A NEW VARIATIONAL APPROACH TO BIFURCATION INTO SPECTRAL GAPS *

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Abstract

We consider a nonlinear equation posed in a Hilbert space H

$$(P) \quad (A - \lambda L)u = N(u).$$

The operators A and L are linear, bounded and self-adjoint. The nonlinear term N satisfy $N(0) = 0$. Assuming that $]a, b[$ is a spectral gap of the combined spectrum $\rho(A, L) := \{\lambda \in I\!R : A - \lambda L : H \to H$ is an isomorphism$\}$ we show that $\lambda = b$ is a bifurcation point for (P). Namely that there exists a sequence $\{(\lambda_n, u_n)\} \subset]a, b[\times H$ of nontrivial solutions of (P) such that $\lambda_n \to b$ and $\|u_n\| \to 0$. For this only mild conditions on N around $u = 0$ are required. A local Lyapunov-Schmidt reduction permits to overcome the strong indefiniteness of the problem. The proof is then based on an original variational approach of mountain pass type.

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1 Introduction

In this paper, we shall be concerned with the nonlinear equation

$$(P) \quad (A-\lambda L)u = N(u) \ \text{ in } \ H$$

where H is a real Hilbert space. We denote by $< \cdot, \cdot >$ and $\| \cdot \|$ respectively its scalar product and norm. The operators A and L are linear, bounded and selfadjoint with

(A1) $< Lu, u >> 0$ for $u \in H \backslash \{0\}$, $\sigma(A) \cap I\!R^+ \neq \emptyset$, $\sigma(A) \cap I\!R^- \neq \emptyset$ and $0 \notin \sigma(A)$.

For the nonlinear term N we assume there exist a $\varepsilon_0 > 0$ and a positive function $\phi \in C^2(B_{\varepsilon_0}, I\!R)$ with $N = \nabla \phi$ on $B_{\varepsilon_0} := \{u \in H : \|u\| \leq \varepsilon_0\}$ which satisfies

(A2) $\dfrac{\phi(u)}{\|u\|^2} \to 0$ as $\|u\| \to 0$,

(A3) there exists $q > 2$ such that $< N(u), u > \leq q\phi(u)$ for all $u \in B_{\varepsilon_0}$.

Let $\rho(A, L) = \{\lambda \in I\!R : A - \lambda L : H \to H$ is an isomorphism $\}$ and $\sigma(A, L) = I\!R \backslash \rho(A, L)$. ¿From (A1), there exist $a, b \in I\!R$, $a < 0 < b$ such that $[a, b] \cap \sigma(A, L) = \{a, b\}$ (see Lemma 2.1). Thus 0 lies in the spectral gap $]a, b[$ of $\sigma(A, L)$. Throughout the paper, we shall refer as problem (P) the issue of finding nontrivial solutions of equation (P) when $\lambda \in]a, b[$.

The aim of our work is to show that under mild assumptions on the operator N around $u = 0$, $b \in \sigma(A, L)$ is a bifurcation point for (P). Namely that there exists a sequence $\{(\lambda_n, u_n)\} \subset]a, b[\times H$ of nontrivial solutions of (P) such that $\lambda_n \to b$ and $\|u_n\| \to 0$. We make no assumption on b. It may be an eigenvalue (of finite or infinite multiplicity) or a point of the continuous spectrum. Note that by (A2), $N(0) = 0$ and thus $(\lambda, 0) \in I\!R \times H$ is always solution of (P).

Our approach is variational. It starts with the observation that $(\lambda, u) \in I\!R \times B_{\varepsilon_0}$ is a solution of (P) if and only if u is a critical point of the functional :

$$J(\lambda, u) = \frac{1}{2} < (A - \lambda L)u, u > -\phi(u).$$

By the spectral theorem for self-adjoint operators, since $0 \notin \sigma(A)$, H splits into two orthogonal subspaces V and W corresponding respectively to the positive and negative part of $\sigma(A)$, namely $H = V \oplus W$. Let P and Q be respectively the orthogonal projections of H on V and W. For $\lambda \in]a, b[$ the quadratic form $< (A - \lambda L)u, u >$ is positive definite on V and negative definite on W (see Lemma 2.1). In the general situation, we consider, both

V and W are allowed to be infinite dimensional. Thus for $\lambda \in]a,b[$, $J(\lambda,\cdot)$ is strongly indefinite and to find a critical point of $J(\lambda,\cdot)$ standard variational procedures, used when $W = \{0\}$, such as the mountain pass theorem cannot be applied.

Another difficulty we shall face searching for a critical point is a possible lack of compactness. It may happen, for example, when the functional is invariant with respect to a group whose orbits are not compact. To deal with some cases of this kind, we introduce the following terminology that we borrow from [25].

Let $O(H)$ denote the group (with respect to composition) of all isometric isomorphisms of H. Given a subgroup G of $O(H)$, $\Theta(u) = \{Tu : T \in G\}$ is the orbit containing $u \in H$ generated by G. A functional $K \in C^1(H, I\!R)$ is called G-invariant if and only if $K(Tu) = K(u)\, \forall\, u \in H, T \in G$. In this case, it follows that $K'(Tu)Tv = K'(u)v\ \forall\, u, v \in H$ and so $T^*\nabla K(Tu) = \nabla K(u)$ $\forall\, u \in H, T \in G$. Thus, ∇K is G-equivariant and we note that $\forall\, u \in H$ and $v \in \Theta(u)$, $K(u) = K(v)$ and $\|\nabla K(u)\| = \|\nabla K(v)\|$.

Definition 1.1 *Given $K \in C^1(H, I\!R)$ and a subgroup G of $O(H)$, we say that K is weakly upper G-compact on H provided that*

1. *K is G-invariant,*

2. *from every bounded sequence $\{u_n\}$ in H such that $K(u_n) \to c > K(0)$ and $\|\nabla K(u_n)\| \to 0$, we can extract a subsequence $\{u_{n_i}\}$ and select elements $v_n \in \Theta(u_n)$, such that $v_{n_i} \rightharpoonup v$ weakly in H where $v \neq 0$ and $\nabla K(v) = 0$.*

To prove that b is a bifurcation point for (P), we exhibit a particular sequence $\{\lambda_n\} \subset]a,b[$, $\lambda_n \to b$ and an associated sequence $\{u_n\} \subset H\backslash\{0\}$ of critical points of $J(\lambda_n,\cdot)$ for which $\|u_n\| \to 0$ as $\lambda_n \to b$. The following condition plays a crucial role to control the norm of u_n, $n \in I\!N$.

For $\delta > 0$ we say that the condition $T(\delta)$ is satisfied if $PL = LP$ and if there exists a $\varepsilon \in]0, \varepsilon_0]$ and a sequence $\{u_n\} \subset H$ with $\|u_n\| = \varepsilon$ such that $\phi(u_n) > 0$ for all $n \in I\!N$ and

$$\lim_{n\to\infty} \frac{< (A - bL)u_n, u_n >}{\phi(u_n)^\delta} = \lim_{n\to\infty} \frac{\|(A - bL)u_n\|^2}{\phi(u_n)^\delta} = 0.$$

We now state our main result.

Theorem 1.1 *Suppose that (A1)-(A3) hold and that $T(\delta)$ is satisfied for a $\delta \geq 1$. Assume also that*

(A4) There exists $K > 0$ such that $\|N(u)\| \leq K\phi(u)^{1-\frac{\delta}{2}}$ for all $u \in B_{\varepsilon_0}$.

(A5) *(i)* *Either N is compact or*

(ii) *for a subgroup G of $O(H)$ and for $\lambda < b$ close to b, $J(\lambda, \cdot)$ is weakly upper G-compact in B_{ε_0}.*

Then, there exists a sequence $\{(\lambda_n, u_n)\} \subset]a, b[\times H$ of nontrivial solutions of (P) such that $\lambda_n \to b^-$ and $||u_n|| \to 0$. In particular, b is a bifurcation point for (P).

Remark. Since, by (A2), $\phi(u) \to 0$, $||N(u)|| \to 0$ and $||N'(u)|| \to 0$ as $\|u\| \to 0$, both ϕ, N, N' are bounded on any ball B_ε, centred at the origin, of radius $0 < \varepsilon < \varepsilon_0$ sufficiently small. Throughout the paper we shall assume that it is already true for $\varepsilon_0 > 0$. Thus (A4) is always satisfied when $T(\delta)$ holds with a $\delta \geq 2$.

Remark. Requiring condition $T(\delta)$ to hold with a $\delta \geq 1$ (or an equivalent condition) is standard in all the works dealing with bifurcation within spectral gaps (see [25]). The purpose of the condition is discussed later in the introduction. For the moment observe that $PL = LP$ implies that V is invariant for A and L. In this case, if there exists an eigenvector $u \in \mathrm{Ker}(A - bL)$ with $\phi(u) > 0$, $T(\delta)$ is trivially satisfied for all $\delta > 0$. However the condition may also be satisfied for some values of δ even when $\mathrm{Ker}(A - bL) = \{0\}$ (see [25]).

An important motivation for studying problem (P) is that it can be viewed as the abstract formulation of several physical models. For example, nonlinear Schrödinger equations of the form :

$$-\Delta u(x) + p(x)u(x) - f(x, u(x)) = \lambda u(x), \quad x \in \mathbb{R}^N \tag{1.1}$$

with p a periodic function in $\mathbb{R}^N$ and f a nonlinear term, can be set in the form of (P). Here the existence of nontrivial solutions reveals the presence of bounded states whose "energy" $\lambda \in \mathbb{R}$ lies in gaps of the spectrum of the linear operator $-\Delta + p$. These bounded states are, so to speak, created by the nonlinear pertubation. In [8] a refined Choquard-Pekar model, relevant in solid state physics, was studied via an abstract formulation of type (P). Equation (P) can also be used to describe nonlinear Dirac equations [10] or even Hamiltonian systems [5, 9, 25]. We refer the reader to [25] where the connection between the abstract formulation (P) and several "concrete" problems (as (1.1)) is established.

We shall now briefly describe what we believe to be the more significant results on problems of type (P) or on the special form (1.1). These results essentially differ by the conditions which are imposed on ϕ (i.e. on N). We distinguish five main conditions :

(N1) ϕ is globally defined on H.

(N2) ϕ is convex.

(N3) ϕ is superquadratic at infinity, namely there exist $p > 2$ and $R > 0$ such that $< N(u), u > \geq p\phi(u)$ for all $u \in H$ with $\|u\| \geq R$.

(N4) There exist C, $D > 0$ such that

$$\|N(u)\| \leq C + D\phi(u), \quad \forall\, u \in H.$$

(N5) N is compact.

A first possible approach is to construct, on a suitable space, a functional having a mountain pass geometry whose critical points correspond to solutions of (P). In that direction, we mention the work of Buffoni, Jeanjean and Stuart [8]. In [8], we look for a solution of (P) when $\lambda \in]a, b[$ is fixed. We use a global Lyapunov-Schmidt reduction to control the part of the solutions in the space W. It leads to study a functional defined only on V. An application of the mountain pass lemma (see [4]) then permits to obtain the desired critical point. This reduction requires ϕ to be globally defined and convex. Subsequently, this approach was extended to study the bifurcation at b by Buffoni [5] and finally by Stuart [25]. In [25], the same conclusion of our Theorem 1.1 is obtained (see Theorem 7.2). In addition to our assumptions, conditions (N1) to (N4) are needed and when N is not compact, the function $< N(u), u > -2\phi(u)$, has to be weakly sequentially lower-semicontinuous.

Among the works relying on an equivalent mountain pass formulation, we also mention [1]. In this paper Alama and Li develop, on a specific class of nonlinear Schrödinger equations with periodic potential, a dual approach in the spirit of [3] (see also [9]). Global conditions and convexity on ϕ are also required. In addition, special features of the class come into play. They imply in particular (N3) and (N4). The paper of Alama and Li deals with the search of solutions for fixed $\lambda \in]a, b[$. Subsequently, their approach was refined in [18] and [19] where the existence of bifurcation points is studied.

A second approach developed to handle (P) consists in searching directly a critical point of $J(\lambda, \cdot)$. Heinz [11, 12] opened this route. He obtained the existence of a solution for any $\lambda \in]a, b[$, using the linking theorem of Benci and Rabinowitz (see [2]) and studied the bifurcation of solutions at $\lambda = b$. His approach was subsequently refined in [13, 14]. Strong assumptions on ϕ are needed in these works. In addition to (N1)-(N4), condition (N5) has to hold. Indeed, the compacity of the nonlinear term is a key ingredient in the Benci-Rabinowitz's theorem.

More recently, substantial improvements along this approach were made by Troestler and Willem [27, 28] (see also [23]). They demonstrate the existence of a nontrivial solution for any $\lambda \in]a,b[$, without assuming convexity nor compacity, for a specific class of nonlinear Schrödinger equations of type (P). Their argument is based upon a generalised linking theorem due to Hofer and Wysocki (see [15]) which had been already used by Esteban and Séré to solve a nonlinear Dirac equation of the form of (P) (see [10]). The approach of [27] was extended by Troestler in [26] to deal with bifurcation. He proved that bifurcation occurs at b without assuming convexity or compacity. However, ϕ still need to be globally defined and superquadratic. Moreover, his arguments to remove the convexity are closely linked with the particular equations he considered.

The third and last approach is actually the oldest. Here, the idea is to use a constrained variational procedure. One looks for solutions of (P) having a small but prescribed norm. The λ now appears as a Lagrange parameter. One finds a sequence of solutions $\{(\lambda_n, u_n)\} \subset]a,b[\times H$ where by construction $\|u_n\| \to 0$ as $n \to \infty$. Then, one checks, a posteriori, that $\lambda_n \to b$. This approach was introduced by Küpper and Stuart [16] and substantial improvements were made by Buffoni and Jeanjean [6, 7]. To our knowledge, [7] is the only result where ϕ needs just to be defined around the origin. Also (N4) and (N5) are removed. We need however both (N2) and (N3).

Let us now sketch the proof of Theorem 1.1. First we show, in Section 2, that solutions (λ, u) of (P) for λ close to b and $\|u\|$ small are of the form $(\lambda, v + g(\lambda, v))$ with $v \in V$. The function g, defined in a neighborhood of $(b, 0) \subset \mathbb{R} \times V$, is obtained by an implicit function theorem. Having done this Lyapunov-Schmidt reduction we may define the functional $F(\lambda, v) := J(\lambda, v + g(\lambda, v))$ on a small ball $B_c(V) := \{u \in V : \|u\| \leq c\}$ of V. In particular F has the property that if v is a critical point of $F(\lambda, \cdot)$ then $v + g(\lambda, v)$ is a critical point of $J(\lambda, \cdot)$. For the reduction we need (A2) but, performing only a local reduction, we manage, in contrast to [8, 25], not to require ϕ convex. Now, because $F(\lambda, \cdot)$ is just defined in $B_c(V)$ we must develop a variational argument within this ball. In Section 3, we show that $F(\lambda, \cdot)$ has in $B_c(V)$ a mountain pass geometry for λ sufficiently close to b. Namely there exists $\lambda_0 \in]a,b[$ such that setting

$$\Gamma_\lambda := \{\gamma \in C([0,1], B_c(V)) / \gamma(0) = 0\ ,\ F(\lambda, \gamma(1)) < 0\},$$

we have that Γ_λ is non void for all $\lambda \in [\lambda_0, b[$ and

$$c(\lambda) := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} F(\lambda, \gamma(t)) > 0.$$

This geometry of $F(\lambda, \cdot)$ is proved using the condition $T(\delta)$ where $\delta \geq 1$. We shall see in Section 4 that, under our assumption (A5), any Palais-Smale sequence for $F(\lambda, \cdot), \lambda \in [\lambda_0, b[$ at the level $c(\lambda) > 0$ (i.e. a $\{v_n\} \subset B_c(V)$ such that $F(\lambda, v_n) \to c(\lambda)$ and $\nabla_v F(\lambda, v_n) \to 0$) leads to a nontrivial critical point of $F(\lambda, \cdot)$. To end the proof of Theorem 1.1 we shall prove that at least for a sequence $\{\lambda_n\} \subset]a, b[$, $\lambda_n \to b$, $F(\lambda_n, \cdot)$ possesses a Palais-Smale sequence in $B_c(V)$ whose "size" goes to zero as $\lambda_n \to b$. For this two main ingredients will be used. The fact the function $\lambda \to c(\lambda)$ is monotone decreasing and estimates on the behavior of $c(\lambda)$ as $\lambda \to b$ obtained as consequence of condition $T(\delta)$ when $\delta \geq 1$ (see Section 3). The proof goes as follow. First, in Section 4, the estimates, combined with the decrease of $\lambda\ddot{!}oc(\lambda)$ permit to establish the existence of a strictly increasing sequence $\{\lambda_n\} \subset]a, b[$, $\lambda_n \to b$, on which both $c(\lambda_n) \to 0$ and $c'(\lambda_n) \to 0$. Here $c'(\lambda)$ denotes the derivative of $c(\lambda)$. Then we prove that for all $n \in I\!N$, $F(\lambda_n, \cdot)$ admits a Palais-Smale sequence at level $c(\lambda_n)$ contained in a ball of $B_c(V)$, centred at the origin, whose radius goes to zero as $c(\lambda_n) \to 0$ and $c'(\lambda_n) \to 0$. The key point here is to explicit a special sequence of minimizing (for $c(\lambda_n)$) paths in Γ_{λ_n} which satisfies some "localisation" properties (see Proposition 4.2) implying the existence of our Palais-Smale sequence.

Remark. The fact that the monotonicity of $\lambda \to c(\lambda)$ plays a role in our proof is reminiscient of Struwe's work on the so-called "monotonicity trick" (see for example [24], Chapter II, Section 9). On various, specific examples, he first showed how the monotonicity of $c(\lambda)$ can be used to derive that an associated family of functionals has a bounded Palais-Smale sequence for almost every value of λ. Recently Struwe's approach has been extended and renewed as to cover general abstract settings [20, 22]. In particular, in [22], the monotonicity condition is no more required. However to obtain a bifurcation result the mere boundedness of the Palais-Smale sequences (indeed here automatically insured since $F(\lambda, \cdot)$ is defined only on $B_c(V)$) is not enough. The idea is to relate precisely the "size" of the Palais-Smale sequence to the quantities $c(\lambda)$ and $c'(\lambda)$ on which the test functions of condition! $T(\delta)$, $\delta \geq 1$ give us informations. We pursue here in the direction of [21] where the behavior of $\lambda \to c(\lambda)$ was first used to study a simpler situation of bifurcation from the infimum of the spectrum.

Remark. If one assume that ϕ is defined on all H and convex it is possible to define $F(\lambda, .)$ on all V and for every $\lambda \in]a, b[$ (see [25] for example). To obtain a critical point for $F(\lambda, \cdot)$ one then faces the problem of a priori bounds on the Palais-Smale sequences. The main motivation for introducing (N3) and (N4) is to ensure that all Palais-Smale sequences for $F(\lambda, \cdot)$ are bounded. In our case, the corresponding difficulty is to avoid that suspected Palais-Smale sequences accumulate on the boundary of $B_c(V)$. However, since the "size" of the Palais-Smale sequence is proved smaller and smaller (as $\lambda_n \to b$), this may not occur.

Remark. A close look at our proofs reveals that the purpose of requiring condition $T(\delta)$ with $\delta \geq 1$ is to ensure (in combinaison with (A2)) a mountain pass geometry for F on $B_c(V)$ and to guarantee a sufficient decrease of $c(\lambda) \to 0$ as $\lambda \to b$.

Remark. If one wants to apply Theorem 1.1 to specific problems, as for example the study of the nonlinear Schrödinger equation (1.1), it is necessary to check that $T(\delta)$ hold for a $\delta \geq 1$. We refer to [25] where this is done for a class of problems of type (1.1) under appropriate conditions on f (see Lemma 9.5). Note also that an application of our Theorem 1.1, in the frame of Hamiltonian systems, will soon be available in [17].

2 Transforming the problem

In this section, using a Lyapunov-Schmidt reduction, we construct an equivalent problem posed on a ball of the subspace V and we give a variational interpretation of this reduction. We also introduce the functional F and we show that it is monotone as a function of $\lambda \in]a, b[$ for λ sufficiently close to b. Before stating the main results of the section we derive some properties of the linear problem associated to problem (P). We also make some observations on the local nature of our assumptions on N.

The spectral theorem for self-adjoint operator asserts that $0 \notin \sigma(A)$ is equivalent to the existence of V, $W = V^{\perp}$ and $\alpha,\ \beta \in]0, +\infty[$ such that

(i) $A(V) \subset V$

(ii) $< Au, u > \geq \beta \|u\|^2,\ \forall\, u \in V$

(iii) $< Au, u > \leq -\alpha\, \|u\|^2,\ \forall\, u \in W.$

Since, by assumption (A1), $\sigma(A) \cap I\!\!R^+ \neq \emptyset$, and $\sigma(A) \cap I\!\!R^- \neq \emptyset$ both V and W are here nontrivial. Now we introduce the following quantities which play a fundamental role in the discussion of (P)

$$a = \sup \left\{ \frac{< Au, u >}{< Lu, u >},\ u \in W,\ u \neq 0 \right\}$$

$$b = \inf \left\{ \frac{< Au, u >}{< Lu, u >},\ u \in V,\ u \neq 0 \right\}$$

$$m(\lambda) = \begin{cases} \alpha(1 + \frac{\lambda}{a}) \text{ for } \lambda \leq 0 \\ \alpha \text{ for } \lambda > 0 \end{cases}$$

$$n(\lambda) = \begin{cases} \beta(1 - \frac{\lambda}{b}) \text{ for } \lambda > 0 \\ \beta \text{ for } \lambda \leq 0 \end{cases}$$

and finally

$$M(\lambda) = \min\{m(\lambda), n(\lambda)\}.$$

These quantities relate to the properties of $A - \lambda L$ as follows.

Lemma 2.1 *Let (A1) be satisfied. Then,* $]a, b[\subset \rho(A, L)$ *and*

1. $a < 0 < b$,
2. $< (A - \lambda L)u, u > \geq n(\lambda)\|u\|^2$, $\forall u \in V$ *and* $\lambda \leq b$,
   $< (A - \lambda L)u, u > \leq -m(\lambda)\|u\|^2$, $\forall u \in W$ *and* $\lambda \geq a$,
3. $\|(A - \lambda L)u\| \geq M(\lambda)\|u\|$, $\forall u \in H$ *and* $a \leq \lambda \leq b$,
4. *If* $PL = LP$ *then* $\{a, b\} \cap \rho(A, L) = \emptyset$.

Proof. See [25]. □

Concerning the assumptions (A2)-(A5) on N, it should be clear that they still hold if we replace $\varepsilon_0 > 0$ by any $\varepsilon \in]0, \varepsilon_0]$. This is less obvious but also true for condition $T(\delta)$. Indeed an easy consequence of (A3) is that

$$\text{for any } t \in [0, 1] \text{ and } u \in B_{\varepsilon_0},\ \phi(tu) \geq t^q \phi(u). \tag{2.1}$$

Finally for further reference we note that by (A1) and (A2),

(2.2) for any $\varepsilon > 0$ sufficiently small, $4\phi(u) \leq < Au, u >, \forall u \in B_\varepsilon(V)$.

Now, we can give the first main result of this section. It is based on the implicit function theorem

Lemma 2.2 *Suppose that (A1)-(A2) are satisfied and* $PL = LP$. *There exists a* $\varepsilon_1 \in]0, \varepsilon_0]$, *an open connected neighbourhood* U *of* 0 *in* W *and a unique function* $g \in C^1(V(b) \times B_{\varepsilon_1}(V), U)$ *where* $B_{\varepsilon_1}(V)$ *is the ball centred in* 0 *of radius* ε_1 *in* V *and* $V(b)$ *an open connected neighborhood of* b, *satisfying the following assertions :*

(i) $g(\lambda, 0) = 0$, $g(V(b) \times B_{\varepsilon_1}(V)) \subset U$ *and for* $(\lambda, v) \in V(b) \times B_{\varepsilon_1}(V)$ *:*

$$Q\nabla_u J(\lambda, v + g(\lambda, v)) = 0. \tag{2.3}$$

(ii) If $(\lambda, v) \in V(b) \times B_{\varepsilon_1}(V)$, *then* $\nabla_u J(\lambda, v + g(\lambda, v)) = P\nabla_u J(\lambda, v + g(\lambda, v))$.

(iii) *If* $(v,w) \in B_{\varepsilon_1}(V) \times U$, *then* $Q\nabla_u J(\lambda, v+w) = 0 \Leftrightarrow w = g(\lambda, v)$.

(iv) *If* u *is a critical point of* $J(\lambda,\cdot)$ *such that* $(\lambda, Pu, Qu) \in V(b)\times B_{\varepsilon_1}(V)\times U$, *then* $Qu = g(\lambda, Pu)$.

Remark. Clearly $\|g(\lambda,v)\| \to 0$ as $\lambda \to b$ and $\|v\| \to 0$ by continuity of g. This is why we can assume without loss of generality that J is well defined at $(\lambda, v + g(\lambda, v))$ in Lemma 2.2.

Proof of Lemma 2.2. We define G in $I\!R \times B_{\frac{\varepsilon_0}{2}}(V) \times B_{\frac{\varepsilon_0}{2}}(W)$ by

$$G(\lambda, v, w) = Q\nabla_u J(\lambda, v + w).$$

It clearly satisfies $G(b, 0, 0) = 0$. Now, an easy computation shows that for any z in W,

$$D_w G(\lambda, v, w)z = (A - \lambda L)z - QN'(v + w)z.$$

Then, by (A2), we obtain :

$$D_w G(b, 0, 0)z = (A - bL)z \quad \text{for } z \in W.$$

Therefore, by Lemma 2.1, it follows that :

$$< D_w G(b, 0, 0)z, z > \leq -m(b)\|z\|^2$$

and thus, $D_w G(b, 0, 0)$ is invertible in W. Applying the implicit function theorem, there exists an open connected neighborhood $\Theta = V(b) \times B_{\varepsilon_1}(V)$ of $(b, 0)$ in $I\!R \times V$, U an open connected neighborhood of 0 in W and a unique C^1−function $g : \Theta \longrightarrow U$ such that for $(\lambda, v, w) \in V(b) \times B_{\varepsilon_1}(V) \times U$, we have :

$$[\nabla_u J(\lambda, v + w) = P\nabla_u J(\lambda, v + w)] \Leftrightarrow w = g(\lambda, v).$$

This proves the assertions (i),(ii),(iii) and (iv) follows from (iii). □

Our next result gives a variational interpretation of the function g. Namely the functional $w \to J(\lambda, v + w)$ for $\lambda \in V(b)$ and $v \in V$ fixed has a unique local maximum in $w = g(\lambda, v)$. From this we deduce, in particular, that $g(\lambda, v) = o(v)$ for v near 0. More precisely, fixing an arbitrary $\bar\lambda \in V(b)$ with $\bar\lambda \in]0, b[$ we have

Proposition 2.3 *Let* $\varepsilon_1 \in]0, \varepsilon_0]$ *be as in Lemma 2.2.*

(i) *There exists a constant* $C > 0$ *and* $\varepsilon_2 \in]0, \varepsilon_1[$ *such that, for any* $\varepsilon \in]0, \varepsilon_2]$, *if* $v \in B_\varepsilon(V)$ *then* $g(\lambda, v) \in B_{C\varepsilon}(W)$. *Moreover for all* $v \in B_\varepsilon(V)$ *with* $\varepsilon > 0$ *sufficiently small*

$$J(\lambda, v + w) \leq J(\lambda, v + g(\lambda, v)), \forall w \in B_{C\varepsilon}(V), \forall\lambda \in [\bar\lambda, b[.$$

(ii) $||g(\lambda, v)|| = o(||v||)$ *as* $\|v\| \to 0$ *uniformly in* $\lambda \in [\bar\lambda, b[$*. In particular for any* $\varepsilon > 0$ *sufficiently small,* $g(\lambda, B_\varepsilon(V)) \subset B_\varepsilon(W), \ \forall \lambda \in [\bar\lambda, b[$.

Proof. (i) Since g is C^1 on $V(b) \times B_{\varepsilon_1}(V)$, for $\varepsilon_2 > 0$ small enough, setting $C = \sup\limits_{[\bar\lambda, b] \times B_{\varepsilon_2}(V)} \|\nabla_v g(\lambda, v)\|$ we can assume that $C < \infty$. Thus

$$\|g(\lambda, v)\| = \|g(\lambda, v) - g(\lambda, 0)\| \le C\|v\|$$

establishing that $g([\bar\lambda, b[\times B_\varepsilon(V)) \subset B_{C\varepsilon}(W)$ for any $\varepsilon \in]0, \varepsilon_2]$. Now, for $v \in B_\varepsilon(V)$ fixed with $\varepsilon \in]0, \varepsilon_2]$, we define $\Phi_{\lambda,v} : B_{C\varepsilon}(W) \to I\!R$ by

$$\Phi_{\lambda,v}(w) = J(\lambda, v + w).$$

Making $\varepsilon_2 > 0$ smaller if necessary we can assume that $v + w \in B_{\varepsilon_0}$ and thus it is well defined. Now setting $\eta := (1 + C^2)^{\frac{1}{2}}\varepsilon > 0$ and $K(\eta) := \sup\limits_{u \in B_\eta} \|N'(u)\|$, we have for $(v, w) \in B_\varepsilon(V) \times B_{C\varepsilon}(W)$, $\lambda \in [\bar\lambda, b[$ and $z \in W$:

$$\begin{aligned} D_w^2 \Phi_{\lambda,v}(z, z) &= <(A - \lambda L)z, z> - <QN'(v + w)z, z> \\ &\le -m(\lambda)\|z\|^2 + K(\eta)\|z\|^2 \\ &\le -m(\bar\lambda)\|z\|^2 + K(\eta)\|z\|^2. \end{aligned}$$

Now, since (A2) implies that $||N'(u)|| \to 0$ as $||u|| \to 0$, we have $K(\eta) \to 0$ as $\varepsilon \to 0$. Thus for $\varepsilon > 0$ small enough $K(\eta) < m(\bar\lambda)$. Consequently $\Phi_{\lambda,v}$ is strictly concave and its (unique) maximum is $w = g(\lambda, v)$ by Lemma 2.2 (iii). This gives (i).

(ii) Take $\varepsilon \in]0, \varepsilon_2]$ such that (i) holds. Setting $\eta := \sup\limits_{u \in B_\varepsilon} \dfrac{\phi(u)}{\|u\|^2} \ge 0$, we have for $(v, w) \in B_\varepsilon(V) \times B_{C\varepsilon}(W)$ and $\lambda \in [\bar\lambda, b[$,

$$\begin{aligned} \Phi_{\lambda,v}(w) - \Phi_{\lambda,v}(0) &= \frac{1}{2} <(A - \lambda L)w, w> - \phi(v + w) + \phi(v) \\ &\le -\frac{1}{2} m(\lambda)\|w\|^2 + \eta\|v\|^2. \end{aligned}$$

Now the variational characterisation of $g(\lambda, v)$ implies that

$$-\frac{1}{2} m(\lambda)\|g(\lambda, v)\|^2 + \eta\|v\|^2 \ge 0$$

which leads to

$$\|g(\lambda, v)\|^2 \le \frac{2\eta\|v\|^2}{m(\bar\lambda)}.$$

Since $\eta \to 0$ as $\varepsilon \to 0$ by (A2) we indeed check that

$$\frac{\|g(\lambda, v)\|}{\|v\|} \to 0 \text{ when } \|v\| \to 0$$

uniformly in $\lambda \in \{b \geq \lambda \geq \bar{\lambda}\}$. Thus (ii) is true and the proof of the proposition is completed. □

For the rest of the paper we now fix a $c \in]0, \frac{1}{2}\varepsilon_1]$. It is choosen sufficiently small so that for $\varepsilon \in]0, c]$ the claims (i) and (ii) of Proposition 2.3 hold and the condition (2.2) is satisfied. As we already said we can assume that $||u_n|| = c$ in condition $T(\delta)$. In view of Lemma 2.2, for any $(\lambda, u) \in [\bar{\lambda}, b[\times H$ solution of (P) with $||u|| < c$, $v = Pu$ is a critical point of the functional $F(\lambda, \cdot)$ defined on the ball $B_c(V)$ by

$$F(\lambda, v) = J(\lambda, v + g(\lambda, v)).$$

Before ending this section we show that the variational characterisation of g of Proposition 2.3 (i) implies that the family of functionals $F(\cdot, v)$ for $\lambda \in [\bar{\lambda}, b[$ has a (strong) monotonicity property.

Proposition 2.4 *Let λ_1, λ_2 be such that $\bar{\lambda} \leq \lambda_2 \leq \lambda_1 < b$. For any $v \in B_c(V)$ we have :*

$$0 \leq J(\lambda_2, v) - J(\lambda_1, v) \leq F(\lambda_2, v) - F(\lambda_1, v).$$

Proof. Observe that by Proposition 2.3 (i) we may write

$$\begin{aligned} F(\lambda_2, v) &= J(\lambda_2, v + g(\lambda_2, v)) \\ &\geq J(\lambda_2, v + g(\lambda_1, v)) \\ &= \frac{1}{2} < (A - \lambda_2 L)v, v > + \frac{1}{2} < (A - \lambda_2 L)g(\lambda_1, v), g(\lambda_1, v) > \\ &- \phi(v + g(\lambda_1, v)). \end{aligned}$$

Thus, we indeed have

$$\begin{aligned} F(\lambda_2, v) - F(\lambda_1, v) &\geq \frac{1}{2} < (\lambda_1 - \lambda_2)Lv, v > \\ &+ \frac{1}{2} < (\lambda_1 - \lambda_2)Lg(\lambda_1, v), g(\lambda_1, v) > \\ &\geq \frac{1}{2} < (\lambda_1 - \lambda_2)Lv, v >= J(\lambda_2, v) - J(\lambda_1, v) \geq 0. \end{aligned}$$ □

3 Mountain pass geometry for $F(\lambda, \cdot)$.

In this section we show that $F(\lambda, \cdot)$, for λ close to b, has a mountain pass geometry. More precisely we shall prove that there exists $\lambda_0 \in [\bar{\lambda}, b[$ such that setting

$$\Gamma_\lambda := \{\gamma \in C([0,1], B_c(V)) \, / \, \gamma(0) = 0 \, , \, F(\lambda, \gamma(1)) < 0\},$$

we have that Γ_λ is non void for all $\lambda \in [\lambda_0, b[$ and

$$c(\lambda) := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} F(\lambda, \gamma(t)) > 0.$$

We also derive a priori estimates on the mountain pass level $c(\lambda)$. We start with the following result which make explicit the behavior of $F(\lambda, \cdot)$ near $v = 0$.

Lemma 3.1 *Assume that (A1)-(A2) hold and that* $\lambda \in [\bar{\lambda}, b[$. *There exists* $\rho(\lambda) > 0$ *such that*

$$F(\lambda, v) \geq \frac{1}{4} n(\lambda) \|v\|^2, \ \ \forall \, v \in B_{\rho(\lambda)}(V).$$

Proof. Let $\lambda \in [\bar{\lambda}, b[$. Note first, that, by Proposition 2.3 (i)

$$F(\lambda, v) \geq J(\lambda, v) = \frac{1}{2} < (A - \lambda L)v, v > - \phi(v). \tag{3.1}$$

Now, by (A2), for any $\eta \in]0, n(\lambda)]$ there exists $d = d(\eta) > 0$ such that

$$\|v\| \leq d \Rightarrow \ \phi(v) \leq \eta \|v\|^2. \tag{3.2}$$

Therefore, from (3.1), it follows that for $\|v\| \leq d$,

$$F(\lambda, v) \geq \frac{1}{2} n(\lambda) \|v\|^2 - \eta \|v\|^2. \tag{3.3}$$

Taking $\eta = \frac{1}{4} n(\lambda)$ and $\rho(\lambda) = \min\{d, c\}$, this completes the proof. □.

Lemma 3.1 show that if, for a $\lambda \in [\bar{\lambda}, b[$, $c(\lambda)$ is defined then $c(\lambda) > 0$. To prove that $c(\lambda)$ is properly defined we need to prove that Γ_λ is non void. This will be done through the construction of a family of test functions for which we benefit from several previous works [12, 13, 14, 25]. We note, however, that the convexity of ϕ was so far a key tool in the construction of these functions (see Lemma 6.2 in [25] for example). To overcome the lack of convexity, we need to substantially modify the existing constructions.

Lemma 3.2 *Assume that (A1)-(A4) hold and that condition $T(\delta)$ is satisfied for a $\delta \geq 1$. Then, there exists a sequence $\{v_n\} \subset V$ which satisfies*

$$\|v_n\| \uparrow c \text{ when } n \to \infty,\ \phi(v_n) > 0 \ \text{ and } \ \lim_{n\to\infty} \frac{< (A-bL)v_n, v_n >}{\phi(v_n)^\delta} = 0.$$

As we just said, a first consequence of the existence of the test functions $\{v_n\} \subset V$ is that Γ_λ is non void for any $\lambda < b$ sufficiently close to b. Indeed

Proposition 3.3 *We define the sequence $\{q_n\} \subset I\!R$ by :*

$$(3.4) \quad q_n = \min\left\{\frac{\alpha\phi(v_n)}{4K(2c)\|L\|c^2}, \frac{\alpha\phi(v_n)^\delta}{32K^2\|L\|c^2}, \frac{\phi(v_n)}{2||L||c^2}, \frac{\alpha}{2||L||}, b-\bar{\lambda}\right\}$$

where α is defined in Section 2, K in (A4) and $K(2c) := \sup_{u\in B_{2c}} \|N'(u)\|$. Then, if $\{v_n\} \subset B_c(V)$ is the sequence obtained in Lemma 3.2, there exists $n_0 \in I\!N$ such that

$$(3.5) \qquad \lambda \in [b-q_n, b[\Rightarrow F(\lambda, v_n) < 0, \ \ \forall\, n \geq n_0.$$

Proof of Lemma 3.2. Since $T(\delta)$ holds there exists a sequence $\{u_n\} \subset H$ such that $\|u_n\| = c$, $\phi(u_n) > 0$, for all $n \in I\!N$ and

$$(3.6) \qquad \lim_{n\to\infty} \frac{< (A-bL)u_n, u_n >}{\phi(u_n)^\delta} = \lim_{n\to\infty} \frac{\|(A-bL)u_n\|^2}{\phi(u_n)^\delta} = 0.$$

Let $v_n = Pu_n \in V$ and $w_n = Qu_n \in W$. Since

$$< (A-bL)u_n, Qu_n > = < (A-bL)Qu_n, Qu_n > \leq -m(b)\|Qu_n\|^2$$

we have that

$$(3.7) \qquad \|(A-bL)u_n\| \geq m(b)\|Qu_n\|.$$

Since ϕ is bounded on B_c we have, using (3.6) and (3.7)

$$\|(A-bL)u_n\| \to 0 \text{ and } \|Qu_n\| \to 0 \text{ when } n \to \infty.$$

This proves that $\|v_n\| \to c$ when $n \to \infty$ and, since $v_n = Pu_n$, we clearly have that $||v_n|| \leq c$, $\forall n \in I\!N$. Now, let us show that

$$(3.8) \qquad \lim_{n\to\infty} \frac{< (A-bL)v_n, v_n >}{\phi(v_n)} = 0.$$

We claim that

$$(3.9) \qquad \text{For } n \in I\!N \text{ large enough, } \ \phi(v_n) \geq \frac{1}{2}\phi(u_n).$$

Indeed, by Taylor-Lagrange's expansion, there exists $\theta_n \in [0,1]$ such that

$$\begin{aligned}\phi(v_n) &= \phi(u_n - Qu_n)\\ &= \phi(u_n) - < N(u_n), Qu_n > + \frac{1}{2} < N'(u_n - \theta_n Qu_n)Qu_n, Qu_n > .\end{aligned}$$

By (A4),

$$(3.10)\; | < N(u_n), Qu_n > | \;\leq\; \|N(u_n)\|\,\|Qu_n\| \;\leq\; K\phi(u_n)^{1-\frac{\delta}{2}}\|Qu_n\|.$$

Also,

$$(3.11)\quad | < N'(u_n - \theta_n Qu_n)Qu_n, Qu_n > | \leq \sup_{u\in B_c} ||N'(u)||\, ||Qu_n||^2.$$

It follows from (3.10) and (3.11), that

$$\phi(v_n) \geq \phi(u_n)\left(1 - K\frac{\|Qu_n\|}{\phi(u_n)^{\frac{\delta}{2}}} - \sup_{u\in B_c} ||N'(u)|| \frac{\|Qu_n\|^2}{2\phi(u_n)}\right).$$

But combining (3.6) and (3.7) and since $\delta \geq 1$ in $T(\delta)$ we have

$$\frac{\|Qu_n\|}{\phi(u_n)^{\frac{\delta}{2}}} \to 0 \text{ and } \frac{\|Qu_n\|^2}{\phi(u_n)} \to 0 \text{ when } n \to \infty.$$

Therefore there does exist $n_0 \in I\!N$ such that for $n \geq n_0$, (3.9) is satisfied. In particular, then $\phi(v_n) > 0$. Now for $n \geq n_0$, by (3.7) and (3.9),

$$\begin{aligned}\frac{< (A - bL)v_n, v_n >}{\phi(v_n)^\delta} &\leq \frac{| < (A - bL)u_n, u_n > |}{(\frac{1}{2}\phi(u_n))^\delta} + \frac{| < (A - bL)Qu_n, Qu_n > |}{(\frac{1}{2}\phi(u_n))^\delta}\\ &\leq 2^\delta\left(\frac{| < (A - bL)u_n, u_n > |}{\phi(u_n)^\delta} + \frac{\|A - bL\|\|Qu_n\|^2}{\phi(u_n)^\delta}\right)\\ &\leq 2^\delta\left(\frac{| < (A - bL)u_n, u_n > |}{\phi(u_n)^\delta} + \frac{C\|(A - bL)u_n\|^2}{\phi(u_n)^\delta}\right)\end{aligned}$$

where $C = \frac{\|A-bL\|}{m(b)^2}$. By $T(\delta)$ the expression above tends to 0 when $n \to \infty$ and this completes the proof of the lemma. □

Proof of Proposition 3.3. We argue by contradiction. Suppose there exists a subsequence of $\{v_n\} \subset V$ (still denoted $\{v_n\}$) such that for a $\lambda_n \geq b - q_n$,

$$(3.12)\qquad F(\lambda_n, v_n) \geq 0.$$

Then, by definition of $F(\lambda_n, v_n)$,

$$0 \leq \frac{1}{2}\left(< (A - \lambda_n L)v_n, v_n > + < (A - \lambda_n L)g_n, g_n >\right) - \phi(v_n + g_n)$$

where $g_n := g(\lambda_n, v_n)$. Combining Lemma 2.1 and the fact that $\phi(u) \geq 0, \forall u \in H$, we deduce that

$$0 \leq < (A - \lambda_n L)v_n, v_n > -m(\lambda_n)\|g_n\|^2$$

or equivalently that

$$m(\lambda_n)\|g_n\|^2 \leq < (A - bL)v_n, v_n > +(b - \lambda_n) < Lv_n, v_n > .$$

Thus,

$$m(\lambda_n)\|g_n\|^2 \leq < (A - bL)v_n, v_n > +(b - \lambda_n)\|L\|c^2$$

and, since $\lambda_n \geq 0$,

$$\|g_n\|^2 \leq \frac{1}{\alpha}\left(< (A - bL)v_n, v_n > +(b - \lambda_n)\|L\|c^2\right). \tag{3.13}$$

Now observe that by Lemma 3.2, for any $\eta > 0$, there exists $n_0 = n_0(\eta) \in I\!N$ such that

$$\frac{< (A - bL)v_n, v_n >}{\phi(v_n)} \leq \eta \ \ \forall n \geq n_0 \tag{3.14}$$

$$\frac{< (A - bL)v_n, v_n >}{\phi(v_n)^\delta} \leq \eta \ \ \forall n \geq n_0 \ \text{ and} \tag{3.15}$$

$$< (A - bL)v_n, v_n > \leq \frac{1}{2}\alpha c^2 \ \ \forall n \geq n_0. \tag{3.16}$$

In the rest of the proof we always assume that $n \geq n_0$. Now, using (3.13), the definition of q_n and the fact that $0 \leq (b - \lambda_n) \leq q_n$, (3.14) implies that

$$\|g_n\|^2 \leq \frac{\eta}{\alpha}\phi(v_n) + \frac{\phi(v_n)}{4K(2c)} = c_0\phi(v_n) \tag{3.17}$$

where $c_0 = \frac{\eta}{\alpha} + \frac{1}{4K(2c)}$. Similarly (3.15) implies that

$$\|g_n\|^2 \leq \frac{\eta}{\alpha}\phi(v_n)^\delta + \frac{\phi(v_n)^\delta}{32K^2} = c_1\phi(v_n)^\delta \tag{3.18}$$

where $c_1 = \frac{\eta}{\alpha} + \frac{1}{32K^2}$. Finally, still from (3.13), we get using (3.16)

$$||g_n||^2 \leq \frac{1}{2}c^2 + \frac{q_n||L||c^2}{\alpha} \leq c^2.$$

Now, as in the proof of Lemma 3.2 (see (3.10) and (3.11)), we have for $\eta > 0$ sufficiently small

$$\begin{aligned}
\phi(v_n + g_n) &\geq \phi(v_n) - K\phi(v_n)^{1-\frac{\delta}{2}}\|g_n\| - \frac{1}{2}K(2c)\|g_n\|^2 \\
&\geq \phi(v_n)\left(1 - K\frac{||g_n||}{\phi(v_n)^{\frac{\delta}{2}}} - K(2c)\frac{||g_n||^2}{2\phi(v_n)}\right) \\
&\geq \phi(v_n)\left(1 - Kc_1^{\frac{1}{2}} - \frac{1}{2}K(2c)c_0\right) \\
&\geq \frac{\phi(v_n)}{2}.
\end{aligned}$$

Thus, taking $\eta > 0$ sufficiently small, it follows that

$$\begin{aligned}
0 \leq F(\lambda_n, v_n) &\leq \frac{1}{2} < (A - \lambda_n L)v_n, v_n > -\phi(v_n + g_n) \\
&\leq \frac{1}{2} < (A - bL)v_n, v_n > + \frac{q_n\|L\|c^2}{2} - \frac{\phi(v_n)}{2} \\
&\leq \frac{1}{2}\eta\phi(v_n) + \frac{\phi(v_n)}{4} - \frac{\phi(v_n)}{2} < 0.
\end{aligned}$$

This contradiction completes the proof. □

Setting $\lambda_0 = b - q_{n_0}$ where q_{n_0} is defined in Proposition 3.3 we have, as a consequence of the above results, that for all $\lambda \in [\lambda_0, b[$ the following holds: (i) Γ_λ is non void and (ii) $\lambda \to c(\lambda)$ is monotone decreasing. Indeed the path γ defined by $\gamma(t) = tv_{n_0}$ for $t \in [0,1]$, belongs to Γ_λ for all $\lambda \in [\lambda_0, b[$; this gives (i). Now for $\lambda_0 \leq \lambda_1 \leq \lambda_2 < b$ we have, by Proposition 2.4, for any $\gamma \in \Gamma_{\lambda_1}$, $F(\lambda_2, \gamma(t)) \leq F(\lambda_1, \gamma(t))$. Thus, $\Gamma_{\lambda_1} \subset \Gamma_{\lambda_2}$ and, from the definition of $c(\lambda)$, it follows that $c(\lambda_2) \leq c(\lambda_1)$.

We will end this section by deriving some a priori estimates on $c(\lambda)$.

Proposition 3.4 *For the sequence $\{q_n\} \subset I\!R$ defined in Proposition 3.3 we have*

(i) There exists $n_0 \in I\!N$ such that for $n \geq n_0$ and $\lambda \in [b - q_n, b[$:

$$0 < c(\lambda) \leq K(q)(< (A - \lambda L)v_n, v_n >)^{\frac{q}{q-2}}\phi(v_n)^{\frac{-2}{q-2}}$$

where $K(q) > 0$ is a constant depending only on q.

(ii) Setting

$$\alpha_n = \begin{cases} b - \frac{<(A-bL)v_n, v_n>}{4\|L\|c^2} & \text{if } < (A - bL)v_n, v_n >> 0 \\ b - \frac{q_n}{n} & \text{if } < (A - bL)v_n, v_n >= 0, \end{cases}$$

we have $\alpha_n \to b$ *when* $n \to \infty$ *and*

$$\lim_{n\to\infty} \frac{c(\alpha_n)}{(b-\alpha_n)^{\theta+1}} = 0$$

where $\theta = \frac{2}{q-2}(1-\frac{1}{\delta})$.

Proof. (i) Remark that, by definition of $c(\lambda)$, Lemmas 3.1 and Proposition 3.3, we have for λ in $[b-q_n, b[$

$$0 < c(\lambda) \leq \max_{t\in[0,1]} F(\lambda, tv_n) = F(\lambda, \hat{t}v_n) \tag{3.19}$$

for a $\hat{t} \in]0,1]$. Setting $g_n = g(\lambda, \hat{t}v_n)$, we have

$$\begin{aligned} F(\lambda,\hat{t}v_n) &= \frac{1}{2}\Big((\hat{t})^2 < (A-\lambda L)v_n, v_n > + < (A-\lambda L)g_n, g_n >\Big) \\ &- \phi(\hat{t}v_n + g_n) \end{aligned} \tag{3.20}$$

and thus, since $\phi(\hat{t}v_n + g_n) \geq 0$, (3.19) implies that

$$0 \leq \hat{t}^2 \left(< (A-bL)v_n, v_n > +(b-\lambda) < Lv_n, v_n >\right) + < (A-\lambda L)g_n, g_n > .$$

Therefore, for $\lambda \geq b - q_n \geq 0$, we obtain that

$$\alpha \left\| \frac{g_n}{\hat{t}} \right\|^2 \leq < (A-bL)v_n, v_n > +(b-\lambda) < Lv_n, v_n > . \tag{3.21}$$

By Lemma 3.2, there exists $n_0 \in I\!N$ such that for all $n \geq n_0$,

$$< (A-bL)v_n, v_n > \leq \frac{\alpha}{4K(2c)}\phi(v_n) \tag{3.22}$$

$$< (A-bL)v_n, v_n > \leq \frac{\alpha}{32K^2}\phi(v_n)^{\delta}. \tag{3.23}$$

$$< (A-bL)v_n, v_n > \leq \frac{1}{2}\alpha c^2. \tag{3.24}$$

Combining (3.21) and (3.22), we deduce that

$$\left\| \frac{g_n}{\hat{t}} \right\|^2 \leq \frac{\phi(v_n)}{4K(2c)} + \frac{q_n\|L\|c^2}{\alpha} \leq \frac{1}{2K(2c)}\phi(v_n). \tag{3.25}$$

Combining (3.21) and (3.23), we deduce that

$$\left\| \frac{g_n}{\hat{t}} \right\|^2 \leq \frac{\phi(v_n)^{\delta}}{32K^2} + \frac{q_n\|L\|c^2}{\alpha} \leq \frac{1}{16K^2}\phi(v_n)^{\delta}. \tag{3.26}$$

Finally, from (3.21) and (3.24) we have

$$\left\|\frac{g_n}{\hat{t}}\right\|^2 \leq \frac{1}{2}c^2 + \frac{q_n||L||c^2}{\alpha} \leq c^2.$$

Now, by (2.1), since $0 \leq \hat{t} \leq 1$ we have

$$\phi(\hat{t}v_n + g_n) \geq (\hat{t})^q \phi(v_n + \frac{g_n}{\hat{t}}). \tag{3.27}$$

As in the proof of Lemma 3.2, for a $\theta_n \in [0,1]$ using (3.25) and (3.26):

$$\begin{aligned}
\phi(v_n + \frac{g_n}{\hat{t}}) &= \phi(v_n) + < N(v_n), \frac{g_n}{\hat{t}} > + \frac{1}{2} < N'(v_n + \theta\frac{g_n}{\hat{t}})\frac{g_n}{\hat{t}}, \frac{g_n}{\hat{t}} > \\
&\geq \phi(v_n) - K\phi(v_n)^{1-\frac{\delta}{2}} \|\frac{g_n}{\hat{t}}\| - \frac{1}{2}K(2c)\|\frac{g_n}{\hat{t}}\|^2 \\
&\geq \phi(v_n)\left(1 - K\frac{||\frac{g_n}{\hat{t}}||}{\phi(v_n)^{\frac{\delta}{2}}} - K(2c)\frac{||\frac{g_n}{\hat{t}}||^2}{2\phi(v_n)}\right) \\
&\geq \frac{\phi(v_n)}{2}.
\end{aligned}$$

Thus (3.27) yields

$$\phi(\hat{t}v_n + g_n) \geq \frac{(\hat{t})^q}{2}\phi(v_n). \tag{3.28}$$

Now for $n \geq n_0$, (3.19), (3.20) and (3.28) lead to

$$\begin{aligned}
c(\lambda) &\leq \frac{(\hat{t})^2}{2} < (A - \lambda L)v_n, v_n > - \frac{(\hat{t})^q}{2}\phi(v_n) \\
&\leq \max_{t \in [0,1]} \left\{ \frac{t^2}{2} < (A - \lambda L)v_n, v_n > - \frac{t^q}{2}\phi(v_n) \right\} \\
&= K(q)\frac{< (A - \lambda L)v_n, v_n >^{\frac{q}{q-2}}}{\phi(v_n)^{\frac{2}{q-2}}}
\end{aligned}$$

where $K(q) = \frac{1}{2}\left((\frac{2}{q})^{\frac{2}{q-2}} - (\frac{2}{q})^{\frac{q}{q-2}}\right) > 0$. This proves (i).
(ii) By definition of α_n, q_n and by Lemma 3.2 clearly for any $m \in I\!N$ large enough, $0 < b - q_n < \alpha_n$. Moreover $\alpha_n \to b$ when $n \to \infty$. Now, remark that

$$\begin{aligned}
< (A - \alpha_n L)v_n, v_n > &\leq < (A - bL)v_n, v_n > + (b - \alpha_n)\|L\|c^2 \\
&\leq 5(b - \alpha_n)\|L\|c^2
\end{aligned} \tag{3.29}$$

if $< (A - bL)v_n, v_n >> 0$ and

$$< (A - \alpha_n L)v_n, v_n > \leq \frac{q_n}{n}\|L\|c^2 \quad \text{if } < (A - bL)v_n, v_n >= 0. \tag{3.30}$$

Setting $\gamma = \frac{2}{(q-2)\delta}$ and $\theta = \frac{2}{(q-2)}(1-\frac{1}{\delta})$, we have by Point (i),

$$c(\alpha_n) \leq K(q) < (A - \alpha_n L)v_n, v_n >^{1+\theta} \left(\frac{< (A - \alpha_n L)v_n, v_n >}{\phi(v_n)^\delta} \right)^\gamma . \tag{3.31}$$

Hence, using (3.29), it follows that

$$c(\alpha_n) \leq K(q)(5(b-\alpha_n)\|L\|c^2)^{1+\theta} (\frac{< (A - \alpha_n L)v_n, v_n >}{\phi(v_n)^\delta})^\gamma \tag{3.32}$$

if $< (A - bL)v_n, v_n >> 0$. In the same way, using (3.30), if
$< (A - bL)v_n, v_n >= 0$,

$$c(\alpha_n) \leq K(q)((b-\alpha_n)\|L\|c^2)^{1+\theta} (\frac{q_n\|L\|c^2}{n\phi(v_n)^\delta})^\gamma . \tag{3.33}$$

Now, by Lemma 3.2 and the definition of α_n, if
$< (A - bL)v_n, v_n >> 0$,

$$\frac{< (A - \alpha_n L)v_n, v_n >}{\phi(v_n)^\delta} \to 0 \text{ when } n \to \infty \tag{3.34}$$

and if $< (A - bL)v_n, v_n >= 0$,

$$\frac{q_n\|L\|c^2}{n\phi(v_n)^\delta} \leq \frac{\alpha}{32K^2 n} \to 0 \text{ when } n \to \infty \, . \tag{3.35}$$

Combining (3.32), (3.34) and (3.33), (3.35) we obtain (ii). The proof of the proposition is now completed. □

4 Existence of a bifurcating sequence for (P)

In this last section we prove Theorem 1.1. To overcome the lack of a priori bounds on the Palais-Smale sequences of $F(\lambda, \cdot)$, $\lambda \in [\lambda_0, b[$ we need to develop an original variational approach. We start with the following result

Proposition 4.1 *There exists a strictly increasing sequence* $\{\lambda_n\} \in [\lambda_0, b[$, $\lambda_n \to b$ *such that* $c(\lambda_n) \to 0$ *and* $c'(\lambda_n) \to 0$.

Proof. First note that since $\lambda \to c(\lambda)$ is non increasing, by Proposition 3.4 (ii), $c(\lambda) \to 0$ when $\lambda \to b^-$. Another consequence of the monotonicity is that $c'(\lambda)$ exists almost everywhere. We claim that there is $\lambda_n \to b^-$ with $c'(\lambda_n) \to 0$. Seeking a contradiction, we assume that

$$a_0 := \liminf_{\lambda \to b^-} (-c'(\lambda)) > 0.$$

Since $\lambda \to c(\lambda)$ is non increasing and positive we have for $\lambda < b$ sufficiently close to b,

$$\begin{aligned} c(\lambda) &= c(\lambda) - \lim_{h \to b^-} c(h) \\ &\geq \lim_{h \to b^-} \int_\lambda^h -c'(t)\, dt \\ &\geq \lim_{h \to b^-} \int_\lambda^h \frac{a_0}{2}\, dt \\ &= \frac{a_0}{2}(b - \lambda). \end{aligned}$$

Thus, making the choice $\lambda = \alpha_n$ (for $n \in I\!N$ large) we obtain that

$$\lim_{n \to \infty} \frac{c(\alpha_n)}{(b - \alpha_n)} \geq \frac{a_0}{2}$$

and this contradicts the a priori estimates of Proposition 3.4 (ii). $\square$

The next result is the key point of our variational approach. Let $\lambda \in]\lambda_0, b[$ be an arbitrary but fixed value where $c'(\lambda)$ exists. Let $\{\lambda_m\} \subset]a, \lambda[$ be a strictly increasing sequence with $\lambda_m \to \lambda$. Finally let $\beta(\lambda) > 0$ be such that

$$\beta^2(\lambda) := \frac{4}{\beta}[(2 + 3b)c(\lambda) - bc'(\lambda)].$$

Proposition 4.2 *For any $\eta > 0$ there exists a sequence of paths $\{\gamma_m\} \subset \Gamma_\lambda$ such that, for $m \in I\!N$ sufficiently large*

(i) $\frac{1}{2} < L\gamma_m(t), \gamma_m(t) > \leq -c'(\lambda) + 3\eta$ *when*

$$F(\lambda, \gamma_m(t)) \geq c(\lambda) - \eta(\lambda - \lambda_m), \tag{4.1}$$

(ii) $\max_{t \in [0,1]} F(\lambda, \gamma_m(t)) \leq c(\lambda) + (-c'(\lambda) + 2\eta)(\lambda - \lambda_m).$

Moreover making the choice $\eta = c(\lambda) > 0$ we have when (4.1) hold

$$||\gamma_m(t)|| \leq \beta(\lambda).$$

Proof. Let $\{\gamma_m\} \subset \Gamma_\lambda$ be an arbitrary sequence such that

$$\max_{t \in [0,1]} F(\lambda_m, \gamma_m(t)) \leq c(\lambda_m) + \eta(\lambda - \lambda_m). \tag{4.2}$$

We note that such sequence exists since $\Gamma_{\lambda_m} \subset \Gamma_\lambda$ for all $m \in I\!N$. Now let $m_0 = m_0(\eta, \lambda)$ be such that, for all $m \geq m_0$,

$$0 \leq \frac{c(\lambda_m) - c(\lambda)}{\lambda - \lambda_m} \leq -c'(\lambda) + \eta. \tag{4.3}$$

When (4.1) is satisfied, it follows that for $m \geq m_0$,

$$
\begin{aligned}
0 &\leq \frac{F(\lambda_m, \gamma_m(t)) - F(\lambda, \gamma_m(t))}{\lambda - \lambda_m} \\
&\leq \frac{c(\lambda_m) + \eta(\lambda - \lambda_m) - c(\lambda) + \eta(\lambda - \lambda_m)}{\lambda - \lambda_m} \\
&= \frac{c(\lambda_m) - c(\lambda)}{\lambda - \lambda_m} + 2\eta \\
&\leq -c'(\lambda) + 3\eta. \qquad (4.4)
\end{aligned}
$$

But, using Proposition 2.4, we also have

$$
\begin{aligned}
0 \leq \frac{1}{2} < L\gamma_m(t), \gamma_m(t) > &= \frac{J(\lambda_m, \gamma_m(t)) - J(\lambda, \gamma_m(t))}{\lambda - \lambda_m} \\
&\leq \frac{F(\lambda_m, \gamma_m(t)) - F(\lambda, \gamma_m(t))}{\lambda - \lambda_m}
\end{aligned}
$$

which yields, together with (4.4),

$$
\frac{1}{2} < L\gamma_m(t), \gamma_m(t) > \leq -c'(\lambda) + 3\eta
$$

when (4.1) is satisfied. This proves (i). Now, by Proposition 2.4, (4.2), (4.3) and since $\lambda_m \uparrow \lambda$, we have that for $m \geq m_0$,

$$
\begin{aligned}
\max_{t \in [0,1]} F(\lambda, \gamma_m(t)) \leq \max_{t \in [0,1]} F(\lambda_m, \gamma_m(t)) &\leq c(\lambda_m) + \eta(\lambda - \lambda_m) \\
&\leq c(\lambda) + (2\eta - c'(\lambda))(\lambda - \lambda_m).
\end{aligned}
$$

Thus (ii) also holds. Now if we choose $\eta = c(\lambda) > 0$, then when (4.1) is satisfied and $m \in I\!N$ is sufficiently large

$$
\frac{1}{2} < L\gamma_m(t), \gamma_m(t) > \leq -c'(\lambda) + 3c(\lambda) \qquad (4.5)
$$

and

$$
F(\lambda, \gamma_m(t)) \leq 2c(\lambda). \qquad (4.6)
$$

Since $J(\lambda, \gamma_m(t)) \leq F(\lambda, \gamma_m(t))$ by Proposition 2.3 (i), we have using the definition of $J(\lambda, \cdot)$

$$
\begin{aligned}
\frac{1}{2} < A\gamma_m(t), \gamma_m(t) > &\leq F(\lambda, \gamma_m(t)) + \frac{\lambda}{2} < L\gamma_m(t), \gamma_m(t) > \\
&+ \phi(\gamma_m(t)).
\end{aligned}
$$

It implies, using (2.2), that

$$\frac{1}{4} < A\gamma_m(t), \gamma_m(t) > \leq \; F(\lambda, \gamma_m(t)) + \frac{b}{2} < L\gamma_m(t), \gamma_m(t) >$$

from which it follows that

$$\frac{\beta}{4}||\gamma_m(t)||^2 \leq \; F(\lambda, \gamma_m(t)) + \frac{b}{2} < L\gamma_m(t), \gamma_m(t) > .$$

Thus using (4.5) and (4.6) we deduce that

$$\frac{\beta}{4}||\gamma_m(t)||^2 \leq 2c(\lambda) + b(-c'(\lambda) + 3c(\lambda))$$

and the proposition is proved. □

We use Proposition 4.2 in the following way. Suppose that for a $\lambda \in [\lambda_0, b[$, $c(\lambda)$ and $c'(\lambda)$ are defined and sufficiently small so that $4\beta(\lambda) \leq c$. Then

Proposition 4.3 *Setting*

$$F_\alpha^\lambda := \Big\{ v \, \in \, B_{2\beta(\lambda)}(V) \, / \, |F(\lambda, v) - c(\lambda)| \leq \alpha \Big\}$$

we have

$$\inf \Big\{ \|\nabla_v F(\lambda, v)\| / v \, \in \, F_\alpha^\lambda \Big\} = 0 \;\; \textit{for any } \alpha > 0. \tag{4.7}$$

Proof. Seeking a contradiction we assume that (4.7) does not hold. Thus, we can choose a $a > 0$ such that for any $v \in F_a^\lambda$

$$\|\nabla_v F(\lambda, v)\| \geq a \;\; \text{and} \;\; 0 < a < \frac{1}{2}c(\lambda).$$

Then, a classical deformation argument says that there exist $\mu \in]0, a[$ and a C^1- map $\tau : \; B_c(V) \to \; B_c(V)$ such that

$$\tau(v) = v \text{ if } \; |F(\lambda, v) - c(\lambda)| \geq a, \tag{4.8}$$

$$F(\lambda, \tau(v)) \leq F(\lambda, v) \;\; \forall \, v \, \in \, B_c(V). \tag{4.9}$$

Moreover for $v \, \in \, B_{\beta(\lambda)}(V)$ with $F(\lambda, v) \leq c(\lambda) + \mu$,

$$F(\lambda, \tau(v)) \leq c(\lambda) - \mu. \tag{4.10}$$

Now consider the sequence $\{\gamma_m\} \subset \Gamma_\lambda$ obtained in Proposition 4.2 where the choice $\eta = c(\lambda) > 0$ is made. Fix a $k \in I\!\!N$ sufficiently large so that

$$(2c(\lambda) - c'(\lambda))(\lambda - \lambda_k) < \mu. \tag{4.11}$$

¿From (4.11), if (4.1) is satisfied in Proposition 4.2 we have that

$$\|\gamma_k(t)\| \leq \beta(\lambda) \text{ and } F(\lambda, \gamma_k(t)) \leq c(\lambda) + \mu. \tag{4.12}$$

Thus, from (4.10) and (4.12) it follows that

$$F(\lambda, \tau(\gamma_k(t))) \leq c(\lambda) - \mu. \tag{4.13}$$

Now, if (4.1) is not satisfied, then

$$F(\lambda, \gamma_k(t)) < c(\lambda) - c(\lambda)(\lambda - \lambda_k) \tag{4.14}$$

which implies together with (4.9) that

$$F(\lambda, \tau(\gamma_k(t))) \leq F(\lambda, \gamma_k(t)) < c(\lambda) - c(\lambda)(\lambda - \lambda_k). \tag{4.15}$$

Therefore, on one hand combining (4.13) and (4.15) we get

$$\max_{t \in [0,1]} F(\lambda, \tau(\gamma_k(t))) < c(\lambda). \tag{4.16}$$

On the other hand, since $a < \frac{1}{2}c(\lambda)$, by (4.8), $\tau(\gamma_k(\cdot))$ belongs to Γ_λ . This contradiction proves the proposition. □

Gathering the results obtained in Propositions 4.1, 4.2 and 4.3 we deduce that there exists a sequence $\{\lambda_n\}$ with $\beta(\lambda_n) \to 0$ as $\lambda_n \to b$ such that, for any $n \in I\!N$, $F(\lambda_n, \cdot)$ has a Palais-Smale sequence at the level $c(\lambda_n)$ contained in the ball a radius $2\beta(\lambda_n)$ centred at the origin. By definition, this means that for any fixed $n \in I\!N$ there exists a sequence $\{v_m\} \subset B_{2\beta(\lambda_n)}(V)$ such that

$$F(\lambda_n, v_m) \to c(\lambda_n) > 0 \text{ and } \nabla_v F(\lambda_n, v_m) \to 0 \text{ as } m \to \infty. \tag{4.17}$$

The proof of Theorem 1.1 will be completed once we have shown

Proposition 4.4 *For any $n \in I\!N$, $J(\lambda_n, \cdot)$ has a nontrivial critical point u_{λ_n} with $||u_{\lambda_n}|| \leq 4\beta(\lambda_n)$.*

Proof. By definition of $F(\lambda, .)$, setting $u_m := v_m + g(\lambda_n, v_m)$ we have by (4.17) that, as $m \to \infty$,

$$J(\lambda_n, u_m) = F(\lambda_n, v_m) \to c(\lambda_n) > 0$$

and

$$\nabla_u J(\lambda_n, u_m) = \nabla_v F(\lambda_n, v_m) \to 0.$$

Now, if (A5)(ii) holds, the proof is straighforward. Indeed, by definition, there exist $u_{\lambda_n} \in H \backslash \{0\}$ and $w_m \in \Theta(u_m)$ such that, up to a subsequence,

$w_m \rightharpoonup u_{\lambda_n}$ and $\nabla_u J(\lambda_n, u_{\lambda_n}) = 0$. Also $||u_{\lambda_n}|| \leq \liminf ||w_m|| \leq 4\beta(\lambda_n)$. If (A5)(i) holds, namely if N is compact, we proceed as follows. Passing to a subsequence we may assume that $u_m \rightharpoonup u_{\lambda_n}$ and $||N(u_m) - w_{\lambda_n}|| \to 0$ where $u_{\lambda_n},\ w_{\lambda_n} \in H$. Since $\lambda_n \in]a, b[$, $A - \lambda_n L$ is invertible and so there exists z_{λ_n} such that $(A - \lambda_n L)z_{\lambda_n} = w_{\lambda_n}$. Now,

$$\begin{aligned} M(\lambda_n)\|u_m - z_{\lambda_n}\| &\leq \|(A - \lambda_n L)(u_m - z_{\lambda_n})\| \\ &= \|(A - \lambda_n L)u_m - w_{\lambda_n}\| \\ &= \|\nabla_u J(\lambda_n, u_m) + N(u_m) - w_{\lambda_n}\| \\ &\leq \|\nabla_u J(\lambda_n, u_m)\| + \|N(u_m) - w_{\lambda_n}\|. \end{aligned}$$

Hence, since $M(\lambda_n) > 0$, $||u_m - z_{\lambda_n}|| \to 0$ and thus $u_m \to u_{\lambda_n}$. By continuity, it follows that $||u_{\lambda_n}|| \leq 4\beta(\lambda_n)$ and that $\nabla_u J(\lambda_n, u_{\lambda_n}) = \lim \nabla_u J(\lambda_n, u_m) = 0$. Also $J(\lambda_n, u_{\lambda_n}) = c(\lambda_n) > 0$ and in particular $u_{\lambda_n} \neq 0$. □

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