On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer type problem set on $\mathbb{R}^N$

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Abstract

Using the "monotonicity trick" introduced by M. Struwe we derive a generic theorem. It says that for a wide class of functionals, having a Mountain-Pass geometry, almost every functionals in this class has a bounded Palais-Smale sequence at the Mountain-Pass level. Then we show how the generic theorem can be used to obtain, for a given functional, a special Palais-Smale sequence possessing extra properties which help to insure its convergence. Subsequently these abstract results are applied to prove the existence of a positive solution for a problem of the form

$$\left.\begin{array}{l} -\Delta u + Ku = f(x,u) \\ u \in H^1(\mathbb{R}^N), K > 0. \end{array}\right\} \qquad (P)$$

We assume that the functional associated to (P) has a Mountain-Pass geometry. Our results cover the case where the nonlinearity f satisfies (i) $f(x,s)s^{-1} \to a \in]0,\infty]$ as $s \to +\infty$ and (ii) $f(x,s)s^{-1}$ is non decreasing as a function of $s \geq 0$, a.e. $x \in \mathbb{R}^N$.

1 Introduction

A first aim of this paper is to study, for a large class of functionals having a Mountain Pass geometry, the existence of a bounded Palais-Smale sequence at the Mountain-Pass level. Proving the existence of such sequences is a preliminary step when one wants to show that the functionals have a critical point. More precisely let X be a Banach space and denote by X^{-1} its dual. By saying that a functional $I \in C^1(X, \mathbb{R})$ possesses a Mountain Pass geometry, a MP geometry for short, we mean that there are two points (v_1, v_2) in X such that setting

$$\Gamma = \{\gamma \in C([0,1], X), \gamma(0) = v_1, \gamma(1) = v_2\}$$

there holds

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > \max\{I(v_1), I(v_2)\}.$$

Also a Palais-Smale sequence of I at the level $c \in \mathbb{R}$ is by definition a sequence $\{u_n\} \subset X$ satisfying $I(u_n) \to c$ and $I'(u_n) \to 0$ in X^{-1}.

It is well known that if I possesses a MP geometry, the value $c \in \mathbb{R}$, called the Mountain-Pass level, is a good candidate for being a critical value of I. Indeed assume in addition that the $(PS)_c$ condition holds, namely that all Palais-Smale sequences for I at the level $c \in \mathbb{R}$ possess a convergent subsequence. Then there exists $u \in X$ satisfying $I(u) = c$ and $I'(u) = 0$. This is a celebrated result, known as the Mountain-Pass theorem, due to Ambrosetti and Rabinowitz [3]. Observing the proof given in [3], or alternatively using the Ekeland's variational principle [13], one sees that the MP geometry directly implies the existence of a Palais-Smale sequence $\{u_n\} \subset X$ for I at the level $c \in \mathbb{R}$. Thus to find a critical point it is sufficient to establish that this particular sequence has a convergent subsequence. Traditionally this is done in two steps. First one proves that $\{u_n\}$ is bounded and this implies (assuming that X is reflexive) the existence of a $u \in X$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in X. Next one shows that $u_n \to u$ strongly in X and by continuity of I and I', u then satisfies $I(u) = c$ and $I'(u) = 0$. Note that in many cases one is interested in finding a (non-trivial) critical point of I but not necessarily at the MP level. Then instead of proving that $u_n \to u$ strongly in X it is sufficient to show that $I'(u) = 0$ (with $I(u) \neq I(0)$). See [10, 17, 21, 30] for some examples.

Concerning the first step, namely the problem of finding conditions on I insuring the existence of a bounded Palais-Smale sequence, a BPS sequence for short, at the MP level most of the work we know deals with specific situations. We mean by this that the functional I is introduced in order that its critical points correspond to (weak) solutions of a given PDE or Hamiltonian type problem. Then particular properties of the underlying problem can directly and crucially be used to prove the existence of a BPS sequence (see [18, 32] for example). A more systematic approach is due to Ghoussoub [14] where his ideas of using dual sets to localize the critical points of the functionals are often a strong help to conlude the existence of a BPS sequence. Let us also mention the work of Cerami [11] which leads to prove that there always exists a sequence $\{u_n\} \subset X$ satisfying $I(u_n) \to c$ and $||I'(u_n)||(1+||u_n||) \to 0$. For this Palais-Smale sequence, called a Cerami's sequence, the additional information that $||I'(u_n)||(1+||u_n||) \to 0$ has in several situations been successfully used to establish that $\{u_n\}$ is bounded. However probably the most significant contribution is due to Struwe (see also [22]). He introduce a general technique often referred to as the "monotonicity trick"(see [24, 25]) which not only in a Mountain-Pass setting [2, 4] but also in minimization problems [27] or in linking type situation [26] has been used by Struwe and others to solve difficult variational problems. Most of these problems had in common the difficulty to establish the existence of a BPS sequence.

Unfortunately Struwe's approach has only been used so far on specific examples. Thus it is not always clear to distinguish what is the core of the approach and what belongs to the specific problem under study. A first achievement of our paper is the derivation of a general abstract result based on Struwe's "monotonicity trick". Clearly, with respect to the existing works, one advantage is the simplicity of the presentation and the "ready to use" aspect of the result. But we also point out that the possibility to obtain a result as general as ours, starting from Struwe's work, was not so obvious [23] (see also 9.5, Chapter II of [24]). Roughly speaking we establish, for a wide class of functionals, a generic result saying that for almost every functionals in this class there exists a BPS sequence at the MP level.

Theorem 1.1 *Let X be a Banach space equipped with the norm $\|.\|$ and let $J \subset \mathbb{R}^+$ be an interval. We consider a family $(I_\lambda)_{\lambda \in J}$ of C^1-functionals on X of the form*

$$I_\lambda(u) = A(u) - \lambda B(u), \ \forall \lambda \in J$$

where $B(u) \geq 0, \forall u \in X$ and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$. We assume there are two points (v_1, v_2) in X such that setting

$$\Gamma = \{\gamma \in C([0,1], X), \gamma(0) = v_1, \gamma(1) = v_2\}$$

there hold, $\forall \lambda \in J$

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\}.$$

Then, for almost every $\lambda \in J$, there is a sequence $\{v_n\} \subset X$ such that

$$(i) \ \{v_n\} \text{ is bounded }, \ (ii) \ I_\lambda(v_n) \to c_\lambda, \ (iii) \ I'_\lambda(v_n) \to 0 \text{ in the dual } X^{-1} \text{ of } X.$$

To derive Theorem 1.1 we have been inspired by [2] and [28]. In particular in [28] the authors obtain a conclusion similar as ours for a special family $(I_\lambda)_{\lambda \in J}$. Their result however is derived using the precise form of the functional and in may not be so apparent that it is in fact very general. In view of Theorem 1.1 a natural question to ask is whether or not the limitation that a BPS sequence exists only for almost every $\lambda \in J$ is essential. The answer is yes and was pointed out to the author by Brezis [9]. Indeed at the end of Section 2 we give an example of a family $(I_\lambda)_{\lambda \in J}$, satisfying all the assumptions of Theorem 1.1, such that for a $\lambda_0 \in J$ all Palais-Smale sequences at the MP level c_{λ_0} are unbounded. Note that for a linking type problem arising in the study of periodic solutions of Hamiltonians systems (see 9.1 Theorem in [24]) the fact that a BPS sequence may not exists for every value of $\lambda \in J$ was proved by Ginzburg [15] and Herman [16].

In many situations one is interested to find a critical point for a given functional, namely for a given value of $\lambda \in J$. Then a first step is to prove the existence of a BPS sequence at the Mountain-Pass level or alternatively at a level different from $I_\lambda(0)$ to avoid finding $u = 0$ as a critical point. We claim that the generic result, Theorem 1.1, is a powerful tool to establish the existence of such sequence. This is particularly true if the problem enjoys some compactness properties.

Corollary 1.2 *Let X be a Banach space equipped with the norm $\|.\|$ and let $I \in C^1(X, \mathbb{R})$ be of the form*

$$I(u) = A(u) - B(u)$$

where B and B' take bounded sets to bounded sets. Suppose there exists $\varepsilon > 0$ such that, for $J = [1-\varepsilon, 1]$, the family $(I_\lambda)_{\lambda \in J}$ defined by

$$I_\lambda(u) = A(u) - \lambda B(u)$$

satisfies the assumptions of Theorem 1.1. Finally assume that for all $\lambda \in J$ any BPS sequences for I_λ at the level $c_\lambda \in \mathbb{R}$ admit a convergent subsequence. Then there exists $\{(\lambda_n, u_n)\} \subset\ [1-\varepsilon, 1] \times X$ with

$$\lambda_n \to 1 \text{ and } \{\lambda_n\} \text{ is increasing}$$

$$I_{\lambda_n}(u_n) = c_{\lambda_n} \text{ and } I'_{\lambda_n}(u_n) = 0 \text{ in } X^{-1}$$

such that, if $\{u_n\} \subset X$ is bounded, there hold,

$$I(u_n) = I_{\lambda_n}(u_n) + (\lambda_n - 1)B(u_n) \to \lim_{n\to\infty} c_{\lambda_n} = c_1$$

$$I'(u_n) = I'_{\lambda_n}(u_n) + (\lambda_n - 1)B'(u_n) \to 0 \text{ in } X^{-1}.$$

The point of Corollary 1.2 is that if $\{u_n\}$ is bounded, it is a BPS sequence for I at the level c_1. Clearly Corollary 1.2 is a direct consequence of Theorem 1.1 if we prove that the map $\lambda \to c_\lambda$ is continuous from the left. This is done in Lemma 2.3.

At this point however one may wonder about the interest of Corollary 1.2. Indeed the existence of a Palais-Smale sequence for I at the MP level was already known and the only remaining problem was, as it is now, to show that it is a bounded sequence. So what progress have we made ? In reality our position is now much more advantageous since with respect to a standard Palais-Smale sequence, the sequence $\{u_n\}$ given in Corollary 1.2 possesses properties which are very useful when one tries to establish that it is bounded. The difference is that instead of starting from a sequence of approximate critical points of I (as can be view a standard Palais-Smale sequence) we now start from a sequence of exact critical points of nearby functionals. The fact that u_n is an exact critical point often provides additional informations on the sequence $\{u_n\}$ which help to show that it is bounded. For example imagine that I is defined on a Sobolev space and that its critical points (as those of I_{λ_n}) correspond to solutions of a PDE problem. Then they possess stronger regularity properties than normally do elements of the ambient space. Also a use of a maximum principle can often garanties a given sign for u_n, $\forall n \in \mathbb{N}$ (see Section 3 for an application of this idea). Moreover there sometimes exist constraints that u_n must satisfies. Just think of all situations where the solutions of a PDE problem satisfy a Pohozaev's type identity. More globally, for $\lambda \in \mathbb{R}$, let

$$K_\lambda = \{u \in X : I_\lambda(u) = c_\lambda \text{ and } I'_\lambda(u) = 0\}$$

and suppose that $\cup_{\lambda\in[1-\varepsilon,1]}K_\lambda$ is bounded for a $\varepsilon > 0$. Then, if for all $\lambda \in [1-\varepsilon, 1]$ any BPS sequence for I_λ at the level $c_\lambda \in \mathbb{R}$ admit a convergent subsequence, the functional I has a critical point.

The idea of constructing Palais-Smale sequences that possess some extra properties which might help to ensure their boundedness, or more generally their convergence, is an old topic. Among some significant contributions in that direction let us mention [5, 20] where Morse type informations on the sequence prove crucial to ensure its compactness. Also this is the central issue in [14].

Remark 1.3 It should be clear that possibilities of using Theorem 1.1 to construct a special, up to boundedness, Palais-Smale sequence for a given functional exist in a large variety of situations. A particular important case is the following. Let X be a Banach space with norm $\|.\|$ and $I \in C^1(X, \mathbb{R})$ be such that $I(0) = 0$. Assume that there are two positive constants r, ρ and $v \in X$ with $\|v\| > \rho$ satisfying

$$I(u) \geq r \ \text{ if } \|u\| = \rho \ \text{ and } \ I(v) \leq 0.$$

Under these hypotheses I has a Mountain-Pass geometry and we denote by $c \geq r$ the MP level. Now writing I as

$$I(u) = I(u) - 0||u||^2$$

we see that there exists $\varepsilon > 0$ such that the family

$$I_\lambda(u) = I(u) - \lambda||u||^2 \text{ for } \lambda \in [0, \varepsilon]$$

satisfies the assumptions of Theorem 1.1. Indeed $I_\lambda(v) \leq 0$ for all $\lambda \geq 0$ and since for $\|u\| = \rho$

$$I_\lambda(u) = I(u) - \lambda||u||^2 \geq r - \lambda\rho^2$$

the claim holds as soon as $\varepsilon < r\rho^{-2}$. Thus, when in addition the family $(I_\lambda)_{\lambda\in[0,\varepsilon]}$ satisfies the compactness conditions of Corollary 1.2, we obtain, up to boundedness, a special Palais-Smale sequence for I at the level $\widetilde{c} := \lim_{\lambda\to 0^+} c_\lambda$. Note that if the map $\lambda \to c_\lambda$ is discontinuous at $\lambda = 0$ we may have $\widetilde{c} < c$. But clearly also $\widetilde{c} > 0 = I(0)$. Thus as far as the search of a non-trivial critical point is concerned we can forget that $\widetilde{c}$ and c may be different. $\square$

In a second part of the paper we apply the abstract results of Section 2 to study the existence of solutions of the problem

$$\left.\begin{array}{c} -\Delta u(x) + Ku(x) = f(x, u(x)) \\ u \in H^1(\mathbb{R}^N), K > 0. \end{array}\right\} \qquad (P)$$

Because we shall look for positive solutions we may assume without restriction that $f(x,s) = 0, \forall s < 0$, a.e. $x \in \mathbb{R}^N$. We require f to satisfy

(H1) (i) $f : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ is a Caratheodory function.

(ii) $f(.,s) \in L^\infty(\mathbb{R}^N)$ and $f(.,s)$ is 1-periodic in x_i, $1 \le i \le N$.

(H2) There is $p \in]2, \frac{2N}{N-2}[$ if $N \ge 3$ and $p > 2$ if $N = 1, 2$ such that $\lim_{s\to\infty} f(x,s)s^{1-p} = 0$, uniformly for $x \in \mathbb{R}^N$.

(H3) $f(x,s)s^{-1} \to 0$ if $s \to 0$, uniformly in $x \in \mathbb{R}^N$.

(H4) There is $a \in]0, \infty]$ such that $f(x,s)s^{-1} \to a$ if $s \to \infty$, uniformly in $x \in \mathbb{R}^N$.

Let $G : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$G(x,s) = \frac{1}{2} f(x,s)s - F(x,s) \text{ with } F(x,s) = \int_0^s f(x,t)\,dt.$$

We shall also use

(A1) $G(x,s) \ge 0, \forall s \ge 0$, a.e. $x \in \mathbb{R}^N$ and there is $\delta > 0$ such that

$$f(x,s)s^{-1} \ge K - \delta \Longrightarrow G(x,s) \ge \delta.$$

(A2) There is $D \in [1, \infty[$ such that, a.e. $x \in \mathbb{R}^N$,

$$G(x,s) \le DG(x,t),\ \forall (t,s) \in \mathbb{R}^+ \times \mathbb{R}^+ \text{ with } s \le t.$$

Theorem 1.4

(i) Assume that (H1)-(H4) and (A1) hold with $a < \infty$ in (H4). Then if $K \in]0, a[$ there exists a non-trivial positive solution of (P).

(ii) Assume that (H1)-(H4) and (A2) hold with $a = \infty$ in (H4). Then there exists a non-trivial positive solution of (P).

Remark 1.5 If $f(x,s)s^{-1}$ is a non decreasing function of $s \ge 0$, a.e. $x \in \mathbb{R}^N$ both (A1) and (A2) are satisfied. In particular then (A2) holds with $D = 1$. Note also that (A2) implies (A1) and thus the assumption on G is weaker when the nonlinearity is asymptotically linear. Finally observe that (H2) always hold when $a < \infty$ in (H4). □

Theorem 1.4 will be proved using a variational procedure in the spirit of Corollary 1.2. For the moment note that formally each critical point of the functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + Ku^2)\, dx - \int_{\mathbb{R}^N} F(x,u)\, dx$$

is a solution of problem (P). Also by the weak maximum principle it is a positive solution of (P). As we shall see, when hypotheses (H1)-(H4) hold and $K \in]0, a[$, I possesses a Mountain Pass geometry.

The existence of solutions of (P) or of closely related problems have been extensively studied these last years (see [7, 31, 33]). In the special case where f is autonomous, namely when the nonlinearity does not depend explicitly on $x \in \mathbb{R}^N$, the existence of one solution of (P) (and even infinity many) was proved by Berestycki and Lions [7] under hypotheses (H1)-(H4). To obtain the existence of one solution they develop a subtle Lagrange multiplier procedure which ultimately relies on the Pohozaev's identity for (P). The lack of compactness due to the translational invariance of (P) is regained working in the subspace of $H^1(\mathbb{R}^N)$ of radially symmetric functions. In the general case where f is not autonomous, Pohozaev's identity provides no informations and, in the previous work, in addition to (H1)-(H4), it is usually assumed that

(SQC) $\quad \exists \mu > 2$ such that $0 \leq \mu F(x,s) \leq f(x,s)s$, $\forall s \geq 0$, a.e. $x \in \mathbb{R}^N$.

The condition (SQC), from now on referred as the superquadraticity condition, was originally introduced in [3] and is still present in most work involving the search of critical points of Mountain-Pass type. Roughly speaking the role of (SQC) is to insure that all Palais-Smale sequences for I at the MP level are bounded.

In Theorem 1.4 we replace (SQC) by (A1) if $a < \infty$ or by (A2) if $a = \infty$ in (H4). A simple calculation shows that (SQC) implies that $f(x,.)$ must increase at least as $s^{\mu-1}$ for $s \to \infty$. So when $a < \infty$ it is not possible that (SQC) holds. When $a = \infty$ it may happen that (SQC) is satisfied but our requirements on f does not imply it. For example (SQC) is not true for the nonlinearity $f(x,s) = f(s) = s\ln(s+1)$ for $s \geq 0$ which satisfies (H1)-(H4) and (A2).

To our knowledge when $a = \infty$ in (H4) there is no general result on (P) without assuming the (SQC) condition. We believe however that the method applied in [1] to deal with an equation of the type (P), set on a bounded domain of $\mathbb{R}^N$, could be extended to cover the case of $\mathbb{R}^N$. However, in addition to (H1)-(H4), it is required in [1] that $f(x,s)s^{-1}$ is convex and this is substantially stronger than (A2). When $a < \infty$ in (H4) we just know two results [31, 33] which can be compared to Theorem 1.4. In [33], (P) is studied assuming that f is radial as a function of $x \in \mathbb{R}^N$. A similar hypothesis is present in [31] on a problem related to (P) arising from a model of self-trapping of an electro-magnetic wave. There, as in many papers dealing with a nonlinearity which is not superquadratic, is used an abstract critical point theorem due to Bartolo, Benci and Fortunato [6] which is based on the work of Cerami [11]. Thanks to the radial assumption the problems are somehow set on $\mathbb{R}$ and possess a much stronger compactness. It is not clear to us how the arguments developed in [31, 33] could be extended to treat a general problem on $\mathbb{R}^N$. Also in addition to (H1)-(H4) the assumptions that $f(x,s)s^{-1}$ is non decreasing and that $G(x,s) \to +\infty$ as $s \to \infty$, a.e. $x \in \mathbb{R}^N$ are needed both in [31] and [33]. Finally f has to satisfy a superquadraticity condition for $s \geq 0$ small. Namely for some $\delta > 0$ there is a $\mu > 2$ such that

$$0 \leq \mu F(x,s) \leq f(x,s)s, \quad \forall s \in [0,\delta], \text{ a.e. } x \in \mathbb{R}^N.$$

For all these reasons we believe that Theorem 1.4, both in the cases $a = \infty$ and $a < \infty$ that we treat in a unified way, strongly generalize the previous existence results.

Let us now sketch the proof of Theorem 1.4. We start by noticing that I is of the form

$$I(u) = A(u) - B(u)$$

with $A(u) \to +\infty$ as $\|u\| \to \infty$ and $B(u) \geq 0$, $\forall u \in H^1(\mathbb{R}^N)$. Then, thanks to Lemmas 3.1 and 3.2, we show that the family of functionals defined by

$$I_\lambda(u) = A(u) - \lambda B(u), \; \lambda \in [1,2]$$

satisfies the assumptions of Theorem 1.1. Thus we get that for almost every $\lambda \in [1,2]$ there exists a bounded sequence $\{v_m\} \subset H^1(\mathbb{R}^N)$ such that

$$I_\lambda(v_m) \to c_\lambda \quad \text{and} \quad I'_\lambda(v_m) \to 0 \text{ in } H^{-1}(\mathbb{R}^N).$$

Using the translational invariance of (P) we establish in Lemma 3.5 that there is a sequence $\{y_m\} \subset \mathbf{Z}^N$ such that $u_m(x) := v_m(x - y_m)$ satisfies $u_m \rightharpoonup u_\lambda \neq 0$ weakly in $H^1(\mathbb{R}^N)$ with $I_\lambda(u_\lambda) \leq c_\lambda$ and $I'_\lambda(u_\lambda) = 0$. From the weak maximum principle we get that $u_\lambda \geq 0$ a.e. $x \in \mathbb{R}^N$. At this point we have proved the existence of a sequence $\{(\lambda_n, u_n)\} \subset [1,2] \times H^1(\mathbb{R}^N)$ with $u_n \geq 0$ a.e. $x \in \mathbb{R}^N$ such that

- $\lambda_n \to 1$ and $\{\lambda_n\}$ is decreasing.
- $u_n \neq 0$, $I_{\lambda_n}(u_n) \leq c_{\lambda_n}$ and $I'_{\lambda_n}(u_n) = 0$.

In Lemma 3.6, assuming that $\{u_n\} \subset H^1(\mathbb{R}^N)$ is bounded we show how to obtain a non-trivial critical point of I corresponding to a positive solution of (P). To prove the boundedness of $\{u_n\}$ we develop an original approach, relying somehow on P.L. Lions work [19] on the concentration compactness principle, which, we believe, could be applied to a large variety of problem where (SQC) does not hold. The proof is by contradiction assuming that $\|u_n\| \to \infty$. Then setting $w_n = u_n\|u_n\|^{-1}$ (and using if necessary the translational invariance of (P)) there is a subsequence of $\{w_n\}$ with $w_n \rightharpoonup w$ in $H^1(\mathbb{R}^N)$ satisfying one of the two following alternatives.

1. (non-vanishing) $\exists \alpha > 0$, $R < \infty$ such that

$$\lim_{n\to\infty} \int_{B_R} w_n^2 \, dx \geq \alpha > 0,$$

2. (vanishing)

$$\lim_{n\to\infty} \sup_{y\in \mathbb{Z}^N} \int_{y+B_R} w_n^2 \, dx = 0, \ \forall R < \infty.$$

We shall prove that none of the two cases can occurs and this will give us the desired contradiction. If we assume that $\{w_n\}$ does not vanish then $w \neq 0$. To eliminate this alternative we distinguish the cases $a < \infty$ and $a = \infty$ in (H4). When $a < \infty$ we show in Lemma 3.7 that $w \neq 0$ satisfies the equation

$$-\Delta w + Kw = aw, \ x \in \mathbb{R}^N.$$

Since the operator $-\Delta$ has no eigenvector in $H^1(\mathbb{R}^N)$ this is a contradiction. When $a = \infty$ we show in Lemma 3.8 that the condition $f(x,s)s^{-1} \to \infty$ as $s \to \infty$ a.e. $x \in \mathbb{R}^N$ prevents the set $\Omega = \{x \in \mathbb{R}^N : w(x) > 0\}$ to have a non-zero Lebesgue measure. But this is the case since $w \neq 0$. To eliminate alternative (2) we distinguish again the cases $a < \infty$ and $a = \infty$. Noticing that, $\forall n \in \mathbb{N}$,

$$\int_{\mathbb{R}^N} G(x, u_n) \, dx \leq \frac{c_{\lambda_n}}{\lambda_n} \leq c$$

we show in Lemma 3.9 that when $a < \infty$ and (A1) holds the integrals goes to $+\infty$. Finally when $a = \infty$ we show in Lemma 3.10 that the vanishing of $\{w_n\}$ is incompatible with the "nice" radial behaviour of I which is insured (A2). Having proved the boundedness of $\{u_n\} \subset H^1(\mathbb{R}^N)$ the proof of Theorem 1.4 is completed.

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Notation

Throughout the article the letter C will denote various positive constants whose exact value may change from line to line but are not essential to the analysis of the problem. Also if we take a subsequence of a sequence $\{u_n\}$ we shall denote it again $\{u_n\}$.

2 Abstract Results

In this section we give the proof of Theorem 1.1 and we show that it is sharp in the sense that a BPS sequence cannot be found for every $\lambda \in J$. Since $J \subset \mathbb{R}^+$ and $B(u) \geq 0$, $\forall u \in X$, the map $\lambda \to c_\lambda$ is non increasing. Thus c'_λ, the derivative of c_λ with respect to λ, exists almost everywhere. Theorem 1.1 will be proved if we establish that the existence of c'_λ implies that I_λ has a BPS sequence at the level c_λ.

Let $\lambda \in J$ be an arbitrary but fixed value where c'_λ exists. Let $\{\lambda_n\} \subset J$ be a strictly increasing sequence such that $\lambda_n \to \lambda$.

Proposition 2.1 *There exist a sequence of paths $\{\gamma_n\} \subset \Gamma$ and $K = K(c'_\lambda) > 0$ such that*

(i) $||\gamma_n(t)|| \leq K$ *if* $\gamma_n(t)$ *satisfies*

$$I_\lambda(\gamma_n(t)) \geq c_\lambda - (\lambda - \lambda_n). \tag{2.1}$$

(ii) $\max_{t\in[0,1]} I_\lambda(\gamma_n(t)) \leq c_\lambda + (-c'_\lambda + 2)(\lambda - \lambda_n)$.

Proof. Let $\{\gamma_n\} \subset \Gamma$ be an arbitrary sequence such that

$$\max_{t\in[0,1]} I_{\lambda_n}(\gamma_n(t)) \leq c_{\lambda_n} + (\lambda - \lambda_n). \tag{2.2}$$

Note that such sequence exists since the class of paths Γ is independent of λ. We shall prove that, for $n \in \mathbb{N}$ sufficiently large, $\{\gamma_n\}$ is a sequence as we are looking for. When $\gamma_n(t)$ satisfies (2.1) we have

$$\begin{aligned} \frac{I_{\lambda_n}(\gamma_n(t)) - I_\lambda(\gamma_n(t))}{\lambda - \lambda_n} &\leq \frac{c_{\lambda_n} + (\lambda - \lambda_n) - c_\lambda + (\lambda - \lambda_n)}{\lambda - \lambda_n} \\ &= \frac{c_{\lambda_n} - c_\lambda}{\lambda - \lambda_n} + 2. \end{aligned}$$

Since c'_λ exists, there is $n(\lambda) \in \mathbb{N}$ such that $\forall n \geq n(\lambda)$

$$-c'_\lambda - 1 \leq \frac{c_{\lambda_n} - c_\lambda}{\lambda - \lambda_n} \leq -c'_\lambda + 1 \tag{2.3}$$

and thus $\forall n \geq n(\lambda)$,

$$\frac{I_{\lambda_n}(\gamma_n(t)) - I_\lambda(\gamma_n(t))}{\lambda - \lambda_n} \leq -c'_\lambda + 3.$$

Consequently

$$B(\gamma_n(t)) = \frac{I_{\lambda_n}(\gamma_n(t)) - I_\lambda(\gamma_n(t))}{\lambda - \lambda_n} \leq -c'_\lambda + 3.$$

Also

$$\begin{aligned}
A(\gamma_n(t)) &= I_{\lambda_n}(\gamma_n(t)) + \lambda_n B(\gamma_n(t)) \\
&\leq c_{\lambda_n} + (\lambda - \lambda_n) + \lambda_n(-c'_\lambda + 3) \\
&\leq C.
\end{aligned}$$

Using our assumption that either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$, the uniform boundedness of $A(\gamma_n(t))$ and $B(\gamma_n(t))$ proves (i). To prove (ii) observe that from (2.3), we have, $\forall n \geq n(\lambda)$

$$c_{\lambda_n} \leq c_\lambda + (-c'_\lambda + 1)(\lambda - \lambda_n). \tag{2.4}$$

Using (2.2), (2.4) and since

$$I_{\lambda_n}(v) \geq I_\lambda(v),\ \forall v \in X$$

we get

$$\begin{aligned}
I_\lambda(\gamma_n(t)) &\leq I_{\lambda_n}(\gamma_n(t)) \\
&\leq c_{\lambda_n} + (\lambda - \lambda_n) \\
&\leq c_\lambda + (-c'_\lambda + 2)(\lambda - \lambda_n).
\end{aligned}$$

Thus Point (ii) also holds. ♠

Roughly speaking Proposition 2.1 says that there exists a sequence of paths $\{\gamma_n\} \subset \Gamma$ such that

$$\max_{t \in [0,1]} I_\lambda(\gamma_n(t)) \to c_\lambda$$

for which, for all $n \in \mathbb{N}$ sufficiently large, starting from a level strictly below c_λ, all the "top" of the path is contained in the ball centred at the origin of fixed radius $K = K(c'_\lambda) > 0$. Now for $\alpha > 0$ we define

$$F_\alpha = \{u \in X : ||u|| \leq K + 1 \text{ and } |I_\lambda(u) - c_\lambda| \leq \alpha\}$$

where the constant $K > 0$ is given in Proposition 2.1.

Proposition 2.2 *For all $\alpha > 0$*

$$\inf\{||I'_\lambda(u)|| : u \in F_\alpha\} = 0. \tag{2.5}$$

Proof. Seeking a contradiction we assume that (2.5) does not hold. Then there exists $\alpha > 0$ such that for any $u \in F_\alpha$ one has

$$||I'_\lambda(u)|| \geq \alpha \tag{2.6}$$

and without loss of generality we can assume that

$$0 < \alpha < \frac{1}{2}\left[c_\lambda - \max\{I_\lambda(v_1), I_\lambda(v_2)\}\right].$$

A classical deformation argument then says that there exist $\varepsilon \in]0, \alpha[$ and a homeomorphism $\eta : X \to X$ such that

$$\eta(u) = u, \ \text{ if } |I_\lambda(u) - c_\lambda| \geq \alpha \tag{2.7}$$

$$I_\lambda(\eta(u)) \leq I_\lambda(u), \ \forall u \in X \tag{2.8}$$

$$I_\lambda(\eta(u)) \leq c_\lambda - \varepsilon, \ \forall u \in X \text{ satisfying } ||u|| \leq K \text{ and } I_\lambda(u) \leq c_\lambda + \varepsilon. \tag{2.9}$$

Let $\{\gamma_n\} \subset \Gamma$ be the sequence obtained in Proposition 2.1. We choose and fix $m \in \mathbb{N}$ sufficiently large in order that

$$(-c'_\lambda + 2)(\lambda - \lambda_m) \leq \varepsilon. \tag{2.10}$$

Clearly by (2.7), $\eta(\gamma_m) \in \Gamma$. Now if $u = \gamma_m(t)$ satisfies

$$I_\lambda(u) \leq c_\lambda - (\lambda - \lambda_m)$$

then (2.8) implies that

$$I_\lambda(\eta(u)) \leq c_\lambda - (\lambda - \lambda_m). \tag{2.11}$$

On the other hand if $u = \gamma_m(t)$ satisfies

$$I_\lambda(u) > c_\lambda - (\lambda - \lambda_m)$$

then Proposition 2.1 and (2.10) implies that u is such that $||u|| \leq K$ with $I_\lambda(u) \leq c_\lambda + \varepsilon$. Applying (2.9) one has

$$I_\lambda(\eta(u)) \leq c_\lambda - \varepsilon \leq c_\lambda - (\lambda - \lambda_m). \tag{2.12}$$

Thus combining (2.11) and (2.12) we get

$$\max_{t \in [0,1]} I_\lambda(\eta(\gamma_m(t))) \leq c_\lambda - (\lambda - \lambda_m)$$

which contradicts the variational characterisation of c_λ. ♠

Proof of Theorem 1.1. Since Proposition 2.2 is true there exists a Palais-Smale sequence for I_λ at the level $c_\lambda \in \mathbb{R}$ which is contained in the ball of radius $K + 1$ centred at the origin. This proves the theorem. ♠

Lemma 2.3 *The map $\lambda \to c_\lambda$ is continuous from the left.*

Proof. Seeking a contradiction we assume there are $\lambda_0 \in J$ and $\{\lambda_n\} \subset J$ with $\lambda_n < \lambda_0$, $\forall n \in \mathbb{N}$ and $\lambda_n \to \lambda_0$ for which

$$c_{\lambda_0} < \lim_{n\to\infty} c_{\lambda_n}.$$

Let $\delta = \lim_{n\to\infty} c_{\lambda_n} - c_{\lambda_0} > 0$. By definition of c_{λ_0} there is $\gamma_0 \in \Gamma$ such that

$$\max_{t\in[0,1]} I_{\lambda_0}(\gamma_0(t)) < c_{\lambda_0} + \frac{\delta}{3}.$$

Using the fact that, $I_\lambda(u) = I_{\lambda_0}(u) + (\lambda_0 - \lambda)B(u)$, $\forall \lambda \in J$, $\forall u \in X$, we get, $\forall \lambda < \lambda_0$

$$\max_{t\in[0,1]} I_{\lambda}(\gamma_0(t)) < c_{\lambda_0} + \frac{\delta}{3} + (\lambda_0 - \lambda)\max_{t\in[0,1]} B(\gamma_0(t)).$$

But B being continuous we have $\max_{t\in[0,1]} B(\gamma_0(t)) \leq C$ for a $C > 0$ and thus, for any $n \in \mathbb{N}$ sufficiently large,

$$\max_{t\in[0,1]} I_{\lambda_n}(\gamma_0(t)) < c_{\lambda_0} + \frac{2\delta}{3}.$$

We reach a contradiction noticing that by definition of c_{λ_n}

$$\max_{t\in[0,1]} I_{\lambda_n}(\gamma_0(t)) \geq c_{\lambda_n}. \qquad \spadesuit$$

We end this section by presenting a family $(I_\lambda)_{\lambda\in J}$ for which there does not exist a BPS sequence for every $\lambda \in J$. As we already mentioned this example was provided to us by Brezis [9] and it shows that Theorem 1.1 is sharp. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x, y) = x^2 - (x-1)^3 y^2.$$

The space $\mathbb{R}^2$ is equipped with the Euclidean norm $\|(x, y)\| = \sqrt{x^2 + y^2}$. Around the origin F behaves as $\|(x, y)\|^2$. Moreover taking $x > 0$ sufficiently large we see that $F(x, 1) < 0$. In particular F has a Mountain-Pass geometry and as we notice in Remark 1.3 there exists $\varepsilon > 0$ such that the family of functions $(F_\lambda)_{\lambda\in[0,\varepsilon]}$ defined by

$$F_\lambda(x, y) = F(x, y) - \lambda(x^2 + y^2)$$

satisfies the assumptions of Theorem 1.1. In fact it is even possible to assume that $\lambda \in [-\varepsilon, \varepsilon]$. Let us show that there is no BPS sequence for $F = F_0$ at the Mountain-Pass level. We have

$$\begin{cases} F_x = 2x - 3(x-1)^2 y^2 \\ F_y = -2(x-1)^3 y. \end{cases}$$

Thus any sequence $\{(x_n, y_n)\} \subset \mathbb{R}^2$ such that $\|F'(x_n, y_n)\| \to 0$ must satisfy

$$2x_n - 3(x_n - 1)^2 y_n^2 \to 0 \tag{2.13}$$
$$(x_n - 1)^3 y_n \to 0. \tag{2.14}$$

Without restriction we can assume that $x_n \to x \in [-\infty, \infty]$ and $y_n \to y \in [-\infty, \infty]$. We distinguish two cases

- $x_n \not\to 1$. Then from (2.14) we get that $y_n \to 0$ and since

$$(x_n - 1)^2 y_n^2 = \Big[(x_n - 1)^3 y_n\Big]^{2/3} y_n^{4/3} \to 0$$

  it follows from (2.13) that $x_n \to 0$.

- $x_n \to 1$. Then from (2.13), $(x_n - 1)^2 y_n^2 \to 2/3$ and in particular $|y_n| \to \infty$.

In the first case $F(x_n, y_n) \to 0$ and in the second $F(x_n, y_n) \to 1$. We deduce that the Mountain-Pass level for F is $c = 1$ and that there is no BPS sequence for F at this level. Analyzing the Palais-Smale sequences of F_λ for $\lambda \in [-\varepsilon, \varepsilon] \setminus \{0\}$ we find that there always exists a critical point at the Mountain-Pass level $c_\lambda = (1-\lambda)(1-\lambda^{1/3})^2$. We have $c_\lambda \to 1$ as $\lambda \to 0$ and thus c_λ is continuous on $[-\varepsilon, \varepsilon]$. Moreover

$$c'_\lambda = (1-\lambda^{1/3})(\frac{5}{3}\lambda^{1/3} - 1 - \frac{2}{3}\lambda^{-2/3}) \text{ for } \lambda \in]-\varepsilon, \varepsilon[\setminus\{0\}$$

and thus c'_λ exists for all $\lambda \in]-\varepsilon, \varepsilon[\setminus\{0\}$. On the contrary we can check that c'_λ for $\lambda = 0$ does not exist as we already knew from Theorem 1.1.

3 Applications

The main aim of this section is to prove Theorem 1.4 applying the abstract variational approach of Section 2. In the proofs that follow, we shall routinely take $N \geq 3$. The proofs for $N = 1$ or $N = 2$ are not more complicated. Our working space is the Sobolev space $H^1(\mathbb{R}^N)$ equipped with the norm

$$\|u\| = \left\{ \int_{\mathbb{R}^N} (|\nabla u|^2 + K u^2)\, dx \right\}^{\frac{1}{2}}$$

which, since $K > 0$, is equivalent to the usual one. We denote by $\|\cdot\|_p$, for each $p \in [1, \infty]$, the standard norm of the Lebesgue space $L^p(\mathbb{R}^N)$. As we mentioned in the introduction proving Theorem 1.4 amounts to find a non-trivial critical point of the functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)\, dx.$$

A proof that, under (H1)-(H3), I is a C^1-functional is given in Proposition 2.1 of [12]. Let us show that I has a Mountain-Pass geometry. Since $I(0) = 0$ this is a consequence of two following results.

Lemma 3.1 *Assume that (H1)-(H3) hold. Then* $I(u) = \frac{1}{2}\|u\|^2 + o(\|u\|^2)$ *as* $u \to 0$.

Proof. By (H3) we know that $f(x,s)s^{-1} \to 0$ as $s \to 0$ uniformly in $x \in \mathbb{R}^N$. Thus for any $\varepsilon > 0$ it follows by (H2) that there exists a $C_\varepsilon > 0$ such that

$$f(x,s) \leq \varepsilon s + C_\varepsilon s^{p-1}, \forall s \geq 0, \text{ a.e. } x \in \mathbb{R}^N \tag{3.1}$$

or equivalently that

$$F(x,s) \leq \frac{\varepsilon}{2}s^2 + \frac{C_\varepsilon}{p}s^p, \forall s \geq 0, \text{ a.e. } x \in \mathbb{R}^N. \tag{3.2}$$

We deduce that

$$\int_{\mathbb{R}^N} F(x,u)\,dx \leq \frac{\varepsilon}{2}\|u\|^2 + C\|u\|_p^p$$

and this implies that $\int_{\mathbb{R}^N} F(x,u)\,dx = o(\|u\|^2)$ as $u \to 0$. ♠

Lemma 3.2 *Assume that (H1), (H2), (H4) hold and that $K \in]0,a[$. Then we can find a $v \in H^1(\mathbb{R}^N)$, $v \neq 0$ satisfying $I(v) \leq 0$.*

Proof. Without loss of generality we can assume that $a < \infty$ in (H4). The proof is based in the construction of a family of testing functions that we borrow from [33] (see also [29]). Let

$$d^2(N) = \int_{\mathbb{R}^N} e^{-2|x|^2}dx \text{ and } D(N) = 4[d(N)]^{-2}\int_{\mathbb{R}^N} |x|^2e^{-2|x|^2}dx.$$

For $\alpha > 0$ we set

$$w_\alpha(x) = [d(N)]^{-1}\alpha^{\frac{N}{4}}e^{-\alpha|x|^2}.$$

Straightforward calculations shows that

$$\|w_\alpha\|_2 = 1 \text{ and } \|\nabla w_\alpha\|_2^2 = \alpha D(N).$$

Thus in particular if we fix $\alpha \in (0, \dfrac{a-K}{D(N)})$ we get that

$$\|\nabla w_\alpha\|_2^2 < (a-K). \tag{3.3}$$

On the other hand by (H4)

$$\lim_{s\to\infty}\frac{F(x,s)}{s^2} = \frac{a}{2}, \text{ uniformly in } x \in \mathbb{R}^N.$$

and since, for every $x \in \mathbb{R}^N$, $tw_\alpha(x) \to +\infty$ as $t \to +\infty$ it follows that

$$\lim_{t\to+\infty}\frac{F(x,tw_\alpha)}{t^2w_\alpha^2} = \frac{a}{2}, \text{ a.e. } x \in \mathbb{R}^N.$$

Now observe that (H1),(H3) and (H4) implies the existence of a constant $C < \infty$ such that $\forall s \geq 0$, a.e. $x \in \mathbb{R}^N$,

$$0 \leq \frac{F(x,s)}{s^2} \leq C. \tag{3.4}$$

Thus using (3.4) it follows by Lebesgue's theorem that

$$\lim_{t \to +\infty} \int_{\mathbb{R}^N} \frac{F(x, tw_\alpha)}{t^2}\, dx = \frac{a}{2} \int_{\mathbb{R}^N} w_\alpha^2 \, dx = \frac{a}{2}.$$

Now using (3.3) we get

$$\begin{aligned} \lim_{t \to +\infty} \frac{I(tw_\alpha)}{t^2} &= \frac{1}{2}\|\nabla w_\alpha\|_2^2 + \frac{K}{2}\|w_\alpha\|_2^2 - \lim_{t \to +\infty} \int_{\mathbb{R}^N} \frac{F(x, tw_\alpha)}{t^2}\, dx \\ &= \frac{1}{2}\|\nabla w_\alpha\|_2^2 + \frac{K}{2} - \frac{a}{2} < 0 \end{aligned}$$

and the lemma is proved. ♠

We define on $H^1(\mathbb{R}^N)$ the family of functionals

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \lambda \int_{\mathbb{R}^N} F(x,u)\, dx, \ \lambda \in [1,2].$$

Lemma 3.3 *Assume that (H1)-(H4) hold. The family (I_λ) with $\lambda \in [1,2]$ satisfies the hypotheses of Theorem 1.1. In particular for almost every $\lambda \in [1,2]$ there exists a bounded sequence $\{v_m\} \subset H^1(\mathbb{R}^N)$ satisfying*

$$I_\lambda(v_m) \to c_\lambda \text{ and } I'_\lambda(v_m) \to 0 \text{ in } H^{-1}(\mathbb{R}^N).$$

Proof. For the $v \in H^1(\mathbb{R}^N)$ obtained in Lemma 3.2, $I_\lambda(v) \leq 0$ for all $\lambda \geq 1$ since

$$\int_{\mathbb{R}^N} F(x,u)\, dx \geq 0, \ \forall u \in H^1(\mathbb{R}^N).$$

Also from Lemma 3.1 we know that

$$\int_{\mathbb{R}^N} F(x,u)\, dx = o(\|u\|^2) \text{ as } u \to 0.$$

Thus setting

$$\Gamma = \left\{\gamma \in C([0,1], H^1(\mathbb{R}^N)), \gamma(0) = 0 \text{ and } \gamma(1) = v\right\}$$

we have, $\forall \lambda \in [1,2]$,

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > 0.$$

An application of Theorem 1.1 now completes the proof. ♠

In the rest of the paper we shall often use the following terminology. Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be an arbitrary bounded sequence. If it is possible to translate each u_n in $\mathbb{R}^N$ such that the translated sequence (still denoted $\{u_n\}$) satisfies, up to a subsequence, $\exists \alpha > 0$, $R < \infty$ such that

$$\lim_{n \to \infty} \int_{B_R} u_n^2 \, dx \geq \alpha > 0$$

we say that $\{u_n\}$ does not vanish. If it is not the case then necessarily one has

$$\lim_{n \to \infty} \sup_{y \in \mathbf{Z}^N} \int_{y + B_R} u_n^2 \, dx = 0, \ \forall R < \infty$$

and in this case we say that $\{u_n\}$ vanish.

Lemma 3.4 *Assume that (H1)-(H3) hold. Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be an arbitrary bounded sequence which vanish. Then*

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} G(x, u_n) \, dx = 0.$$

Proof. It is known that if $\{u_n\} \subset H^1(\mathbb{R}^N)$ vanish then $u_n \to 0$ strongly in $L^q(\mathbb{R}^N)$ for all $q \in]2, \frac{2N}{N-2}[$. A proof of this result is given in Lemma 2.18 of [12]. It is a special case of Lemma I.1 of [19]. Now by (3.1) and (3.2) we know that $\forall \varepsilon > 0$, $\exists \, C_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^N} f(x, u_n) u_n \, dx \leq \varepsilon \|u_n\|_2^2 + C_\varepsilon \|u_n\|_p^p$$

$$\int_{\mathbb{R}^N} F(x, u_n) \, dx \leq \frac{\varepsilon}{2} \|u_n\|_2^2 + \frac{C_\varepsilon}{p} \|u_n\|_p^p.$$

Thus if $\{u_n\} \subset H^1(\mathbb{R}^N)$ vanish both

$$\int_{\mathbb{R}^N} f(x, u_n) u_n \, dx \to 0 \ \text{ and } \ \int_{\mathbb{R}^N} F(x, u_n) \, dx \to 0$$

and the lemma follows from the definition of G. ♠

Lemma 3.5 *Assume that (H1)-(H4) and either (A1) or (A2) hold. Let $\lambda \in [1, 2]$ be fixed. Then for all bounded sequences $\{v_m\} \subset H^1(\mathbb{R}^N)$ satisfying*

- $0 < \lim_{m \to \infty} I_\lambda(v_m) \leq c_\lambda$
- $I'_\lambda(v_m) \to 0$ *in* $H^{-1}(\mathbb{R}^N)$

there exists $\{y_m\} \subset \mathbf{Z}^N$ such that, up to a subsequence, $u_m(x) := v_m(x - y_m)$ satisfies $u_m \rightharpoonup u_\lambda \neq 0$ with $I_\lambda(u_\lambda) \leq c_\lambda$ and $I'_\lambda(u_\lambda) = 0$.

Proof. Since $\{v_m\} \subset H^1(\mathbb{R}^N)$ is bounded we have

$$\int_{\mathbb{R}^N} G(x, v_m)\, dx = I_\lambda(v_m) - \frac{1}{2} I'_\lambda(v_m)v_m \to \lim_{m\to\infty} I_\lambda(v_m) > 0.$$

Thus we see, by Lemma 3.4, that $\{v_m\} \subset H^1(\mathbb{R}^N)$ does not vanish and there is $\{y_m\} \subset \mathbf{Z}^N$ such that, up to a subsequence, $u_m(x) := v_m(x - y_m)$ satisfies: $\exists \alpha > 0$, $R < \infty$ such that

$$\lim_{m\to\infty} \int_{B_R} u_m^2\, dx \geq \alpha > 0. \tag{3.5}$$

Moreover since problem (P) is invariant under the translation group associated to the periodicity of $f(., s)$ we still have

- $0 < \lim_{m\to\infty} I_\lambda(u_m) \leq c_\lambda$
- $I'_\lambda(u_m) \to 0$ in $H^{-1}(\mathbb{R}^N)$.

We have, up to a subsequence, $u_m \rightharpoonup u_\lambda$ for a $u_\lambda \in H^1(\mathbb{R}^N)$ and to complete the proof of the lemma we just need to show, that $u_\lambda \neq 0$, $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) \leq c_\lambda$.

Step 1 $\quad u_\lambda \neq 0$

Since (3.5) hold we get by the compactness of the Sobolev embedding $H^1(B_R) \hookrightarrow L^2(B_R)$ that

$$\|u_\lambda\|_2^2 \geq \int_{B_R} u_\lambda^2\, dx = \lim_{m\to\infty} \int_{B_R} u_m^2\, dx \geq \alpha > 0.$$

Thus $u_\lambda \neq 0$ and Step 1 is completed.

Step 2 $\quad I'_\lambda(u_\lambda) = 0$

Noting that $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$ it suffices to check that $I'_\lambda(v)\varphi = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Let $(\cdot,\cdot)$ denote the inner product on $H^1(\mathbb{R}^N)$ associated to our chosen norm. Then

$$I'_\lambda(u_m)\varphi - I'_\lambda(u_\lambda)\varphi = (u_m - u_\lambda, \varphi) - \int_{\mathbb{R}^N} (f(x, u_m) - f(x, u_\lambda))\, \varphi\, dx \to 0$$

since $u_m \rightharpoonup u_\lambda$ weakly in $H^1(\mathbb{R}^N)$ and strongly in $L^q_{loc}(\mathbb{R}^N)$ for $q \in [2, \frac{2N}{N-2}[$. Thus recalling that $I'_\lambda(u_m) \to 0$ we indeed have that $I'_\lambda(u_\lambda) = 0$.

Step 3 $\quad I_\lambda(u_\lambda) \leq c_\lambda$

Observe that either (A1) or (A2) imply that

$$G(x, s) \geq 0,\ \forall s \geq 0,\ \text{ a.e. } x \in \mathbb{R}^N. \tag{3.6}$$

Thus using Fatou's Lemma we get using Step 2

$$\begin{aligned} c_\lambda &\geq \lim_{m\to\infty}\left[I_\lambda(u_m) - \frac{1}{2}I'_\lambda(u_m)u_m\right] \\ &= \lim_{m\to\infty}\int_{\mathbb{R}^N} G(x,u_m)\,dx \\ &\geq \int_{\mathbb{R}^N} G(x,u_\lambda)\,dx \\ &= I_\lambda(u_\lambda) - \frac{1}{2}I'_\lambda(u_\lambda)u_\lambda = I_\lambda(u_\lambda). \end{aligned}$$

This ends the proof of the lemma. ♠

At this point combining Lemmas 3.3 and 3.5 we deduce the existence of a sequence $\{(\lambda_n, u_n)\} \subset [1,2] \times H^1(\mathbb{R}^N)$ with $u_n \geq 0$ a.e. $x \in \mathbb{R}^N$ such that

- $\lambda_n \to 1$ and $\{\lambda_n\}$ is decreasing
- $u_n \neq 0,\ I_{\lambda_n}(u_n) \leq c_{\lambda_n}$ and $I'_{\lambda_n}(u_n) = 0$.

Since

$$\frac{1}{2}\|u_n\|^2 - \lambda_n \int_{\mathbb{R}^N} F(x,u_n)\,dx \leq c_{\lambda_n} \quad \text{and} \quad \|u_n\|^2 = \lambda_n \int_{\mathbb{R}^N} f(x,u_n)u_n\,dx$$

we have in particular that

$$\int_{\mathbb{R}^N} G(x,u_n)\,dx \leq \frac{c_{\lambda_n}}{\lambda_n}.$$

Clearly $\frac{c_{\lambda_n}}{\lambda_n}$ is increasing and bounded by $c = c_1$. It follows that

$$\int_{\mathbb{R}^N} G(x,u_n)\,dx \leq c,\ \forall n \in \mathbb{N}. \tag{3.7}$$

Lemma 3.6 *Assume that (H1)-(H4) and either (A1) or (A2) hold. If the sequence* $\{u_n\} \subset H^1(\mathbb{R}^N)$ *given above is bounded there exists* $u \neq 0$ *such that* $I'(u) = 0$. *In particular* u *is a non-trivial positive solution of (P).*

Proof. First notice that

$$I'(u_n)v = I'_{\lambda_n}(u_n)v + (\lambda_n - \lambda)\int_{\mathbb{R}^N} f(x,u_n)v\,dx \to 0,\ \forall v \in H^1(\mathbb{R}^N).$$

Now knowing that

$$I(u_n) = I_{\lambda_n}(u_n) + (\lambda_n - \lambda)\int_{\mathbb{R}^N} F(x,u_n)\,dx$$

we distinguish two cases. Either $\limsup_{n\to\infty} I_{\lambda_n}(u_n) > 0$ or $\limsup_{n\to\infty} I_{\lambda_n}(u_n) \leq 0$. In the first case we get $\limsup_{n\to\infty} I(u_n) > 0$ and the result follows from Lemma 3.5. In the second case we define the sequence $\{z_n\} \subset H^1(\mathbb{R}^N)$ by $z_n = t_n u_n$ with $t_n \in [0,1]$ satisfying

$$I_{\lambda_n}(z_n) = \max_{t\in[0,1]} I_{\lambda_n}(tu_n). \tag{3.8}$$

(If for a $n \in \mathbb{N}$, t_n defined by (3.8) is not unique we choose the smaller possible value). By construction $\{z_n\} \subset H^1(\mathbb{R}^N)$ is bounded. Moreover on one hand $I'_{\lambda_n}(z_n)z_n = 0,\ \forall n \in \mathbb{N}$ and thus

$$\lambda_n \int_{\mathbb{R}^N} G(x, z_n)\, dx = I_{\lambda_n}(z_n) - \frac{1}{2} I'_{\lambda_n}(z_n)z_n = I_{\lambda_n}(z_n). \tag{3.9}$$

On the other hand it is easily seen, following the proof of Lemma 3.1, that $I'_{\lambda_n}(u)u = ||u||^2 + o(\|u\|^2)$ as $u \to 0$, uniformly in $n \in \mathbb{N}$. Thus, since $I'_{\lambda_n}(u_n) = 0$, there is $\alpha > 0$ such that $||u_n|| \geq \alpha, \forall n \in \mathbb{N}$. Recording that $\limsup_{n\to\infty} I_{\lambda_n}(u_n) \leq 0$, we then obtain from Lemma 3.1 and (3.8) that $\liminf_{n\to\infty} I_{\lambda_n}(z_n) > 0$ and from (3.9) it follows that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} G(x, z_n)\, dx = \liminf_{n\to\infty} I_{\lambda_n}(z_n) > 0.$$

Lemma 3.4 then shows that $\{z_n\}$ does not vanish and so neither $\{u_n\}$. At this point we conclude repeating Steps 1 and 2 of the proof of Lemma 3.5. ♠

In view of Lemma 3.6 to complete the proof of Theorem 1.4 we just need to check that $\{u_n\} \subset H^1(\mathbb{R}^N)$ is bounded. This is the purpose of our last four lemmas. Seeking a contradiction we assume that $\|u_n\| \to \infty$ and define the sequence $\{w_n\} \subset H^1(\mathbb{R}^N)$ by

$$w_n = \frac{u_n}{\|u_n\|}.$$

Clearly $\|w_n\| = 1$ and thus $w_n \rightharpoonup w$ up to a subsequence. Either $\{w_n\} \subset H^1(\mathbb{R}^N)$ vanish or it does not vanish. Using (A1) when $a < \infty$ or (A2) when $a = \infty$ in (H4) we shall prove that none of these alternatives can occur and this contradiction will prove that $\{u_n\} \subset H^1(\mathbb{R}^N)$ is bounded. Assume first that $\{w_n\} \subset H^1(\mathbb{R}^N)$ does not vanish. Then, as in the proof of Lemma 3.5, using if necessary the translation invariance of problem (P), we get that $w_n \rightharpoonup w \neq 0$. Also we can assume without loss of generality that $w_n \to w$ a.e. $x \in \mathbb{R}^N$. At this point the proof bifurcates to cover separately the cases $a < \infty$ and $a = \infty$ in (H4).

Lemma 3.7 *Assume that (H1)-(H4) hold with $a < \infty$ in (H4) and that $K \in]0, a[$. Then the non-vanishing of $\{w_n\} \subset H^1(\mathbb{R}^N)$ is impossible.*

Proof. We shall prove $0 \neq w \in H^1(\mathbb{R}^N)$ satisfies the eigenvalue problem

$$-\Delta w(x) + Kw(x) = aw(x),\ x \in \mathbb{R}^N. \tag{3.10}$$

This gives us the desired contradiction since it is well known that the operator $-\Delta$ has no eigenvalue in $H^1(\mathbb{R}^N)$. To prove that (3.10) holds it suffices to check that, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} [\nabla w \nabla \varphi + Kw\varphi]\, dx = \int_{\mathbb{R}^N} [aw\varphi]\ dx. \tag{3.11}$$

Recall that $I'_{\lambda_n}(u_n) = 0$. Thus we have

$$-\Delta u_n + Ku_n = \lambda_n f(x, u_n) \text{ in } H^{-1}(\mathbb{R}^N).$$

Consequently $\{w_n\} \subset H^1(\mathbb{R}^N)$ satisfies

$$-\Delta w_n + Kw_n = \lambda_n \frac{f(x, u_n)}{u_n} w_n \text{ in } H^{-1}(\mathbb{R}^N),$$

and this implies that, $\forall \varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} [\nabla w_n \nabla \varphi + Kw_n\varphi]\, dx = \int_{\mathbb{R}^N} \left[\lambda_n \frac{f(x, u_n)}{u_n} w_n \varphi\right] dx. \tag{3.12}$$

Since $w_n \rightharpoonup w$ weakly in $H^1(\mathbb{R}^N)$ we have, $\forall \varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} [\nabla w_n \nabla \varphi + Kw_n\varphi]\, dx \to \int_{\mathbb{R}^N} [\nabla w \nabla \varphi + Kw\varphi]\, dx. \tag{3.13}$$

We claim that

$$\lambda_n \frac{f(x, u_n)}{u_n} w_n \to aw,\ \text{a.e. } x \in \mathbb{R}^N. \tag{3.14}$$

To prove (3.14) it is convenient to distinguish the cases $w(x) = 0$ and $w(x) \neq 0$ (without loss of generality we can assume that $w \neq 0$ is defined everywhere on $\mathbb{R}^N$). Let $x \in \mathbb{R}^N$ be such that $w(x) = 0$. Using the assumptions (H1),(H3) and (H4) we see that there exists $C < \infty$ such that

$$0 \leq \frac{f(x, s)}{s} \leq C,\ \forall s \geq 0,\ \text{a.e. } x \in \mathbb{R}^N. \tag{3.15}$$

Thus since $\{\lambda_n\} \subset \mathbb{R}$ is bounded and $w_n(x) \to w(x)$ a.e. $x \in \mathbb{R}^N$ we have for such $x \in \mathbb{R}^N$

$$0 \leq \lambda_n \frac{f(x, u_n(x))}{u_n(x)} w_n(x) \leq \lambda_n C w_n(x) \to 0 = aw(x).$$

Now let $x \in \mathbb{R}^N$ be such that $w(x) \neq 0$. Then we necessarily have $u_n(x) \to \infty$ and thus, using (H4), we get since $\lambda_n \to 1$

$$\lambda_n \frac{f(x, u_n(x))}{u_n(x)} \to a.$$

Consequently also in this case

$$\lambda_n \frac{f(x, u_n(x))}{u_n(x)} w_n(x) \to a w(x) \tag{3.16}$$

and (3.14) is established. Now let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be arbitrary but fixed and let $\Omega \subset \mathbb{R}^N$ be a compact set such that *supp* $\varphi \subset \Omega$. By the compactness of the Sobolev embedding $H^1(\Omega) \hookrightarrow L^1(\Omega)$ we have $w_n \to w$ strongly in $L^1(\Omega)$. Thus in particular there is a $h \in L^1(\Omega)$ such that $w_n(x) \leq h(x)$ a.e. $x \in \Omega$ (see Theorem IV.9 in [8]) and using again (3.15) we have

$$0 \leq \lambda_n \frac{f(x, u_n)}{u_n} w_n \leq C w_n \leq Ch, \text{ a.e. } x \in \Omega. \tag{3.17}$$

Now (3.14) and (3.17) allows to apply Lebesgue theorem and we get

$$\int_{\mathbb{R}^N} \left[\lambda_n \frac{f(x, u_n)}{u_n} w_n \varphi \right] dx \to \int_{\mathbb{R}^N} [a w \varphi] \, dx. \tag{3.18}$$

Since (3.18) holds for an arbitrary $\varphi \in C_0^\infty(\mathbb{R}^N)$, combining (3.13) and (3.18) we indeed get (3.11). Thus (3.10) holds and the lemma is proved. ♠

Lemma 3.8 *Assume that (H1)-(H4) hold with $a = \infty$ in (H4). Then the non-vanishing of $\{w_n\} \subset H^1(\mathbb{R}^N)$ is impossible.*

Proof. From

$$-\Delta u_n + K u_n = \lambda_n f(x, u_n)$$

we deduce that

$$-\Delta w_n + K w_n = \lambda_n \frac{f(x, u_n)}{||u_n||}. \tag{3.19}$$

Multiplying (3.19) by an arbitrary $v \in H^1(\mathbb{R}^N)$ and integrating it follows that

$$\int_{\mathbb{R}^N} [\nabla w_n \nabla v + K w_n v] \, dx = \lambda_n \int_{\mathbb{R}^N} \frac{f(x, u_n)}{||u_n||} v \, dx.$$

Thus if $w_n \rightharpoonup w$, we have, $\forall v \in H^1(\mathbb{R}^N)$,

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{f(x, u_n)}{||u_n||} v \, dx = \int_{\mathbb{R}^N} [\nabla w \nabla v + K w v] \, dx$$

and in particular setting $v = w$ we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{f(x,u_n)}{||u_n||} w \, dx = ||w||^2 < \infty. \tag{3.20}$$

But on $\Omega = \{x \in \mathbb{R}^N : w(x) \neq 0\}$ we have, since $a = \infty$,

$$\frac{f(x,u_n)}{||u_n||} w = \frac{f(x,u_n)}{u_n} w_n w \to +\infty, \ a.e. \ x \in \mathbb{R}^N.$$

Thus taking into account that $|\Omega| > 0$ and using Fatou's Lemma we deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{f(x,u_n)}{||u_n||} w \, dx = +\infty.$$

This contradicts (3.20). ♠

Now we shall prove that the vanishing of $\{w_n\} \subset H^1(\mathbb{R}^N)$ is forbidden. Here also we distinguish the cases $a < \infty$ and $a = \infty$ and (H4).

Lemma 3.9 *Assume that (H1)-(H4) hold with $a < \infty$ in (H4). Then if (A1) hold the vanishing of $\{w_n\} \subset H^1(\mathbb{R}^N)$ is impossible.*

Proof. We have

$$-\Delta u_n + K u_n = \lambda_n f(x,u_n).$$

Thus

$$-\Delta w_n + K w_n = \lambda_n \frac{f(x,u_n)}{u_n} w_n. \tag{3.21}$$

Multiplying (3.21) by w_n and integrating we get

$$\int_{\mathbb{R}^N} \left[|\nabla w_n|^2 + K w_n^2 \right] dx = \lambda_n \int_{\mathbb{R}^N} \frac{f(x,u_n)}{u_n} w_n^2 \, dx$$

and we deduce from the normalization of $\{w_n\} \subset H^1(\mathbb{R}^N)$ that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{f(x,u_n)}{u_n} w_n^2 \, dx = 1. \tag{3.22}$$

We define for $\delta > 0$ given in (A1)

$$\Omega_n = \{x \in \mathbb{R}^N : \frac{f(x,u_n)}{u_n} \leq K - \frac{\delta}{2}\}.$$

Then since, $1 = ||w_n||^2 = ||\nabla w_n||_2^2 + K||w_n||_2^2$, we have

$$\begin{aligned}\int_{\Omega_n} \frac{f(x,u_n)}{u_n} w_n^2\, dx &\leq (K - \frac{\delta}{2}) \int_{\Omega_n} w_n^2\, dx \\ &\leq \frac{1}{K}(K - \frac{\delta}{2}).\end{aligned}$$

Consequently we see, using (3.22), that necessarily

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N \setminus \Omega_n} \frac{f(x,u_n)}{u_n} w_n^2\, dx > 0. \tag{3.23}$$

We claim that

$$\limsup_{n\to\infty} |\mathbb{R}^N \setminus \Omega_n| = \infty. \tag{3.24}$$

Seeking a contradiction we assume that

$$\limsup_{n\to\infty} |\mathbb{R}^N \setminus \Omega_n| < \infty. \tag{3.25}$$

Note that by (3.15)

$$\int_{\mathbb{R}^N \setminus \Omega_n} \frac{f(x,u_n)}{u_n} w_n^2\, dx \leq C \int_{\mathbb{R}^N \setminus \Omega_n} w_n^2\, dx. \tag{3.26}$$

But, since $\{w_n\} \subset H^1(\mathbb{R}^N)$ vanishs, we have taking into account (3.25)

$$\lim_{n\to\infty} \int_{\mathbb{R}^N \setminus \Omega_n} w_n^2\, dx \to 0$$

and thus (3.26) contradicts (3.23). The contradiction proves that (3.24) is true. Now observe that by (A1), $G(x,s) \geq 0,\ \forall s \geq 0, a.e.\ x \in \mathbb{R}^N$ and thus, $\forall n \in \mathbb{N}$,

$$\begin{aligned}\int_{\mathbb{R}^N} G(x,u_n)\, dx &= \int_{\Omega_n} G(x,u_n)\, dx + \int_{\mathbb{R}^N \setminus \Omega_n} G(x,u_n)\, dx \\ &\geq \int_{\mathbb{R}^N \setminus \Omega_n} G(x,u_n)\, dx.\end{aligned}$$

Taking into account (3.7) we deduce that, for all $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^N \setminus \Omega_n} G(x,u_n)\, dx \leq C. \tag{3.27}$$

But on $\mathbb{R}^N \setminus \Omega_n$ we have $\dfrac{f(x,u_n)}{u_n} \geq K - \dfrac{\delta}{2}$ and thus by (A1)

$$G(x,u_n) \geq \delta, \quad a.e.\ x \in \mathbb{R}^N \setminus \Omega_n. \tag{3.28}$$

Combining (3.24) and (3.28) we get a contradiction with (3.27). ♠

Lemma 3.10 *Assume that (H1)-(H4) hold with $a = \infty$ in (H4). Then if (A2) hold the vanishing of $\{w_n\} \subset H^1(\mathbb{R}^N)$ is impossible.*

Proof. We use again the sequence $\{z_n\} \subset H^1(\mathbb{R}^N)$ introduce in Lemma 3.6. We claim that, under our assumptions and since we assume that $||u_n|| \to \infty$,

$$\lim_{n\to\infty} I_{\lambda_n}(z_n) = +\infty. \tag{3.29}$$

Seeking a contradiction we assume that for a $M < \infty$

$$\liminf_{n\to\infty} I_{\lambda_n}(z_n) \leq M \tag{3.30}$$

and we define, for the corresponding subsequence, $\{k_n\} \subset H^1(\mathbb{R}^N)$ by

$$k_n = \sqrt{4M}\frac{u_n}{\|u_n\|}.$$

Now, since $\{k_n\} \subset H^1(\mathbb{R}^N)$ vanishs and is bounded, from the proof of Lemma 3.4, we get that

$$\int_{\mathbb{R}^N} F(x, k_n)\, dx \to 0.$$

It follows that, for $n \in \mathbb{N}$ sufficiently large,

$$I_{\lambda_n}(k_n) = 2M - \lambda_n \int_{\mathbb{R}^N} F(x, k_n)\, dx \geq \frac{3}{2}M. \tag{3.31}$$

Since k_n and z_n corresponds, for all $n \in \mathbb{N}$, to the same direction we see using the definition of z_n that (3.31) contradicts (3.30). Thus (3.29) hold. Now we have $I'_{\lambda_n}(z_n)z_n = 0, \forall n \in \mathbb{N}$, and thus

$$I_{\lambda_n}(z_n) = I_{\lambda_n}(z_n) - \frac{1}{2}I'_{\lambda_n}(z_n)z_n = \lambda_n \int_{\mathbb{R}^N} G(x, z_n)\, dx. \tag{3.32}$$

Combining (3.29) and (3.32) we see that

$$\int_{\mathbb{R}^N} G(x, z_n)\, dx \to +\infty.$$

But from (A2) and (3.7) we also have

$$\int_{\mathbb{R}^N} G(x, z_n)\, dx \leq D \int_{\mathbb{R}^N} G(x, u_n)\, dx \leq C. \tag{3.33}$$

This contradiction proves the lemma. ♠

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