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A Note on a Mountain Pass Characterization of Least Energy Solutions

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Abstract

We consider the equation

$$-u'' = g(u),\ u(x) \in H^1(\mathbb{R}). \tag{0.1}$$

Under general assumptions on the nonlinearity g we prove that the, unique up to translation, solution of (0.1) is at the mountain pass level of the associated functional. This result extends a corresponding result for least energy solutions when (0.1) is set on $\mathbb{R}^N$.

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1 Introduction

This note is a continuation of our study [7] on the nonlinear scalar field equations:

$$-\Delta u = g(u), \quad u(x) \in H^1(\mathbb{R}^N). \tag{1.1}$$

In [7], we showed, when $N \geq 2$, that least energy solutions of (1.1) admit a mountain pass characterization under the conditions:

(g0) $g(s) \in C(\mathbb{R}, \mathbb{R})$ is continuous and odd.

(g1) $-\infty < \liminf_{s\to 0} \frac{g(s)}{s} \leq \limsup_{s\to 0} \frac{g(s)}{s} < 0$ for $N \geq 3$,
$\lim_{s\to 0} \frac{g(s)}{s} \in (-\infty, 0)$ for $N = 2$.

(g2) When $N \geq 3$, $\lim_{s\to\infty} \frac{g(s)}{|s|^{\frac{N+2}{N-2}}} = 0$.
When $N = 2$, for any $\alpha > 0$

$$\lim_{s\to\infty} \frac{g(s)}{e^{\alpha s^2}} = 0.$$

(g3) There exists $s_0 > 0$ such that $G(s_0) > 0$, where

$$G(s) = \int_0^s g(\tau)\, d\tau.$$

More precisely, under the above conditions, we observe that the natural functional corresponding to (1.1):

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \int_{\mathbb{R}^N} G(u)\, dx \in C^1(H^1(\mathbb{R}^N), \mathbb{R}), \tag{1.2}$$

has a mountain pass geometry and that defining the mountain pass value by

$$b = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) \tag{1.3}$$

where

$$\Gamma = \{\gamma(t) \in C([0,1], H^1(\mathbb{R}^N));\ \gamma(0) = 0,\ I(\gamma(1)) < 0\},$$

the following result holds.

Theorem 1.1 ([7]) *Assume $N \geq 2$ and* (g0)–(g3). *Then least energy solutions of* (1.1) *have a mountain pass characterization, that is,*

$$b = m, \tag{1.4}$$

where $b > 0$ *is defined in* (1.3) *and*

$$m = \inf\{I(u);\, u \in H^1(\mathbb{R}) \setminus \{0\} \text{ is a solution of } (1.1)\}. \tag{1.5}$$

Moreover, for any least energy solution $\omega(x)$ *of* (1.1)*, there exists a path* $\gamma \in \Gamma$ *such that*

$$\omega \in \gamma([0,1]) \quad \text{and} \quad \max_{t\in[0,1]} I(\gamma(t)) \le I(\omega). \tag{1.6}$$

We remark that

(i) Exactly under the conditions (g0)–(g3), the existence of a least energy solution of (1.1) is shown in Berestycki-Lions [2] (for $N \ge 3$) and Berestycki-Gallouët-Kavian [3] (for $N = 2$). It is also observed that these conditions are almost necessary for the existence of a solution.

(ii) To get a mountain pass characterization, we remark that we do not assume the monotonicity of

$$s \mapsto \frac{g(s)}{s};\ (0,\infty) \to \mathbb{R}.$$

(iii) In the study of non-autonomous elliptic problems, informations on the least energy level of associated autonomous problems often play a crucial role. For a discussion of this feature, on problems of the type

$$-\Delta u = g(x,u), \quad u(x) \in H^1(\mathbb{R}^N),$$

we refer to [6]. For an application to singular perturbation problems for nonlinear Schrödinger equations of the form:

$$-\varepsilon^2 \Delta u + V(x)u = g(u), \quad u(x) \in H^1(\mathbb{R}^N),$$

we refer to [8, 10].

In view of Theorem 1.1, it is natural to ask if there is a corresponding result when $N = 1$. The purpose of this note is to study this problem and to show the following:

Theorem 1.2 *Assume* $N = 1$, (g0) *and*

(g1') $c_0 = -\lim_{s\to 0} \frac{g(s)}{s} \in (0,\infty)$.

(g2') *There exists* $s_0 > 0$ *such that*

$$\begin{aligned} G(s) &< 0 \quad \text{for all } s \in (0,s_0), \\ G(s_0) &= 0, \\ g(s_0) &> 0. \end{aligned}$$

Then

(i) (1.1) *has a unique positive solution* $\omega(x)$*, up to translation. Moreover, after a suitable translation,* $\omega(x)$ *satisfies*

$$\begin{aligned} \omega(0) &= s_0, \\ \omega_x(x) &> 0 \quad \text{in } (-\infty, 0), \\ \omega_x(x) &< 0 \quad \text{in } (0, \infty), \\ \omega(-x) &= \omega(x). \end{aligned}$$

(ii) *The set of all solutions of* (1.1) *is* $\{\pm\omega(x-t);\, t \in \mathbb{R}\} \cup \{0\}$*. In particular all (non trivial) solutions of* (1.1) *are least energy solutions.*

(iii) *The corresponding functional* $I(u)$ *defined in* (1.2) *has a mountain pass geometry and*

$$b = m, \tag{1.7}$$

($b > 0$ *and* m *are defined in* (1.3), (1.5)*). In addition for any least energy solution* $\omega(x)$ *of* (1.1) *there exists a path* $\gamma \in \Gamma$ *such that* (1.6) *hold.*

Remark 1.3 *In* [2]*, Berestycki and Lions showed that for a locally Lipschitz continuous function* $g(s)$ *satisfying* $g(0) = 0$*,* (g2') *is a necessary and sufficient condition for the existence of a non-zero solution of*

$$\begin{aligned} -u_{xx} &= g(u), \quad \text{in } \mathbb{R}, \\ u(x) &\to 0 \quad \text{as } x \to \pm\infty, \\ u(x_0) &> 0 \quad \text{for some } x_0 \in \mathbb{R}. \end{aligned}$$

Moreover under (g2') *they show the existence of a unique* (*up to translation*) *positive solution. It was also shown that the solution and its derivative decay exponentially if we assume* (g1') *in addition to* (g2').

Theorem 1.2 has connections with the work on second order autonomous Hamiltonian systems by variational method. See, for example, Ambrosetti-Bertotti [1], Bolotin [4], Caldiroli [5], Rabinowitz-Tanaka [9] and references therein. Actually for proving Theorem 1.2 we do use techniques and results developed in [9, 5]. However, the main focus when dealing with systems is on the existence of homoclinic orbits. In particular, in [1, 4, 5, 9] Hamiltonian systems of the type

$$\ddot{q} + \nabla V(q) = 0, \tag{1.8}$$

$$q(t) \to 0 \quad \text{as } t \to \pm\infty \tag{1.9}$$

are studied and the existence of a nontrivial homoclinic solution is shown under the following assumptions:

(v0) $V \in C^2(\mathbb{R}^N, \mathbb{R})$,

(v1) $V(0) = 0$ and $V''(0)$ is negative definite, (v2) $\Omega = \{x \in \mathbb{R}^N;\, V(x) < 0\} \cup \{0\}$ is bounded and $\nabla V(q) \neq 0$ for all $x \in \partial\Omega$.

In our paper, in contrast, our focus in not as much on the existence (and uniqueness) but on proving the mountain pass characterization of least energy solutions. In that direction we remark that functional $I(q) : H^1(\mathbb{R}, \mathbb{R}^N) \to \mathbb{R}$ defined below has a mountain pass geometry and that it can be used to find homoclinic solutions of (1.8)–(1.9).

$$I(q) = \int_{-\infty}^{\infty} \frac{1}{2}|\dot{q}|^2 - V(q)\, dt. \tag{1.10}$$

However, mountain pass characterization for least energy solutions is not true for systems. Indeed we have

Theorem 1.4 *When $N \geq 2$, there exists a potential $V(q)$ satisfying (v0)–(v2) and*

$$b > m,$$

where b is the mountain pass value and m is the least energy level for $I(q)$ given in (1.10).

We shall give an example of such potential at the end of the paper.

The proof of Theorem 1.2 crucially relies on the conservation of total energy, a consequence of the fact that (1.1) is autonomous. Points (i)-(ii) are somehow classical to establish. To prove (1.7) we first use an idea from Caldiroli [5] to obtain a path satisfying (1.6) and this proves that $b \leq m$. The proof that $b \geq m$ (remember we do not know in advance if b is a critical value) relies on a result by Rabinowitz and Tanaka [9].

2 Uniqueness of homoclinic and periodic solutions

From now on we assume $N = 1$ and (g0), (g1'), (g2'). Since we do not assume a Lipschitz condition on $g(s)$, some justifications are required for the uniqueness of homoclinic and periodic solutions of (1.1).

By (g2'), we can find $\delta_0 > 0$, $\ell_0 \in (0, \delta_0]$ such that

(i)

$$sg(s) \leq -\frac{c_0}{2}s^2 \quad \text{for } |s| \leq 2\delta_0, \tag{2.1}$$

$$g(s) > 0 \quad \text{for } s \in [s_0 - 2\delta_0, s_0]. \tag{2.2}$$

(ii) In particular, $G(s) = \int_0^s g(\tau)\, d\tau$ is monotone in $[0, 2\delta_0]$ and $[s_0 - 2\delta_0, s_0]$.

(iii)

For any $C \in [G(\ell_0), 0)$, the equation $G(s) = C$ has exactly 2 solutions. One of them — denoted by $\mu_0(C)$ — belongs to $(0, \delta_0]$ and the other — denoted by $\mu_1(C)$ — belongs to $[s_0 - \delta_0, s_0)$. (2.3)

Proposition 2.1 *For any $C \in [G(\ell_0), 0]$, the following problem has a unique solution $u(x)$ up to translation.*

$$\begin{aligned} -u_{xx} &= g(u) \qquad \text{in } \mathbb{R}, \\ \frac{1}{2}|u_x|^2 + G(u) &= C \qquad \text{in } \mathbb{R}, \\ u(\mathbb{R}) \cap (0, s_0] &\neq \emptyset. \end{aligned}$$

Moreover, if $C = 0$, $u(x)$ is a homoclinic solution emanating from 0. If $C \in [G(\ell_0), 0)$, it is a periodic solution.

Proof. We first deal with the case $C \in [G(\ell_0), 0)$. It is easily seen that $u(x) \in [\mu_0(C), \mu_1(C)]$ for all x. By (2.1) and (2.2), $u(x)$ cannot be constant and we can find a maximal interval $(a, b) \subset \mathbb{R}$, where $u_x(x) > 0$. We remark that

$$u_x = \sqrt{2}\sqrt{C - G(u)} \quad \text{in } (a, b).$$

Thus setting $F_C(s) = \int_{2\delta_0}^{s} \frac{1}{\sqrt{2}\sqrt{C-G(\tau)}}\, d\tau : (\mu_0(C), \mu_1(C)) \to \mathbb{R}$, we have

$$\frac{d}{dx} F_C(u(x)) = 1 \quad \text{in } (a, b). \tag{2.4}$$

Thus, we have for a suitable constant A

$$F_C(u(x)) = x + A \quad \text{in } (a, b).$$

We remark that F_C is a strictly increasing function and that $-\infty < F_C(\mu_0(C)) < F_C(\mu_1(C)) < \infty$ since $g(\mu_0(C)) < 0$, $g(\mu_1(C)) > 0$ (this follows from (2.1) and (2.2)). By the maximality of the interval, we observe that $u(a) = \mu_1(C)$, $u(b) = \mu_1(C)$.

In a maximal interval (a', b'), where $u_x < 0$, we can repeat the same argument and observe that

$$\frac{d}{dx} F_C(u(x)) = -1 \quad \text{in } (a', b') \tag{2.5}$$

and that $u(a') = \mu_1(C)$ and $u(b') = \mu_0(C)$. Thus in any maximal interval (α, β), where $u_x(x) \neq 0$, we have either (2.4) or (2.5). This shows that $u(x)$ is periodic and unique up to translation.

We argue in a similar way when $C = 0$. Setting $F_0(s) = \int_{2\delta_0}^{s} \frac{1}{\sqrt{2}\sqrt{-G(\tau)}}\, d\tau : (0, s_0) \to \mathbb{R}$, we have

$$\frac{d}{dt} F_0(f(x)) = \pm 1. \tag{2.6}$$

We also observe that $-\infty = F_0(0) < F_0(s_0) < \infty$ by (g2'). Thus any maximal interval, where $u_x > 0$ ($u_x < 0$ resp.), must be of a form $(-\infty, a)$ with $a < \infty$ ((b, ∞) with $-\infty < b$ resp.) and thus $u(x)$ is a homoclinic solution. Uniqueness also follows from (2.6). ∎

Corollary 2.2 *The set of all solutions of* (1.1) *is*

$$\{\pm\omega(x-y);\ y \in \mathbb{R}\} \cup \{0\},$$

where $\omega(x)$ *is the unique positive homoclinic solution of* (1.1) *satisfying* $\omega(0) = s_0$.

Proof. Let $u(x)$ be a non-zero solution of (1.1). Since $u(x)$ and $u_x(x)$ decay to 0 exponentially as $|x| \to \infty$, we have

$$\frac{1}{2}|u_x|^2 + G(u) = 0 \quad \text{in } \mathbb{R}.$$

We assume that $u(x)$ is positive somewhere and take a maximal interval (a, b) ((a', b') resp.), where $u > 0$ and $u_x > 0$ ($u_x < 0$ resp.). As in the proof of Proposition 2.1, we can see that $(a, b) = (-\infty, x_0)$ ($(a', b') = (x_0, \infty)$ resp.) for some $x_0 \in \mathbb{R}$. Thus $u(x)$ is a positive homoclinic solution and $u(x) = \omega(x - y)$ for a suitable $y \in \mathbb{R}$, by uniqueness. In the same way, we can also see that if $u(x)$ is negative somewhere, then $u(x)$ is a negative homoclinic solution and $u(x) = -\omega(x-y)$ holds for a suitable $y \in \mathbb{R}$. ∎

3 A mountain pass geometry for $I(u)$ and proof that $b = m$

In what follows we use the notation:

$$\begin{aligned} ||u||^2_{L^2(A)} &= \int_A |u|^2\, dx, \\ ||u||^2_{H^1(A)} &= \int_A |u_x|^2 + |u|^2\, dx \end{aligned}$$

for $A \subset \mathbb{R}$.

First we claim that $I(u)$ has a mountain pass geometry under (g0), (g1'), (g2'). We say that $I(u)$ has a *mountain pass geometry* if

(i) $I(0) = 0$.

(ii) There exists $\rho_0 > 0$ such that

$$\begin{aligned} I(u) &\geq 0 \quad \text{for all } ||u||_{H^1(\mathbb{R})} \leq \rho_0, && (3.1) \\ \inf_{||u||_{H^1(\mathbb{R})} = \rho_0} I(u) &> 0. && (3.2) \end{aligned}$$

(iii) There exists $u_0 \in H^1(\mathbb{R})$ such that

$$||u_0||_{H^1(\mathbb{R})} \geq \rho_0 \quad \text{and} \quad I(u_0) < 0. \qquad (3.3)$$

In fact, $I(0) = 0$ is trivial. By (2.1) we have

$$-G(s) \geq \frac{c_0}{4} s^2 \quad \text{for } |s| \leq 2\delta_0.$$

Choosing $\rho_0 > 0$ so that $||u||_{H^1(\mathbb{R})} \leq \rho_0$ implies $||u||_{L^\infty(\mathbb{R})} \leq 2\delta_0$, we have

$$\begin{aligned} \inf_{||u||_{H^1(\mathbb{R})}=\rho_0} I(u) &\geq \inf_{||u||_{H^1(\mathbb{R})}=\rho_0} \left(\frac{1}{2}||u_x||^2_{L^2(\mathbb{R})} + \frac{c_0}{4}||u||^2_{L^2(\mathbb{R})} \right) \\ &\geq \min\{\frac{1}{2}, \frac{c_0}{4}\}\rho_0^2. \end{aligned}$$

Thus (3.1)–(3.2) holds. Choosing $u_0(x) = \tilde{h}_L(x)$, where $\tilde{h}_L(x)$ is given in (3.6) below, we can also see that (3.3) holds. Thus $I(u)$ has a mountain pass geometry.

Let $\omega(x)$ be the unique positive homoclinic solution of (1.1) such that $\omega(0) = s_0$. We shall construct a path $\gamma \in \Gamma$ such that

$$\omega \in \gamma([0,1]), \tag{3.4}$$

$$\max_{t\in[0,1]} I(\gamma(t)) \leq I(\omega). \tag{3.5}$$

For this we use an idea from Caldiroli [5] and define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} \omega(x) & \text{in } [0,\infty), \\ x^4 + s_0 & \text{in } [-\varepsilon_0, 0), \\ \varepsilon_0^4 + s_0 & \text{in}(-\infty, -\varepsilon_0]. \end{cases}$$

Here $\varepsilon_0 > 0$ is chosen so that

$$\frac{1}{2}|h_x(x)|^2 - G(h(x)) = 8x^6 - G(x^4 + s_0) < 0 \quad \text{for } x \in (-\varepsilon_0, 0].$$

By (g2'), we can choose such an $\varepsilon_0 > 0$. For $y \in \mathbb{R}$, we also define

$$\tilde{h}_y(x) = \begin{cases} h(x-y) & \text{for } x \geq 0, \\ \tilde{h}_y(-x) & \text{for } x < 0. \end{cases} \tag{3.6}$$

We easily observe that

$$\begin{aligned} ||\tilde{h}_y||_{H^1(\mathbb{R})} &\to 0 \quad \text{as } y \to -\infty, \\ ||\tilde{h}_y||_{H^1(\mathbb{R})} &\to \infty \quad \text{as } y \to \infty, \\ \tilde{h}_0(x) &= \omega(x), \\ I(\tilde{h}_y) &\leq I(\omega) \quad \text{for all } y \in \mathbb{R}, \\ I(\tilde{h}_y) &\to -\infty \quad \text{as } y \to \infty. \end{aligned}$$

Thus, choosing a large $L > 1$ such that $I(\tilde{h}_L) < 0$ and defining

$$\gamma(t)(x) = \tilde{h}_{\varphi(t)}(x),$$

where $\varphi : (0,1] \to \mathbb{R}$ is a continuous function such that $\lim_{t\to 0} \varphi(t) = -\infty$ and $\varphi(1) = L$, we have (3.4) and (3.5). Summarizing our study, we get that $b \leq m$, where b and m are defined in (1.3) and (1.5).

At this point to end the proof of Theorem 1.2 we just have to show that $b \geq m$. We shall now prove this inequality.

For every $\gamma \in \Gamma$ one has that $||\gamma(0)||_\infty = 0$ and $||\gamma(1)||_\infty > s_0$. Hence, by continuity, there exists $t_0 \in (0,1)$ such that $||\gamma(t_0)|| = s_0$. Thus

$$\gamma([0,1]) \cap X_0 \neq \emptyset, \tag{3.7}$$

where

$$X_0 = \{u \in H^1(\mathbb{R}); ||u||_\infty = s_0\}.$$

This implies

$$b \geq \inf_{u \in X_0} I(u). \tag{3.8}$$

Now thanks to a result by Rabinowitz and Tanaka, Theorem 2.2 in [9], there exists a homoclinic orbit $\bar{u} \in X_0$ such that $I(\bar{u}) = \inf_{u\in X_0} I(u)$. Then, by (ii) of Theorem 1.2, $\bar{u} = \omega(\cdot + t_0)$ for some $t_0 \in \mathbb{R}$ and therefore $b \geq I(\bar{u}) = I(\omega) = m$. ∎

4 A counter example for systems — Proof of Theorem 1.4

We end the paper by showing that the mountain pass characterization of least energy solutions established in Theorem 1.2 do not hold any more if (1.1) is replaced by a system. In this case the uniqueness (up to translation) of solutions (see (ii) of Theorem 1.2) may fail, and, as we shall see, the minimizer in X_0 obtained in [9] may not be a least energy solution. According to the proof of $b \geq m$ this shows that in general $b > m$.

To construct a counter example, we use an idea from singular Hamiltonian systems. Let

$$\ddot{q} + \nabla V(q) = 0. \tag{4.1}$$

Choose $e \in \mathbb{R}^N \setminus \{0\}$, $N \geq 2$ and let $V(q) \in C^2(\mathbb{R}^N \setminus \{e\}, \mathbb{R})$ be a potential satisfying for a small $h > 0$

(V1) $V(q) \leq 0$ for all $q \in \mathbb{R}^N$ and $V(q) = 0$ holds if and only if $q = 0$.

(V2) $V(q) = -1$ in $\mathbb{R}^N \setminus (B_h(0) \cup B_h(e))$.

(V3) $V''(0)$ is negative definite.

(V4) $V(q) = -\frac{1}{|q-e|^2}$ for $q \in B_{h/2}(e)$.

Here we use the notation: $B_h(y) = \{x \in \mathbb{R}^N;\, |x-y| < h\}$ for $y \in \mathbb{R}^N$. By the result of Tanaka [11], we know that (4.1) has at least one non trivial homoclinic orbit $q_0(t)$. We remark that $q_0(t)$ satisfies

$$|q_0(t)| \le |e| + h \quad \text{for all } t \in \mathbb{R}.$$

Indeed, suppose that $t_0 \in \mathbb{R}$ satisfies $|q_0(t_0)| = \max_{t\in\mathbb{R}} |q_0(t)|$. Then $\frac{1}{2}\frac{d^2}{dt^2}|q_0(t_0)|^2 \le 0$ and thus we have

$$(\ddot{q}_0(t_0), q_0(t_0)) + |\dot{q}_0(t_0)|^2 = -(\nabla V(q_0(t_0)), q_0(t_0)) - 2V(q_0(t_0)) \le 0.$$

Here we used the fact that $\frac{1}{2}|\dot{q}|^2 + V(q_0(t)) \equiv 0$. By (V2) it can take place only in $B_h(0) \cup B_h(e)$ and we have $|q_0(t_0)| \le |e| + h$.

Set $\delta_0 = \min |q_0(t) - e|$ and modify $V(q)$ in $B_{\delta_0/2}(e)$ to get a smooth potential $\tilde{V}(q)$ satisfying $\tilde{V}(q) < 0$ for all $q \ne 0$. Next for a large $R > 0$ we set

$$\tilde{V}_R(q) = \begin{cases} \tilde{V}(q) & \text{if } |q| \le R, \\ (|q| - R)^2 - 1 & \text{if } |q| \ge R, \end{cases}$$

and we consider the following Hamiltonian system

$$\ddot{q} + \nabla \tilde{V}_R(q) = 0. \tag{4.2}$$

Let us show that for sufficiently large $R > 1$ the mountain pass level b_R associated with (4.2) satisfies $b_R > m$ where m is the least energy level associated with (4.2). Indeed, $q_0(t)$ is still a solution of (4.2) and thus the least energy level for (4.2) is bounded from above by a constant independent of R. On the contrary for the mountain pass level b_R in a similar way to (3.7)–(3.8), we have

$$b_R \ge 2 \inf_{q \in X_R} \int_0^\infty \frac{1}{2}|\dot{q}|^2 - \tilde{V}_R(q)\, dt,$$

where

$$X_R = \{q(t) \in H^1(0,\infty;\mathbb{R}^N);\, |q(0)| = R+1, |q(t)| \le R+1 \text{ for all } t \in [0,\infty)\}.$$

Let $q \in X_R$ be any function, we choose $[s_0, s_1] \subset (0, \infty)$ so that

$$|q(s_0)| = R,\ |q(s_1)| = R/2,\ |q(t)| \in [R/2, R] \text{ for } t \in [s_0, s_1].$$

Since we may assume $V(q) = -1$ in $B_R(0) \setminus B_{R/2}(0)$, we have

$$\begin{aligned} \int_0^\infty \frac{1}{2}|\dot{q}|^2 - V(q)\, dt &\ge \int_{s_0}^{s_1} \frac{1}{2}|\dot{q}|^2 - V(q)\, dt = \int_{s_0}^{s_1} \frac{1}{2}|\dot{q}|^2 + 1\, dt \\ &\ge \int_{s_0}^{s_1} \sqrt{2}|\dot{q}|\, dt \ge \sqrt{2}|q(s_1) - q(s_0)| \\ &\ge \frac{R}{\sqrt{2}}. \end{aligned}$$

This shows that $b_R \to \infty$ as $R \to \infty$ and thus $b_R > m$ for $R > 0$ large enough. ∎

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