cdot)$ *if* $I(\lambda, u_n) \to a$ *and* $\partial_u I(\lambda, u_n) \to 0$ *in the dual space of* X *as* $n \to \infty$*.*

Struwe (see [4, Chapter II, Section 9], [1], [5]) showed in specific examples how monotonic structure can be used to infer that c in (1.2) is monotone and hence differentiable almost everywhere on J, and deduced the existence of a *bounded* Palais-Smale sequence at level $c(\lambda)$ (a $BPSS\lambda$), for almost all $\lambda \in J$. Then Jeanjean [2] developed an abstract version of Struwe's method under the assumption that $I(\lambda, u) = A(u) - \lambda B(u)$, where $B(u) \geq 0$ for all u, and drew the same conclusion. (Jeanjean's hypotheses also imply monotonicity, and hence almost-everywhere differentiability, of c).

The present purpose is simply to point out (Theorem 2.1) how monotonicity and almost-everywhere differentiability of c may be redundant in this context, because of a classical theorem of Denjoy.

Lemma 1.1. *For any real-valued function c on J, the set D of points $\lambda_0 \in J$ for which there exists a strictly increasing sequence $\{\lambda_n\} \subset J$, with*

$$\lambda_n \to \lambda_0 \text{ and } \frac{c(\lambda_n) - c(\lambda_0)}{\lambda_0 - \lambda_n} \leq M(\lambda_0) \tag{1.3}$$

for some $M(\lambda_0) < \infty$, has full measure in J.

Proof. Let $\lambda_0 \in J$. If $\lambda_0 \notin D$ then (1.3) fails for every strictly increasing sequence $\lambda_n \nearrow \lambda_0$ and every constant M. Therefore both Dini left derivatives of the function c at λ_0 are $-\infty$. But according to a theorem of Denjoy [3, Theorem (4.4), page 270], the set of such points has zero Lebesgue measure. □

2 Bounded Palais-Smale Mountain-Pass Sequences

An example of Brezis (see [2]) shows that, even when $\mathfrak{I}$ has mountain-pass geometry and satisfies (**H**) below, there need not exist a $BPSS\lambda$ for *every* value of λ.

(**H**) When $\{(\lambda_n, u_n)\} \subset J \times X$, with $\{\lambda_n\}$ strictly increasing, is such that $\lambda_n \nearrow \lambda_0 \in J$ and the sequences

$$-I(\lambda_0, u_n), \quad I(\lambda_n, u_n) \text{ and } \frac{I(\lambda_n, u_n) - I(\lambda_0, u_n)}{\lambda_0 - \lambda_n} \text{ are all bounded above,}$$

then $\{\|u_n\|\}$ is bounded and, for $\epsilon > 0$, there exists $N > 0$ such that

$$I(\lambda_0, u_n) \leq I(\lambda_n, u_n) + \epsilon \text{ for all } n \geq N.$$

Theorem 2.1. *Suppose that* (**H**) *holds and that $\mathfrak{I}$ has mountain-pass geometry. Then for each $\lambda_0 \in D$ (defined in Lemma 1.1), $I(\lambda_0, \cdot)$ has a bounded Palais-Smale sequence $\{v_n\} \subset X$ at level $c(\lambda_0)$. Recall that D has full measure in J.*

It will be clear from the proof that mountain-pass geometry *almost everywhere* is all that is required for the theorem. We need two lemmas. Let $\lambda_0 \in D$ and let $\{\lambda_n\}$ be a strictly increasing sequence such that (1.3) hold.

Lemma 2.1. *There exists a sequence of paths $\{\gamma_n\} \subset \Gamma$ and $K = K(\lambda_0) > 0$ such that*

(i) $\|\gamma_n(t)\| \leq K$ *when*

$$I(\lambda_0, \gamma_n(t)) \geq c(\lambda_0) - (\lambda_0 - \lambda_n). \tag{2.4}$$

(ii) For any $\epsilon > 0$, $\max_{t \in [0,1]} I(\lambda_0, \gamma_n(t)) \leq c(\lambda_0) + \epsilon$ when $n \in \mathbb{N}$ is sufficiently large.

Proof. Let $\{\gamma_n\} \subset \Gamma$ be such that

$$\max_{t\in[0,1]} I(\lambda_n, \gamma_n(t)) \le c(\lambda_n) + (\lambda_0 - \lambda_n). \tag{2.5}$$

Then for all points t satisfying (2.4)

$$\begin{aligned} \frac{I(\lambda_n,\gamma_n(t)) - I(\lambda_0,\gamma_n(t))}{\lambda_0-\lambda_n} &\le \frac{c(\lambda_n) + (\lambda_0-\lambda_n) - c(\lambda_0) + (\lambda_0 - \lambda_n)}{\lambda_0 - \lambda_n} \\ &\le M(\lambda_0) + 2. \end{aligned} \tag{2.6}$$

Now (2.4) gives that $I(\lambda_0, \gamma_n(t))$ is bounded below. By (2.5) and (1.3), $I(\lambda_n, \gamma_n(t))$ is bounded above. Hence, by (**H**), (i) holds.

To prove (ii), let $t_n \in [0, 1]$ be such that

$$\max_{t\in[0,1]} I(\lambda_0, \gamma_n(t)) = I(\lambda_0, \gamma_n(t_n)) \text{ and let } d_n := I(\lambda_0, \gamma_n(t_n)) - I(\lambda_n, \gamma_n(t_n)).$$

¿From the definition of $c(\lambda_0)$, (2.4) holds for $t = t_n$ and so $\{\gamma_n(t_n)\}$ is bounded by part (i). Therefore

$$\begin{aligned} \max_{t\in[0,1]} I(\lambda_0,\gamma_n(t)) &= I(\lambda_0,\gamma_n(t_n)) = I(\lambda_n,\gamma_n(t_n)) + d_n \\ &\le \max_{t\in[0,1]} I(\lambda_n,\gamma_n(t)) + d_n \le c(\lambda_n) + (\lambda_0-\lambda_n) + d_n \\ &= c(\lambda_0) + (c(\lambda_n) - c(\lambda_0)) + (\lambda_0 - \lambda_n) + d_n. \end{aligned}$$

Let $\epsilon > 0$. By Lemma 1.1, $c(\lambda_n) - c(\lambda_0) \le \frac{\epsilon}{3}$ for $n \in \mathbb{N}$ sufficiently large. Also $d_n \le \frac{\epsilon}{3}$ by (**H**). This completes the proof. □

Now for $a > 0$ let

$$F_a = \{u \in X : \|u\| \le K + 1 \text{ and } |I(\lambda_0, u) - c(\lambda_0)| \le a\}$$

where the constant $K > 0$ is given by Lemma 2.1

Lemma 2.2. *For all $a > 0$,*

$$\inf\{\|\partial_u I(\lambda_0, u)\| : u \in F_a\} = 0. \tag{2.7}$$

Proof. Seeking a contradiction, we assume that (2.7) does not hold. Then, because of the mountain-pass geometry of $\mathfrak{I}$, $a > 0$ can be chosen such that for any $u \in F_a$

$$\|\partial_u I(\lambda_0, u)\| \ge a \text{ and } 0 < a < \frac{1}{2}\left[c(\lambda_0) - \max\{I(\lambda_0, v_1), I(\lambda_0, v_2)\}\right]. \tag{2.8}$$

A classical deformation argument says there exist $\alpha \in]0, a[$ and a homeomorphism $\eta : X \to X$ such that

$$\eta(u) = u \;\text{ if } |I(\lambda_0, u) - c(\lambda_0)| \ge a \tag{2.9}$$

$$I(\lambda_0, \eta(u)) \le I(\lambda_0, u) \;\text{ for all } u \in X \tag{2.10}$$

and

$$I(\lambda_0, \eta(u)) \le c(\lambda_0) - \alpha \;\text{ for all } u \in X \text{ with } \|u\| \le K \text{ and } I(\lambda_0, u) \le c(\lambda_0) + \alpha. \tag{2.11}$$

Let $\{\gamma_n\} \subset \Gamma$ be the sequence obtained in Lemma 2.1. By Lemma 2.1(ii) we can choose and fix $m \in \mathbb{N}$ sufficiently large that

$$\max_{t\in[0,1]} I(\lambda_0, \gamma_m(t)) \le c(\lambda_0) + \alpha. \tag{2.12}$$

Clearly by (2.8) and (2.9), $\eta \circ \gamma_m \in \Gamma$. Now if $u = \gamma_m(t)$ with $I(\lambda_0, u) \leq c(\lambda_0) - (\lambda_0 - \lambda_m)$, then (2.10) implies that

$$I(\lambda_0, \eta(u)) \leq c(\lambda_0) - (\lambda_0 - \lambda_m). \tag{2.13}$$

On the other hand if $u = \gamma_m(t)$ with $I(\lambda_0, u) > c(\lambda_0) - (\lambda_0 - \lambda_m)$ then Lemma 2.1(i) and (2.12) implies that u is such that $\|u\| \leq K$ with $I(\lambda_0, u) \leq c(\lambda_0) + \alpha$. Now (2.11) gives that

$$I(\lambda_0, \eta(u)) \leq c(\lambda_0) - \alpha \leq c(\lambda_0) - (\lambda_0 - \lambda_m), \tag{2.14}$$

which, combined with (2.13), yields

$$\max_{t \in [0,1]} I(\lambda_0, \eta \circ \gamma_m(t)) \leq c(\lambda_0) - (\lambda_0 - \lambda_m).$$

This contradicts the variational characterisation of $c(\lambda_0)$ and proves the required result. □

Proof of Theorem 2.1. Let $\lambda_0 \in D$. By Lemma 2.2 with $a = 1/n, \ n \in \mathbb{N}$, there exists a Palais-Smale sequence for $I(\lambda_0, \cdot)$ at the level $c(\lambda_0)$ which is contained in the ball of radius $K+1$ centred at the origin. This proves the theorem. □

Here is an example where (**H**) can be easily verified. Note that the hypotheses involve one-sided inequalities on the behaviour of the functional (not its absolute value) and many variants are possible.

Example 2.1. Suppose $I(\lambda, \cdot) \in C^1(X, \mathbb{R})$ is of the form

$$I(\lambda, u) = A(\lambda, u) - \lambda B(u), \ \lambda \in J,$$

where, for any sequence $\{(\lambda_n, u_n)\} \subset J \times X$ with $\lambda_n \nearrow \lambda_0 \in J$ strictly increasing, $\{I(\lambda_n, u_n)\}$ bounded above and $\{I(\lambda_0, u_n)\}$ bounded below:

(B1) if $\|u_n\| \to \infty$ then $B(u_n) \to +\infty$;

(B2) if $\{u_n\}$ is bounded there exists $M > 0$ such that $B(u_n) \geq -M$ for all $n \in \mathbb{N}$;

(B3) $A(\lambda_0, u_n) - A(\lambda_n, u_n) \leq C(\lambda_0 - \lambda_n)$ uniformly for $n \in \mathbb{N}$ for some $C > 0$.

Then (**H**) holds.

Proof. By (B3)

$$\begin{aligned} I(\lambda_n, u_n) - I(\lambda_0, u_n) &= A(\lambda_n, u_n) - \lambda_n B(u_n) - A(\lambda_0, u_n) + \lambda_0 B(u_n) \\ &\geq -C(\lambda_0 - \lambda_n) + (\lambda_0 - \lambda_n) B(u_n). \end{aligned}$$

Thus

$$\frac{I(\lambda_n, u_n) - I(\lambda_0, u_n)}{\lambda_0 - \lambda_n} \geq -C + B(u_n)$$

and from (B1) we see that $\{u_n\}$ is bounded. Also from (B3),

$$\begin{aligned} I(\lambda_0, u_n) - I(\lambda_n, u_n) &= A(\lambda_0, u_n) - \lambda_0 B(u_n) - A(\lambda_n, u_n) + \lambda_n B(u_n) \\ &\leq C(\lambda_0 - \lambda_n) + (\lambda_n - \lambda_0) B(u_n). \end{aligned}$$

Since $\lambda_n - \lambda_0 < 0$, we conclude, using (B2), that (**H**) hold. □

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