A positive solution for a nonlinear Schrödinger equation on $\mathbb{R}^N$. *

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1 Introduction

In this paper we study the existence of positive solutions for Schrödinger type equations of the form:

$$-\Delta u + V(x)u = f(u), \qquad u \in H^1(\mathbb{R}^N), \tag{1.1}$$

where $N \geq 2$, $f(u) : \mathbb{R} \to \mathbb{R}$ is a nonlinear continuous function and $V(x) \in C(\mathbb{R}, \mathbb{R})$.

If the potential $V(x)$ is constant, namely if (1.1) is autonomous, Berestycki-Lions [1] (for $N = 1$ and $N \geq 3$) and Berestycki-Gallouët-Kavian [2] (for $N = 2$) provide an existence result for a very wide class of nonlinearities (see Theorem 2.1 below). In particular only conditions on $f(s)$ near 0 and ∞ are required. In contrast, when (1.1) is not autonomous, up to our knowledge, all existence results require some global conditions on $f(s)$. For

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example, the following condition — called the global Ambrosetti-Rabinowitz superlinear condition — is often assumed.

$$\exists\, \mu > 2: \ 0 < \mu \int_0^s f(\tau)\, d\tau \le s f(s) \quad \text{for all } s \in \mathbb{R}.$$

In this paper, we consider non autonomous cases and assuming a decay condition (v4) on $\nabla V(x)$ we derive an existence result which do not need global conditions on $f(s)$. More precisely, on the nonlinear term $f \in C(\mathbb{R}^+, \mathbb{R})$, we assume

- (f1) $f(0) = 0$ and $f'(0)$ defined as $\lim_{s \to 0^+} f(s)s^{-1}$ exists,
- (f2) there is $p < \infty$ if $N = 2$, $p < \frac{N+2}{N-2}$ if $N \ge 3$ such that $\lim_{s \to +\infty} f(s)s^{-p} = 0$,
- (f3) $\lim_{s \to +\infty} f(s)s^{-1} = +\infty$,

and on the potential $V \in C(\mathbb{R}^N, \mathbb{R})$,

- (v1) $f'(0) < \inf \sigma(-\Delta + V(x))$, where $\sigma(-\Delta + V(x))$ denotes the spectrum of the self-adjoint operator $-\Delta + V(x) : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, i.e.,

$$\inf \sigma(-\Delta + V(x)) = \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2\, dx}{\int_{\mathbb{R}^N} |u|^2\, dx}$$

- (v2) $V(x) \to V(\infty) \in \mathbb{R}$ as $|x| \to \infty$,
- (v3) $V(x) \le V(\infty)$, a. e. $x \in \mathbb{R}^N$,
- (v4) there exists a function $\phi \in L^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ such that

$$|x||\nabla V(x)| \le \phi(x)^2, \forall x \in \mathbb{R}^N.$$

Our main result is the following:

Theorem 1.1 *Assume $N \ge 2$ and (f1)–(f3), (v1)–(v4). Then (1.1) has a non trivial positive solution.*

Remark 1.2 (i) In case where $V(x) \equiv V(\infty)$, namely when (1.1) is autonomous, Theorem 1.1 is contained in the result of [1]. See also [2].
(ii) Considering, for a constant $L \in \mathbb{R}$, $V + L$ and $f + Ls$ instead of V and f, we may assume, without loss of generality that

- (v5) $f \ge 0$, $f'(0) \ge 0$, $\alpha_0 \equiv \inf \sigma(-\Delta + V(x)) > 0$ and $0 \le f'(0) < \alpha_0$.

We shall make this assumption throughout the paper.
(iii) We remark that $\inf_{x \in \mathbb{R}^N} V(x) \leq \alpha_0 \leq V(\infty)$.

Theorem 1.1 will be proved by a variational approach. Because we look for a positive solution, we may assume without restriction that $f(s) = 0$ for all $s \leq 0$. We associate with (1.1) the functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\, dx - \int_{\mathbb{R}^N} F(u)\, dx$$

where $F(s) = \int_0^s f(t)dt$. We shall work on $H^1(\mathbb{R}^N) \equiv H$ with the norm

$$||u||^2_{H^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx.$$

We also use the notation:

$$||u||_p = \Big(\int_{\mathbb{R}^N} |u|^p \, dx \Big)^{1/p} \quad \text{for all } p \in [1, \infty),$$

and remark that by the definition of α_0,

$$\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 \, dx \geq \alpha_0 ||u||_2^2 \quad \text{for all } u \in H. \tag{1.2}$$

Under, (f1)–(f2) and (v2), (v5) I is a C^1 functional and it is standard that any critical point of I is a nonnegative solution of (1.1).

First we shall prove that under (f1)–(f3) and (v2), (v5), I has a Mountain Pass geometry (a MP geometry in short). Namely, setting

$$\Gamma = \{\gamma \in C([0,1], H), \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\},$$

that $\Gamma \neq \emptyset$ and

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > 0.$$

For such geometry Ekeland's principle implies the existence of a Palais-Smale sequence (a PS sequence in short) at the Mountain Pass level c (the MP level in short) for I. Namely a sequence $\{u_n\} \subset H$ such that

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as } n \to \infty.$$

A crucial step to obtain the existence of a critical point is to show the boundedness of a sequence of this type. It is challenging under our assumptions. To overcome this difficulty we use an indirect approach developed in [4]. For $\lambda \in [\frac{1}{2}, 1]$ we consider the family of functionals $I_\lambda : H \to \mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\, dx - \lambda \int_{\mathbb{R}^N} F(u)\, dx.$$

These functionals have a MP geometry and denoting c_λ the corresponding MP levels we deduce from [4] that there exists a sequence $\{\lambda_j\} \subset [\frac{1}{2}, 1]$ such that

- $\lambda_j \to 1$ as $j \to \infty$.
- I_{λ_j} has a bounded PS sequence $\{u_n^j\}$ at level c_{λ_j}.

We can see that, for all $j \in \mathbb{N}$, $\{u_n^j\}$ converges weakly to a non trivial critical point u_j of I_{λ_j}. If we can prove that the sequence $\{u_j\}$ is bounded, it will follows (arguing as in [4]) that it is a (bounded) PS sequence for I.

To show that $\{u_j\}$ is bounded we need condition (v4) on V. It allows us to make use of a Pohozaev type identity to derive, in Proposition 4.2, the boundedness of $\{u_j\}$. A key point which allows to use the identity is that $\{u_j\}$ is a sequence of exact critical points. It is because we need this property that we follow an approximation procedure to obtain a bounded PS sequence for I, instead of starting directly from an arbitrary PS sequence.

To show that the bounded sequence $\{u_j\}$ converges weakly to a non trivial critical point of I, the "problem at infinity" plays an important role. It is known since the work of P.L. Lions [8]. Let $I^\infty : H \to \mathbb{R}$ be defined by

$$I^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(\infty)u^2)\,dx - \int_{\mathbb{R}^N} F(u)\,dx$$

and set

$$m^\infty = \inf\{I^\infty(u)\,;u \neq 0,\ I^{\infty\prime}(u) = 0\}.$$

We shall prove, that $\{u_j\}$ has a non trivial weak limit if $c < m^\infty$. In turn we derive that $c < m^\infty$ as a consequence of a general result on autonomous problem in $\mathbb{R}^N$ establish in [5, 6]. Roughly speaking we show in [5, 6] that under general assumptions on f, the MP value c^∞ for I^∞ coincides with the least energy level m^∞. Namely, one always have

$$c^\infty = m^\infty. \tag{1.3}$$

Here

$$c^\infty = \inf_{\gamma \in \Gamma^\infty} \max_{t\in[0,1]} I^\infty(\gamma(t))$$

with $\Gamma^\infty = \{\gamma \in C([0,1], H), \gamma(0) = 0 \text{ and } I^\infty(\gamma(1)) < 0\}$. Moreover, in addition to (1.3), we show the existence of a path $\gamma_0 \in \Gamma^\infty$ such that

$$\max_{t\in[0,1]} I^\infty(\gamma_0(t)) = c^\infty(= m^\infty)$$

with $\gamma_0(t)(x) > 0$ for all $x \in \mathbb{R}^N$, $\forall t \in (0,1]$. At this point if we assume that $V(x) \leq V(\infty)$ for all $x \in \mathbb{R}^N$ but $V(x) \not\equiv V(\infty)$, we easily get that $c < m^\infty$ (if $V(x) \equiv V(\infty)$, we recall that Theorem 1.1 is contained in [1] (see also [2])). Having proved $c < m^\infty$ we derive that $u_j \rightharpoonup u \neq 0$ with $I'(u) = 0$ through a precise decomposition of the sequence, as a sum of translated critical points, in the spirit of the pioneering work [8]. Since we only require weak conditions on f, in particular f may not be C^1, we cannot use one of the many decompositions of the literature (see [3] for example). This description being susceptible of others applications we place it in a self contained section.

The paper is organized as follows. In Section 2 we present the results on least energy solutions for autonomous problems which are crucial to insure the compactness of bounded PS sequences. In Section 3 we solve the approximating problems. Section 4 is devoted to the proof of Theorem 1.1. Finally the decomposition of the PS sequences is given in Section 5.

2 Some results on autonomous problems

In this section we recall some facts about autonomous equations of the form

$$-\Delta u = g(u), \qquad u \in H^1(\mathbb{R}^N). \tag{2.1}$$

Here we state results not only for $N \geq 2$ but also for $N = 1$.

A solution v of (2.1) is said to be a *least energy solution* if and only if

$$J(v) = m, \text{ where } m = \inf\{J(u);\, u \in H \setminus \{0\} \text{ is a solution of (2.1)}\}. \tag{2.2}$$

Here $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ is the natural functional corresponding to (2.1)

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \int_{\mathbb{R}^N} G(u)\, dx$$

with $G(s) = \int_0^s g(\tau)\, d\tau$. The following results are due to Berestycki-Lions [1] for $N = 1$ and $N \geq 3$ and Berestycki-Gallouët-Kavian [2] for $N = 2$.

Theorem 2.1 *Assume that*

(g0) $g \in C(\mathbb{R}, \mathbb{R})$ *is continuous and odd.*

(g1) $-\infty < \liminf_{s\to 0} \frac{g(s)}{s} \leq \limsup_{s\to 0} \frac{g(s)}{s} = -\nu < 0$ *for* $N \geq 3$,
$\lim_{s\to 0} \frac{g(s)}{s} = -\nu \in (-\infty, 0)$ *for* $N = 1, 2$.

(g2) When $N \geq 3$, $\lim_{s\to\infty} \frac{|g(s)|}{s^{\frac{N+2}{N-2}}} = 0$.
When $N = 2$, *for any* $\alpha > 0$ *there exists* $C_\alpha > 0$ *such that*

$$|g(s)| \leq C_\alpha e^{\alpha s^2} \quad \textit{for all } s \geq 0.$$

(g3) When $N \geq 2$, there exists $\xi_0 > 0$ such that $G(\xi_0) > 0$.
When $N = 1$, there exists $\xi_0 > 0$ such that

$$G(\xi) < 0 \text{ for all } \xi \in]0, \xi_0[, \quad G(\xi_0) = 0 \quad \text{ and } \quad g(\xi_0) > 0.$$

Then J is well defined and of class C^1. Also $m > 0$ and there exists a least energy solution ω of (2.1) which is a classical solution and satisfies $\omega > 0$ on $\mathbb{R}^N$.

In [5, 6] the authors complemented this result in the following way:

Theorem 2.2 *Assume (g0)–(g3). Then setting*

$$\Gamma_J = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)), \gamma(0) = 0 \text{ and } J(\gamma(1)) < 0\},$$

we have $\Gamma_J \neq \emptyset$ and $b = m$ with

$$b = \inf_{\gamma \in \Gamma_J} \max_{t \in [0,1]} J(\gamma(t)) > 0.$$

Moreover for any least energy solution ω of (2.1) as given by Theorem 2.2 there exists a path $\gamma \in \Gamma$ such that $\gamma(t)(x) > 0$ for all $(t, x) \in (0, 1] \times \mathbb{R}^N$, $\omega \in \gamma([0, 1])$ and

$$\max_{t \in [0,1]} J(\gamma(t)) = b.$$

Remark 2.3 In [5, 6] it is also proved that, under (g1)–(g2), there exists $c_1 > 0$, $\delta_0 > 0$ such that

$$J(u) \geq c_1 ||u||^2_{H^1(\mathbb{R}^N)} \text{ when } ||u||_{H^1(\mathbb{R}^N)} \leq \delta_0.$$

3 Solutions for approximating problems

For $\lambda \in [\frac{1}{2}, 1]$ we consider the family of functionals $I : H \to \mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\, dx - \lambda \int_{\mathbb{R}^N} F(u)\, dx.$$

In Lemma 3.5 we show that for each $\lambda \in [\frac{1}{2}, 1]$, I_λ has a MP geometry. The corresponding MP level is denoted c_λ. The aim of the section is to prove that for almost every $\lambda \in [\frac{1}{2}, 1]$, I_λ possesses a non trivial critical point u_λ such that $I_\lambda(u_\lambda) \leq c_\lambda$.

A first step in this direction is to show that, for almost every $\lambda \in [\frac{1}{2}, 1]$, I_λ possesses a bounded Palais-Smale sequence (a BPS sequence for short) at the level c_λ. For this we shall use some abstract results of [4].

Theorem 3.1 *Let X be a Banach space equipped with a norm $\|.\|_X$ and let $J \subset \mathbb{R}^+$ be an interval. We consider a family $(I_\lambda)_{\lambda \in J}$ of C^1-functionals on X of the form*

$$I_\lambda(u) = A(u) - \lambda B(u), \ \forall \lambda \in J$$

where $B(u) \geq 0, \forall u \in X$ and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\|_X \to \infty$. We assume there are two points v_1, v_2 in X such that

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\} \quad \forall \lambda \in J,$$

where

$$\Gamma = \{\gamma \in C([0,1], X), \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost every $\lambda \in J$, there is a sequence $\{v_n\} \subset X$ such that

(i) $\{v_n\}$ is bounded , (ii) $I_\lambda(v_n) \to c_\lambda$, (iii) $I'_\lambda(v_n) \to 0$ in the dual X^{-1} of X.

Remark 3.2 This result which is Theorem 1.1 in [4] is reminiscient of Struwe's monotonicity trick (see [9]) and can be viewed as its generalization. Since [4], results in the same spirit, namely which establish the existence of BPS sequence for almost every value of a parameter, have been obtained for families of functionals enjoying other homotopy invariance. We mention, for example, [10] for a linking type situation. Also it was subsequently proved in [7] that the condition $B(u) \geq 0$, $\forall u \in X$ can be removed. In this case there is no more a monotone dependence of c_λ upon $\lambda \in J$ (in contrast to Theorem 3.1 where the map $\lambda \to c_\lambda$ is non increasing).

Remark 3.3 In Lemma 2.3 of [4] it is also proved that, under the assumptions of Theorem 3.1, the map $\lambda \to c_\lambda$ is continuous from the left.

We shall use Theorem 3.1 with $X = H$, $|| \cdot ||_X = || \cdot ||_{H^1(\mathbb{R}^N)}$, $J = [\frac{1}{2}, 1]$. First we remark

Lemma 3.4 *For any $\varepsilon > 0$ there exists a $c_\varepsilon > 0$ such that*

$$c_\varepsilon ||\nabla u||_2^2 + (\alpha_0 - \varepsilon)||u||_2^2 \leq \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 \, dx \quad \textit{for all } u \in H.$$

In particular under (v2),(v5)

$$||u||^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 \, dx$$

is equivalent to the norm $|| \cdot ||_{H^1(\mathbb{R}^N)}$.

Proof. For $\delta \in (0,1)$ we consider the following minimizing problem:

$$\mu_\delta = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(1-\delta)|\nabla u|^2 + V(x)u^2\,dx}{||u||_2^2}.$$

We remark that $\mu_\delta \geq \int_{x\in\mathbb{R}^N} V(x) > -\infty$ for all $\delta \in (0,1)$. To prove the lemma it is sufficient to show that $\lim_{\delta\to 0}\mu_\delta \geq \alpha_0$. By definition of μ_δ, there exists $u_\delta \in H$ with $||u_\delta||_{H^1(\mathbb{R}^N)} = 1$ such that

$$(1-\delta)||\nabla u_\delta||_2^2 + \int_{\mathbb{R}^N} V(x)u_\delta^2\,dx \leq (\mu_\delta + \delta)||u_\delta||_2^2. \tag{3.1}$$

From (1.2) it follows that

$$\alpha_0||u_\delta||_2^2 - \delta||\nabla u_\delta||_2^2 \leq (\mu_\delta + \delta)||u_\delta||_2^2,$$

that is,

$$(\alpha_0 - \mu_\delta - \delta)||u_\delta||_2^2 \leq \delta||\nabla u_\delta||_2^2 \leq \delta \to 0 \quad \text{as } \delta \to 0.$$

Thus if $\lim_{\delta\to 0}\mu_\delta < \alpha_0$, we have $||u_\delta||_2 \to 0$ and thus by (3.1) $||\nabla u_\delta||_2 \to 0$. This is in contradiction with $||u_\delta||_{H^1(\mathbb{R}^N)} = 1$ and thus it holds that $\lim_{\delta\to 0}\mu_\delta \geq \alpha_0$. ♠

The following lemma ensures that I_λ has MP geometry.

Lemma 3.5 *Assume that (f1)–(f3), (v1)–(v3) and (v5) hold. Then*

(i) there exists a $v \in H \setminus \{0\}$ *with* $I_\lambda(v) \leq 0$ *for all* $\lambda \in [\frac{1}{2}, 1]$.

(ii)

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(0), I_\lambda(v)\} \quad \textit{for all } \lambda \in [\frac{1}{2}, 1].$$

Here

$$\Gamma = \{\gamma \in C([0,1], H);\ \gamma(0) = 0, \gamma(1) = v\}.$$

Proof. We have for any $u \in H$, $\lambda \in [\frac{1}{2}, 1]$, $I_\lambda(u) \leq I_{1/2}(u) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + V(x)u^2\,dx - \frac{1}{2}\int_{\mathbb{R}^N}F(u)\,dx$. Also, by (f3) it is standard to find a $v \in H \setminus \{0\}$ such that $I_{1/2}(v) \leq 0$. Thus we have (i).

For (ii) we choose $\varepsilon_0 > 0$ such that $\alpha_0 - \varepsilon_0 > f'(0)$. By Lemma 3.4, there exists $c_{\varepsilon_0} > 0$ such that

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + V(x)u^2\,dx - \lambda\int_{\mathbb{R}^N}F(u)\,dx \\ &\geq \frac{1}{2}c_{\varepsilon_0}||\nabla u||_2^2 + \frac{1}{2}(\alpha_0 - \varepsilon_0)||u||_2^2 - \int_{\mathbb{R}^N}F(u)\,dx. \end{aligned}$$

By Remark 2.3,

$$J_0(u) = \frac{1}{2}c_{\varepsilon_0}||\nabla u||_2^2 + \frac{1}{2}(\alpha_0 - \varepsilon_0)||u||_2^2 - \int_{\mathbb{R}^N} F(u)\, dx$$

satisfies

$$\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_0(\gamma(t)) > 0.$$

Thus we have

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)) \geq \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_0(\gamma(t)) > 0.$$

♠

Remark 3.6 It is standard under (f1)–(f2) and (v1) that there exists a $\delta_0 > 0$ independent of $\lambda \in [\frac{1}{2}, 1]$ such that

$$||u||_{H^1(\mathbb{R}^N)} \geq \delta_0 \quad \text{for any non trivial critical point } u \text{ of } I_\lambda.$$

$I_\lambda(u) = \frac{1}{2}||u||^2 - \lambda \int_{\mathbb{R}^N} F(u)\, dx = A(u) - \lambda B(u)$ satisfies $A(u) \to \infty$ as $||u||_{H^1(\mathbb{R}^N)} \to \infty$, $B(u) \geq 0$ for all $u \in H$. Thus from Lemma 3.5 and Theorem 3.1 we get that I_λ has a BPS sequence, at the level c_λ for almost every $\lambda \in [\frac{1}{2}, 1]$. On the convergence of BPS sequences we have the following result:

Lemma 3.7 *Assume that (f1)–(f3), (v2),(v3),(v5) hold and let $\lambda \in [\frac{1}{2}, 1]$ be arbitrary but fixed. Then any bounded Palais-Smale sequence $\{u_n\}$ for I_λ satisfying $\limsup_{n\to\infty} I_\lambda(u_n) \leq c_\lambda$ and $||u_n||_{H^1(\mathbb{R}^N)} \not\to 0$, after extracting a subsequence, converges weakly to a non trivial critical point u_λ of I_λ with $I_\lambda(u_\lambda) \leq c_\lambda$.*

Proof. Since $\{u_n\}$ is bounded, from Theorem 5.1 (see also Remark 5.2) which is establish in Section 5 we know that,

$$I_\lambda(u_n) \to I_\lambda(u_0) + \sum_{k=1}^{l} I_\lambda^\infty(w_\lambda^k), \tag{3.2}$$

with $\ell \geq 0$, u_0 a critical point of I_λ and $I_\lambda^\infty : H \to \mathbb{R}$ given by

$$I_\lambda^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(\infty)u^2)\, dx - \lambda \int_{\mathbb{R}^N} F(u) dx.$$

The w_λ^k, for $k = 1, .., l$ are non-trivial critical points of I_λ^∞. Since any solution of

$$-\Delta u + V(\infty)u = \lambda f(u), \qquad u \in H \tag{3.3}$$

is non negative we can we can regard it as a solution of (2.1) with

$$g(s) = \begin{cases} -V(\infty)s + \lambda f(s), & \text{for } s \geq 0, \\ -g(-s), & \text{for } s < 0. \end{cases}$$

We observe that a least energy solution for (2.1) — which we may assume positive — is also a least energy solution of (3.3) and the converse is also true.

Thus, from Theorem 2.1, we see that any non trivial critical point w_λ of I_λ^∞ satisfies $I_\lambda^\infty(w_\lambda) > 0$ and all we have to do to prove the lemma is to show that $u_0 \neq 0$. Since $||u_n||_{H^1(\mathbb{R}^N)} \not\to 0$ we deduce from Theorem 5.1 that if $u_0 = 0$, then $\ell > 0$ and

$$c_\lambda = \sum_{k=1}^{\ell} I_\lambda^\infty(w_\lambda^k) \geq m_\lambda = \inf\{I_\lambda^\infty(u)\,;u \neq 0,\ I_\lambda^{\infty\prime}(u) = 0\}.$$

In turn we can observe that

$$c_\lambda < m_\lambda. \tag{3.4}$$

To see (3.4) let ω_λ be a least energy solution of

$$-\Delta u + V(\infty)u = \lambda f(u)$$

as provided by Theorem 2.1. Applying Theorem 2.2 to the functional I_λ^∞ we can find a path $\gamma(t) \in C([0,1], H)$ such that $\gamma(t)(x) > 0$, $\forall x \in \mathbb{R}^N$, $\forall t \in (0,1]$, $\gamma(0) = 0$, $I_\lambda^\infty(\gamma(1)) < 0$, $\omega_\lambda \in \gamma([0,1])$ and

$$\max_{t\in[0,1]} I_\lambda^\infty(\gamma(t)) = I_\lambda^\infty(\omega_\lambda).$$

Without restriction we can assume that $V \not\equiv V(\infty)$ in (v3) (otherwise there is nothing to prove). Thus

$$I_\lambda(\gamma(t)) < I_\lambda^\infty(\gamma(t)) \text{ for all } t \in]0,1]$$

and it follows from the definition of c_λ, that

$$c_\lambda \leq \max_{t\in[0,1]} I_\lambda(\gamma(t)) < \max_{t\in[0,1]} I_\lambda^\infty(\gamma(t)) = m_\lambda.$$

♠

Combining Lemmas 3.5, 3.7, Theorem 3.1 and the observation (see Remark 3.6) that $\forall\lambda \in [\frac{1}{2}, 1]$, $I_\lambda(u_n) \to c_\lambda \neq 0$ implies $||u_n||_{H^1(\mathbb{R}^N)} \not\to 0$, we deduce that I_λ has a non trivial critical point for almost every $\lambda \in [\frac{1}{2}, 1]$. We point out that this result is valid without using condition (v4). As a special case we obtain the existence of a sequence $\{(\lambda_j, u_j)\} \subset [\frac{1}{2}, 1] \times H$ with $\lambda_j \to 1$ and $u_j \neq 0$ satisfying $I'_{\lambda_j}(u_j) = 0$ and $I_{\lambda_j}(u_j) \leq c_{\lambda_j}$.

4 Proof of Theorem 1.1

The idea of the proof is to show that the sequence $\{u_j\}$ of critical points of I_{λ_j} obtained in Section 3 is bounded and that it is a Palais-Smale sequence for I satisfying

$\limsup_{j\to\infty} I(u_j) \le c$ and $||u_j||_{H^1(\mathbb{R}^N)} \not\to 0$. Then applying Lemma 3.7 we obtain a non trivial critical point of I and this completed the proof of Theorem 1.1.

To show the boundedness of $\{u_j\} \subset H$ we shall make use of the following Pohozaev type identity. Since its proof is standard we do not provide it. (See for example [1]).

Proposition 4.1 *Let $u(x)$ be a critical point of I_λ with $\lambda \in [\frac{1}{2}, 1]$ arbitrary, then $u(x)$ satisfies*

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \frac{N}{2}\int_{\mathbb{R}^N} V(x)u^2\, dx + \frac{1}{2}\int_{\mathbb{R}^N} \nabla V(x) x u^2\, dx - N\lambda \int_{\mathbb{R}^N} F(u)\, dx = 0. \tag{4.1}$$

Now we apply the above proposition to $\{(\lambda_j, u_j)\} \subset [\frac{1}{2}, 1] \times H$ obtained in the previous section.

Proposition 4.2 *Assume that (f1)–(f3), (v1)–(v5) hold. Then $\{u_j\} \subset H$ is bounded.*

Proof. Since $I_{\lambda_j}(u_j) \le c_{\lambda_j} \le c_{\frac{1}{2}}$ we deduce, from Proposition 4.1, that

$$\int_{\mathbb{R}^N} |\nabla u_j|^2\, dx \le \frac{1}{2}\int_{\mathbb{R}^N} |\nabla V(x)||x| u_j^2\, dx + c_{\frac{1}{2}} N.$$

Thus taking (v4) into account

$$\int_{\mathbb{R}^N} |\nabla u_j|^2\, dx \le \frac{1}{2}\int_{\mathbb{R}^N} u_j^2 \phi^2\, dx + c_{\frac{1}{2}} N. \tag{4.2}$$

Also since $I'_{\lambda_j}(u_j)(\phi^2 u_j) = 0$,

$$\int_{\mathbb{R}^N} \nabla u_j \nabla(\phi^2 u_j)\, dx + \int_{\mathbb{R}^N} V(x) u_j^2 \phi^2\, dx = \lambda_j \int_{\mathbb{R}^N} f(u_j) u_j \phi^2\, dx. \tag{4.3}$$

Now it follows from (f3) that for any $L > 0$ there exists $C(L) > 0$ such that

$$f(s)s \ge Ls^2 - C(L) \quad \text{for all } s \ge 0.$$

We deduce that, for a $\tilde{C}(L) > 0$,

$$\int_{\mathbb{R}^N} f(u_j)u_j\phi^2\, dx \ge L\int_{\mathbb{R}^N} u_j^2\phi^2\, dx - C(L)\int_{\mathbb{R}^N} \phi^2\, dx = L\int_{\mathbb{R}^N} u_j^2\phi^2\, dx - \tilde{C}(L). \tag{4.4}$$

We also have, for a $C > 0$, using (4.2)

$$\begin{aligned}
|\int_{\mathbb{R}^N} \nabla u_j \nabla(\phi^2 u_j)\, dx| &\le \int_{\mathbb{R}^N} |\nabla u_j|^2\phi^2\, dx + 2\int_{\mathbb{R}^N} |\nabla u_j||u_j||\phi||\nabla\phi|\, dx \\
&\le \int_{\mathbb{R}^N} |\nabla u_j|^2\phi^2\, dx + \int_{\mathbb{R}^N} |\nabla u_j|^2|\nabla\phi|^2\, dx + \int_{\mathbb{R}^N} u_j^2\phi^2\, dx \\
&\le (||\phi||_\infty^2 + ||\nabla\phi||_\infty^2)\int_{\mathbb{R}^N} |\nabla u_j|^2\, dx + \int_{\mathbb{R}^N} u_j^2\phi^2\, dx \\
&\le (||\phi||_\infty^2 + ||\nabla\phi||_\infty^2)(\frac{1}{2}\int_{\mathbb{R}^N} u_j^2\phi^2\, dx + c_{\frac{1}{2}}N) + \int_{\mathbb{R}^N} u_j^2\phi^2\, dx \\
&\le C\int_{\mathbb{R}^N} u_j^2\phi^2\, dx + C.
\end{aligned} \tag{4.5}$$

$$\int_{\mathbb{R}^N} V(x)u_j^2\phi^2\,dx \le V(\infty)\int_{\mathbb{R}^N} u_j^2\phi^2\,dx. \tag{4.6}$$

Finally, combining (4.3)- (4.6), we get,

$$L\int_{\mathbb{R}^N} u_j^2\phi^2\,dx - \tilde{C}(L) \le (C+V(\infty))\int_{\mathbb{R}^N} u_j^2\phi^2\,dx + C. \tag{4.7}$$

Taking $L>0$ large enough this shows that

$$\int_{\mathbb{R}^N} u_j^2\phi^2\,dx$$

is bounded and thus, by (4.2), $\int_{\mathbb{R}^N} |\nabla u_j|^2\,dx$ is bounded.

Next we show that

$$||u_j||_2^2 = \int_{\mathbb{R}^n} u_j^2\,dx$$

stays bounded as $j\to\infty$. We argue indirectly and assume

$$r_j \equiv ||u_j||_2^{2/N} \to \infty.$$

We set

$$\tilde{u}_j(x) = u_j(r_j x).$$

Then we have

$$||\nabla\tilde{u}_j||_2^2 = r_j^{2-N}||\nabla u_j||_2^2, \qquad ||\tilde{u}_j||_2^2 = 1. \tag{4.8}$$

In particular, $\{\tilde{u}_j\}$ is a bounded sequence in $H^1(\mathbb{R}^N)$. We can also observe that $\tilde{u}_j(x)$ satisfies

$$-\frac{1}{r_j^2}\Delta\tilde{u}_j + V(r_j x)\tilde{u}_j = \lambda_j f(\tilde{u}_j) \quad \text{in } \mathbb{R}^N. \tag{4.9}$$

Now we claim that

$$\sup_{x\in\mathbb{R}^N} ||\tilde{u}_j||_{L^2(B_1(x))}^2 \equiv \sup_{x\in\mathbb{R}^N}\int_{B_1(x)} \tilde{u}_j^2\,dx \to 0 \quad \text{as } j\to\infty, \tag{4.10}$$

where $B_1(x) = \{y\in\mathbb{R}^N;\ |y-x|\le 1\}$. In fact, it sufficies to show

$$\tilde{u}_j(x+y_j) \to 0 \quad \text{weakly in } H^1(\mathbb{R}^N) \tag{4.11}$$

for any sequence $\{y_j\}\subset\mathbb{R}^N$. Assume $\tilde{u}_j(x+y_j)\to\tilde{u}(x)$ weakly in $H^1(\mathbb{R}^N)$ after extracting a subsequence. We remark that it follows from (v2) that $V(r_j x+y_j)\to V(\infty)$ a.e. in $\mathbb{R}^N$. Then by (4.9) we have

$$V(\infty)\tilde{u} = f(\tilde{u}) \quad \text{in } \mathbb{R}^N.$$

Since $\tilde{u}(x)\in H^1(\mathbb{R}^N)$ and $\xi=0$ is an isolated solution of $V(\infty)\xi = f(\xi)$, we have $\tilde{u}\equiv 0$. This implies (4.11) and thus (4.10). Now we use the following lemma.

Lemma 4.3 *(see [8]) Assume that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and that*

$$\sup_{z\in\mathbb{R}^N}\int_{B_1(z)}|v_n|^2\,dx\to 0.$$

Then $||v_n||_r\to 0$ for $r\in]2,\frac{2N}{N-2}[$ when $N\geq 3$ and for $r\in]2,\infty[$ when $N=1,2$. Here $B_1(z)=\{y\in\mathbb{R}^N,|y-z|\leq 1\}$.

End of the proof of Proposition 4.2. By Lemma 4.3, for p given in (f2) it follows

$$||\tilde{u}_j||_{p+1}\to 0\quad\text{as } j\to\infty.$$

By (f1)–(f2), we have for any $\delta>0$ there exists $C_\delta>0$ such that

$$|f(\xi)-f'(0)\xi||\xi|\leq\delta\xi^2+C_\delta|\xi|^{p+1}\quad\text{for all }\xi\in\mathbb{R}.$$

Thus we have

$$|\int_{\mathbb{R}^N}(f(\tilde{u}_j)-f'(0)\tilde{u}_j)\tilde{u}_j\,dx|\leq\delta||\tilde{u}_j||_2^2+C_\delta||\tilde{u}_j||_{p+1}^{p+1}\to\delta\quad\text{as } j\to\infty.$$

Since $\delta>0$ is arbitrary, we have $\int_{\mathbb{R}^N}(f(\tilde{u}_j)-f'(0)\tilde{u}_j)\tilde{u}_j\,dx\to 0$. We remark that $f'(0)<V(\infty)$ follows from (v5) and Remark 1.2 (iii). Multiplying $\tilde{u}_j$ to (4.9) and itegrating, we have from (4.8) that

$$\begin{aligned}\frac{1}{r_j^2}||\nabla\tilde{u}_j||_2^2 &= -\int_{\mathbb{R}^N}(V(r_jx)-\lambda_jf'(0))\tilde{u}_j^2\,dx+\lambda_j\int_{\mathbb{R}^N}(f(\tilde{u}_j)-f'(0)\tilde{u}_j)\tilde{u}_j\,dx\\ &= -(V(\infty)-f'(0))||\tilde{u}_j||_2^2+o(1)\to-(V(\infty)-f'(0))<0\end{aligned}$$

as $j\to\infty$. This is a contradiction and $||\tilde{u}_j||_2^2$ is bounded as $j\to\infty$. ♠

Remark 4.4 When $N\geq 3$, we can show the boundedness of $||u_j||_2^2$ directly. In fact, we observe that $I'_{\lambda_j}(u_j)u_j=0$. Namely that

$$\int_{\mathbb{R}^N}|\nabla u_j|^2+V(x)u_j^2\,dx=\lambda_j\int_{\mathbb{R}^N}f(u_j)u_j\,dx.$$

By (f1)–(f2), for any $\delta>0$ there exist $C_\delta>0$ such that

$$f(s)\leq(f'(0)+\delta)s+C_\delta s^{\frac{N+2}{N-2}}\text{ for all } s\geq 0.$$

Thus, by (1.2)

$$\begin{aligned}\alpha_0\int_{\mathbb{R}^N}u_j^2\,dx &\leq \int_{\mathbb{R}^N}|\nabla u_j|^2+V(x)u_j^2\,dx\\ &\leq \int_{\mathbb{R}^N}f(u_j)u_j\,dx\\ &\leq (f'(0)+\delta)\int_{\mathbb{R}^N}u_j^2\,dx+C_\delta C||\nabla u_j||_2^{\frac{2N}{N-2}}.\end{aligned}$$

Choosing $f'(0)+\delta<\alpha_0=\inf\sigma(-\Delta+V(x))$, this shows that $||u_j||_2^2$ is bounded.

Lemma 4.5 *Assume that (f1)–(f3), (v1)–(v5) hold. Then the sequence* $\{u_j\} \subset H$ *is a Palais-Smale sequence for* I *satisfying* $\limsup_{j\to\infty} I(u_j) \le c$ *and* $||u_j||_{H^1(\mathbb{R}^N)} \not\to 0$.

Proof. The fact that $||u_j||_{H^1(\mathbb{R}^N)} \not\to 0$ follows from Remark 3.6. Now we have

$$I(u_j) = I_{\lambda_j}(u_j) + (\lambda_j - 1)\int_{\mathbb{R}^N} F(u_j)\,dx. \tag{4.12}$$

Since $\{u_j\} \subset H$ is bounded, $\int_{\mathbb{R}^N} F(u_j)\,dx$ stays bounded as $j \to \infty$. Also we recall that $I_\lambda(u_j) \le c_j$ and that, by Remark 3.3, $\lim_{j\to\infty} c_j = c$. Thus (4.12) gives

$$\limsup_{j\to\infty} I(u_j) \le c.$$

Also, in the dual of H,

$$I'(u_j) = I'_{\lambda_j}(u_j) + (\lambda_j - 1)f(u_j)$$

and thus $\lim_{j\to\infty} I'(u_j) = 0$. ♠

Proof of Theorem 1.1:

By Proposition 4.2 and Lemma 4.5, $\{u_j\} \subset H$ satisfy the assumptions of Lemma 3.7 for $\lambda = 1$. Thus I possesses a non trivial critical point and this proves Theorem 1.1. ♠

We end this section showing the existence of a least energy solution in the setting of Theorem 1.1.

Theorem 4.6 *Under the assumptions of Theorem 1.1, (1.1) has a least energy solution. Namely there exists a solution* $w \in H$ *such that* $I(w) = m$ *where*

$$m = \inf\{I(u)\,;\, u \ne 0,\ I'(u) = 0\}.$$

Proof. Let $\{u_n\} \subset H$ be a sequence of non trivial critical points of I satisfying $I(u_n) \to m$. From the proof of Proposition 4.2 we see, since $\{I(u_n)\}$ is bounded from above, that $\{u_n\} \subset H$ is bounded. Also by Remark 3.6, $||u_n||_{H^1(\mathbb{R}^N)} \not\to 0$.

Thus in particular $m > -\infty$ and $\{u_n\} \subset H$ is a PS sequence of I. Applying Theorem 5.1 we get that

$$I(u_n) \to I(u_0) + \sum_{k=1}^{l} I^\infty(w^k), \tag{4.13}$$

with $l \ge 0$ and u_0 a critical point of I. Now let m_∞ be the least energy level for I^∞. As in the proof of Lemma 3.7 we assume $V \not\equiv V(\infty)$, thus, we have $m < m^\infty$. Since $I^\infty(w^k) \ge m^\infty > 0$ for each k, we deduce $u_0 \ne 0$ and $\ell = 0$ from (4.13). Thus there exists a solution $w(x)$ such that $I(w) = m$. ♠

5 Decomposition of bounded Palais-Smale sequences

We consider functionals $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ of the form

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2\,dx - \int_{\mathbb{R}^N} F(u)\,dx.$$

We assume $f \in C(\mathbb{R})$ and that $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfy (v2) and

(f1') $f(0) = 0$ and $\lim_{s\to 0^+} \frac{f(s)}{s} = 0$,

(f2') there is $p < \infty$ if $N = 1, 2$ and $p < \frac{N+2}{N-2}$ if $N \geq 3$ such that $\lim_{s\to\infty} f(s)|s|^{-p} = 0$,

(v1') $\alpha_0 = \inf \sigma(-\Delta + V(x)) > 0$,

The aim of the section is to derive a description of the bounded Palais-Smale sequences of I in the spirit of [8]. We work on $H \equiv H^1(\mathbb{R}^N)$ with the norm

$$||u||^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx$$

which is equivalent to the standard $H^1(\mathbb{R}^N)$ norm (see Lemma 3.4). Our result is:

Theorem 5.1 *Assume that (f1')–(f2'), (v1'), (v2) hold and let $\{u_n\}$ be a bounded Palais-Smale sequence for I. Then there exists a subsequence of $\{u_n\}$, still denoted $\{u_n\}$, an integer $l \in \mathbb{N} \cup \{0\}$, sequences $\{y_n^k\} \subset \mathbb{R}^N$, $w^k \in H$ for $1 \leq k \leq l$ such that,*

(i) $u_n \rightharpoonup u_0$ *with* $I'(u_0) = 0$,

(ii) $|y_n^k| \to \infty$ *and* $|y_n^k - y_n^{k'}| \to \infty$ *for* $k \neq k'$,

(iii) $w^k \neq 0$ *and* $I^{\infty\prime}(w^k) = 0$ *for* $1 \leq k \leq l$,

(iv) $||u_n - u_0 - \sum_{k=1}^{l} w^k(\cdot - y_n^k)|| \to 0$,

(v) $I(u_n) \to I(u_0) + \sum_{k=1}^{l} I^\infty(w^k)$,

where we agree that in the case $l = 0$ the above holds without w^k, $\{y_n^k\}$.

Remark 5.2 The decomposition provided by Theorem 5.1 is still true assuming just that $\alpha_0 > f'(0)$. To see this it suffices to write I in the form

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + (V(x) - f'(0))u^2\,dx - \int_{\mathbb{R}^N} (F(u) - \frac{1}{2}f'(0)u^2)\,dx$$

Remark 5.3 It is standard under (f1')–(f2') and (v1') that there exists a $\rho_0 > 0$ such that for any non trivial critical point u of I, $||u|| \geq \rho_0$.

Proof of Theorem 5.1 The proof consists of several steps:

Step 1: *Extracting a subsequence if necessary we can assume that $u_n \rightharpoonup u_0$ weakly in H with u_0 a critical point of I.*

Indeed, since $\{u_n\}$ is bounded we may assume that, up to a subsequence, $u_n \rightharpoonup u_0$ weakly in H. Let us prove that $I'(u_0) = 0$. Noting that $C_0^\infty(\mathbb{R}^N)$ is dense in H, it suffices to check that $I'(u_0)\varphi = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. But we have,

$$\begin{aligned} I'(u_n)\varphi - I'(u_0)\varphi &= \int_{\mathbb{R}^N} \nabla(u_n - u_0)\nabla\varphi \, dx + \int_{\mathbb{R}^N} V(x)(u_n - u_0)\varphi \, dx \\ &- \int_{\mathbb{R}^N} (f(u_n) - f(u_0))\, \varphi \, dx \to 0, \end{aligned}$$

since $v_n \rightharpoonup v$ weakly in H and strongly in $L^q_{loc}(\mathbb{R}^N)$ for $q \in [2, \frac{2N}{N-2}[$ if $N \geq 3, q \geq 2$ if $N = 1, 2$. Thus recalling that $I'(u_n) \to 0$ we indeed have $I'(u_0) = 0$.

Now we set $v_n^1 = u_n - u_0$.

Step 2: *Suppose*

$$\sup_{z \in \mathbb{R}^N} \int_{B_1(z)} |v_n^1|^2 \, dx \to 0.$$

Then $u_n \to u_0$ and Theorem 5.1 holds with $l = 0$.

We compute

$$\begin{aligned} I'(u_n)v_n^1 &= \int_{\mathbb{R}^N} \nabla u_n \nabla v_n^1 \, dx + \int_{\mathbb{R}^N} V(x) u_n v_n^1 \, dx - \int_{\mathbb{R}^N} f(u_n) v_n^1 \, dx \\ &= \int_{\mathbb{R}^N} \nabla v_n^1 \nabla v_n^1 \, dx + \int_{\mathbb{R}^N} \nabla u_0 \nabla v_n^1 \, dx + \int_{\mathbb{R}^N} V(x) |v_n^1|^2 \, dx \\ &+ \int_{\mathbb{R}^N} V(x) u_0 v_n^1 \, dx - \int_{\mathbb{R}^N} f(u_n) v_n^1 \, dx. \end{aligned}$$

Thus,

$$\begin{aligned} ||v_n^1||^2 = \int_{\mathbb{R}^N} |\nabla v_n^1|^2 + V(x)|v_n^1|^2 \, dx &= I'(u_n)v_n^1 - \int_{\mathbb{R}^N} \nabla u_0 \nabla v_n^1 \, dx \\ &- \int_{\mathbb{R}^N} V(x) u_0 v_n^1 \, dx + \int_{\mathbb{R}^N} f(u_n) v_n^1 \, dx, \end{aligned}$$

and, since $I'(u_0)v_n^1 = 0$, it follows that

$$||v_n^1||^2 = I'(u_n)v_n^1 + \int_{\mathbb{R}^N} f(u_n) v_n^1 \, dx - \int_{\mathbb{R}^N} f(u_0) v_n^1 \, dx.$$

Now $I'(u_n)v_n^1 \to 0$ since $\{v_n^1\}$ is bounded. Also by (f1')–(f2'), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(s)| \leq \varepsilon|s| + C_\varepsilon |s|^p \quad \text{for all } s \geq 0.$$

Thus, from Hölder inequality,

$$\left| \int_{\mathbb{R}^N} f(u_n) v_n^1 \, dx \right| \leq \varepsilon ||u_n||_2 \, ||v_n^1||_2 + C_\varepsilon ||u_n||_{p+1}^p \, ||v_n^1||_{p+1}$$

and since by Lemma 4.3, $||v_n^1||_{p+1} \to 0$ this shows that

$$\int_{\mathbb{R}^N} f(u_n) v_n^1 \, dx \to 0.$$

In a similar way, we have $\int_{\mathbb{R}^N} f(u_0) v_n^1 \, dx \to 0$. Thus $v_n^1 \to 0$ and Step 2 is completed.

Step 3: *Suppose* $\exists \{z_n\} \subset \mathbb{R}^N$ *such that, for a* $d > 0$,

$$\int_{B_1(z_n)} |v_n^1|^2 \, dx \to d > 0.$$

Then, after extracting a subsequence if necessary, we have for a $w \in H$,

$$(i) \quad |z_n| \to \infty, \quad (ii) \quad u_n(\cdot + z_n) \rightharpoonup w \neq 0, \quad (iii) \quad I^{\infty\prime}(w) = 0.$$

Clearly (i),(ii) are standard and the point is to show (iii). We define $\tilde{u}_n(\cdot) = u_n(\cdot + z_n)$ and observe that, as in Step 1, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$I^{\infty\prime}(\tilde{u}_n)\varphi - I^{\infty\prime}(w)\varphi \to 0.$$

Thus to prove that $I^{\infty\prime}(w) = 0$ it suffices to show that $I^{\infty\prime}(\tilde{u}_n)\varphi \to 0$, for any fixed $\varphi \in C_0^\infty(\mathbb{R}^N)$. We have

$$\begin{aligned} I'(u_n)\varphi(\cdot - z_n) &= \int_{\mathbb{R}^N} \nabla u_n(x) \nabla \varphi(x - z_n) \, dx + \int_{\mathbb{R}^N} V(x) u_n(x) \varphi(x - z_n) \, dx \\ &- \int_{\mathbb{R}^N} f(u_n(x)) \varphi(x - z_n) \, dx \end{aligned}$$

or equivalently

$$\begin{aligned} I'(u_n)\varphi(\cdot - z_n) &= \int_{\mathbb{R}^N} \nabla u_n(y + z_n) \nabla \varphi(y) \, dy + \int_{\mathbb{R}^N} V(y + z_n) u_n(y + z_n) \varphi(y) \, dy \\ &- \int_{\mathbb{R}^N} f(u_n(y + z_n)) \varphi(y) \, dy. \end{aligned}$$

Thus, since $I'(u_n)\varphi(\cdot - z_n) \to 0$, from the definition of $\tilde{u}_n$ it follows that

$$\int_{\mathbb{R}^N} \nabla \tilde{u}_n(y) \nabla \varphi(y) \, dy + \int_{\mathbb{R}^N} V(y + z_n) \tilde{u}_n(y) \varphi(y) \, dy - \int_{\mathbb{R}^N} f(\tilde{u}_n(y)) \varphi(y) \, dy \to 0. \quad (5.1)$$

Also, since $|z_n| \to \infty$, and $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (V(y + z_n) - V(\infty)) \tilde{u}_n(y) \varphi(y) \, dy \to 0. \quad (5.2)$$

Thus we obtain from (5.1), (5.2),

$$\begin{aligned} I^{\infty\prime}(\tilde{u}_n)\varphi &= \int_{\mathbb{R}^N} \nabla\tilde{u}_n(y)\nabla\varphi(y)\,dy + \int_{\mathbb{R}^N} V(\infty)\tilde{u}_n(y)\varphi(y)\,dy \\ &- \int_{\mathbb{R}^N} f(\tilde{u}_n(y))\varphi(y)\,dy \to 0 \end{aligned}$$

and Step 3 is completed.

Step 4: *Assume there exists $m \geq 1$, $\{y_n^k\} \subset \mathbb{R}^N$, $w^k \in H$ for $1 \leq k \leq m$ such that*

$$|y_n^k| \to \infty, \quad |y_n^k - y_n^{k'}| \to \infty \quad \text{if} \quad k \neq k',$$
$$u_n(\cdot + y_n^k) \to w^k \neq 0, \quad \forall\, 1 \leq k \leq m,$$
$$I^{\infty\prime}(w^k) = 0, \quad \forall\, 1 \leq k \leq m.$$

Then

1) If $\sup_{z\in\mathbb{R}^N} \int_{B_1(z)} |u_n - u_0 - \sum_{k=1}^m w^k(\cdot - y_n^k)|^2\,dx \to 0$ *then*

$$||u_n - u_0 - \sum_{k=1}^m w^k(\cdot - y_n^k)|| \to 0$$

2) If $\exists (z_n) \subset \mathbb{R}^N$ *such that, for a* $d > 0$,

$$\int_{B_1(z_n)} |u_n - u_0 - \sum_{k=1}^m w^k(\cdot - y_n^k)|^2\,dx \to d > 0,$$

then, after extracting a subsequence if necessary, the following holds

$$(i) \quad |z_n| \to \infty, \quad |z_n - y_n^k| \to \infty, \text{ for all } 1 \leq k \leq m,$$
$$(ii) \quad u_n(\cdot + z_n) \rightharpoonup w^{m+1} \neq 0, \quad (iii) \quad I^{\infty\prime}(w^{m+1}) = 0.$$

Assume that (1) holds. Then setting $\xi_n = u_n - u_0 - \sum_{k=1}^m w^k(\cdot - y_n^k)$ we have $\xi_n \to 0$ in $L^{p+1}(\mathbb{R}^N)$ and we compute

$$\begin{aligned} I'(u_n)\xi_n &= \int_{\mathbb{R}^N} \nabla\xi_n\nabla\xi_n\,dx + \int_{\mathbb{R}^N} \nabla u_0\nabla\xi_n\,dx + \int_{\mathbb{R}^N} \nabla(\sum_{k=1}^m w^k(\cdot - y_n^k))\nabla\xi_n\,dx \\ &+ \int_{\mathbb{R}^N} V(x)\xi_n^2\,dx + \int_{\mathbb{R}^N} V(x)u_0\xi_n\,dx \\ &+ \int_{\mathbb{R}^N} V(x)(\sum_{k=1}^m w^k(\cdot - y_n^k))\xi_n\,dx - \int_{\mathbb{R}^N} f(u_n)\xi_n\,dx. \end{aligned}$$

Thus

$$\begin{aligned} ||\xi_n||^2 &= I'(u_n)\xi_n - \int_{\mathbb{R}^N} \nabla u_0\nabla\xi_n\,dx - \int_{\mathbb{R}^N} V(x)u_0\xi_n\,dx - \int_{\mathbb{R}^N} \nabla(\sum_{k=1}^m w^k(\cdot - y_n^k))\nabla\xi_n\,dx \\ &- \int_{\mathbb{R}^N} V(x)(\sum_{k=1}^m w(\cdot - y_n^k))\xi_n\,dx + \int_{\mathbb{R}^N} f(u_n)\xi_n\,dx. \end{aligned}$$

Since $I'(u_0)\xi_n = 0$ it follows that

$$\begin{aligned}
||\xi_n||^2 &= I'(u_n)\xi_n - \int_{\mathbb{R}^N} f(u_0)\xi_n\,dx - \sum_{k=1}^m \int_{\mathbb{R}^N} \nabla(w^k(\cdot - y_n^k))\nabla\xi_n\,dx \\
&- \sum_{k=1}^m \int_{\mathbb{R}^N} V(\infty)w^k(\cdot - y_n^k)\xi_n\,dx + \sum_{k=1}^m \int_{\mathbb{R}^N} (V(\infty) - V(x))w^k(\cdot - y_n^k)\xi_n\,dx \\
&+ \int_{\mathbb{R}^N} f(u_n)\xi_n\,dx,
\end{aligned}$$

or equivalently, since $I^{\infty\prime}(w^k) = 0$,

$$\begin{aligned}
||\xi_n||^2 &= I'(u_n)\xi_n - \sum_{k=1}^m \int_{\mathbb{R}^N} f(w^k)\xi_n(\cdot + y_n^k)\,dx \\
&+ \sum_{k=1}^m \int_{\mathbb{R}^N} (V(\infty) - V(x))w^k(\cdot - y_n^k)\xi_n\,dx + \int_{\mathbb{R}^N} (f(u_n) - f(u_0))\xi_n\,dx
\end{aligned}$$

and using repeatedly the fact that $||\xi_n||_{p+1} \to 0$ we deduce that $||\xi_n|| \to 0$.

Now we assume that (2) hold. Clearly (i),(ii) hold. To show (iii) we set $\tilde{u}_n = u_n(\cdot + z_n)$ and observe that for all $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$I^{\infty\prime}(u_n)\varphi - I^{\infty\prime}(w^{m+1})\varphi \to 0.$$

Thus we just have to prove that $I^{\infty\prime}(u_n)\varphi \to 0$ and this is done as in Step 1.

Step 5: *Conclusion*

By Step 1 we know that $u_n \rightharpoonup u_0$ with $I'(u_0) = 0$ and this is (i) of Theorem 5.1. If the assumption of Step 2 holds, then $u_n \to u_0$ and Theorem 5.1 hold with $l = 0$. Otherwise the assumption of Step 3 holds. We set $\{y_n^1\} = \{z_n\}$ and $w^1 = w$. Now if 1) of Step 4 holds with $m = 1$ this proves (ii)–(iv) of Theorem 5.1. If not, 2) of Step 4 must hold and setting $\{y_n^2\} = \{z_n\}$ and $w^2 = w^2$ we iterate Step 4. Clearly all we have to do to end the proof of (i)–(iv) is to show that 1) of Step 4 must occur after a finite number of iterations. But we observe, on one hand, that by the properties of the weak convergence, $\forall m \geq 1$

$$\lim_{n\to\infty} ||u_n||^2 - ||u_0||^2 - \sum_{k=1}^m ||w^k||^2 = \lim_{n\to\infty} ||u_n - u_0 - \sum_{k=1}^m w^k(\cdot - y_n^k)||^2 \geq 0.$$

On the other hand, by Remark 5.3, there is a $\rho_0 > 0$ such that $||w|| \geq \rho_0$ for any non trivial critical point of I^∞. Thus at one point, say for $l \in \mathbb{N}$, 1) of Step 4 will occur.

To complete the proof of Theorem 5.1 we just have to show that

$$I(u_n) \to I(u_0) + \sum_{k=1}^l I^\infty(w^k).$$

Writing $u_n = u_0 + (u_n - u_0)$ we first prove that

$$I(u_n) \to I(u_0) + I^\infty(u_n - u_0). \tag{5.3}$$

Indeed

$$\begin{aligned} I(u_n) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_0|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla(u_n-u_0)|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} \nabla u_0 \nabla(u_n-u_0)\,dx \\ &+ \frac{1}{2}\int_{\mathbb{R}^N} V(x)u_0^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)(u_n-u_0)^2\,dx \\ &+ \int_{\mathbb{R}^N} V(x)u_0(u_n-u_0)\,dx - \int_{\mathbb{R}^N} F(u_n)\,dx, \end{aligned}$$

or equivalently

$$\begin{aligned} I(u_n) &= I(u_0) + I^\infty(u_n-u_0) + \int_{\mathbb{R}^N} \nabla u_0 \nabla(u_n-u_0)\,dx \\ &+ \frac{1}{2}\int_{\mathbb{R}^N} (V(x)-V(\infty))(u_n-u_0)^2\,dx + \int_{\mathbb{R}^N} V(x)u_0(u_n-u_0)\,dx \\ &+ \int_{\mathbb{R}^N} F(u_n-u_0)\,dx + \int_{\mathbb{R}^N} F(u_0)\,dx - \int_{\mathbb{R}^N} F(u_n)\,dx. \end{aligned}$$

Thus all we have to show to prove (5.3) is that

$$\int_{\mathbb{R}^N} \left[F(u_n-u_0) + F(u_0) - F(u_n)\right]\,dx \to 0.$$

But under (f1')–(f2') this is classical (see [3] for example). Now one proves that

$$I^\infty(u_n-u_0) \to \sum_{k=1}^{l} I^\infty(w^k)$$

in the same way and using the observation that I^∞ is autonomous. ♠

References

[1] Berestycki H. and Lions P.L., Nonlinear scalar field equations I, *Arch. Rat. Mech. Anal.* **82** (1983), 313-346.

[2] Berestycki H. , Gallouët T. and Kavian O., Equations de Champs scalaires euclidiens non linéaires dans le plan. *C.R. Acad. Sci; Paris Ser. I Math.* **297** (1983), 5, 307-310.

[3] Coti Zelati V. and Rabinowitz P.H., Homoclinic type solutions for a semilinear elliptic PDE on $\mathbb{R}^N$, *Comm. Pure and Applied Math.*, Vol. **XIV** (1992), 1217-1269.

[4] Jeanjean L., On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on $\mathbb{R}^N$, *Proc. Roy. Soc. Edinburgh*, **129A**, (1999), 787-809.

[5] Jeanjean L., Tanaka K., A remark on least energy solutions in $\mathbb{R}^N$, *Proc Amer. Math. Soc.* **131**, (2003), 2399-2408.

[6] Jeanjean L., Tanaka K., A note on a mountain pass characterization of least energy solutions, Preprint.

[7] Jeanjean L. and Toland J. F., Bounded Palais-Smale Mountain-Pass sequences, *C. R. Acad. Sci. Paris*, **t. 327**, 1, 1998, 23-28.

[8] Lions P.L., The concentration-compactness principle in the calculus of variations. The locally compact case. Part I and II., *Ann. Inst. H. Poincaré, Anal. non-lin.*, **1** (1984), 109-145 and 223-283.

[9] Struwe M., Variational methods, 2nd edn (New York: Springer, 1996).

[10] Szulkin A. and Zou W., Homoclinic orbits for asymptotically linear Hamiltonian systems, *J. Funct. Anal.*, **187**, (2001), 1, 25-41.