Solutions for a quasilinear Schrödinger equation : A dual approach *

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Abstract

We consider quasilinear stationary Schrödinger equations of the form

$$-\Delta u - \Delta(u^2)u = g(x,u), \qquad x \in \mathbb{R}^N. \tag{0.1}$$

Introducing a change of unknown, we transform the search of solutions $u(x)$ of (0.1) into the search of solutions $v(x)$ of the semilinear equation

$$-\Delta v = \frac{1}{\sqrt{1+2f^2(v)}}\, g(x, f(v)), \qquad x \in \mathbb{R}^N, \tag{0.2}$$

where f is suitably chosen. If v is a classical solution of (0.2) then $u = f(v)$ is a classical solution of (0.1). Variational methods are then used to obtain various existence results.

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1 Introduction

In this paper we deal with equations of the form

$$-\Delta u - \Delta(u^2)u = g(x,u), \qquad u \in H^1(\mathbb{R}^N). \tag{1.1}$$

These equations model several physical phenomena but until recently little had been done to prove rigorously the existence of solutions.

A major difficulty associated with (1.1) is the following; one may seek to obtain solutions by looking for critical points of the associated "natural" functional, $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \int_{\mathbb{R}^N} |\nabla u|^2 u^2\, dx - \int_{\mathbb{R}^N} G(x,u)\, dx$$

where $G(x,s) = \int_0^s g(x,t)\, dt$. However except when $N = 1$ this functional is not defined on all $H^1(\mathbb{R}^N)$.

The first existence results for equations of the form of (1.1) are, up to our knowledge, due to [12, 8]; papers to which we refer for a presentation of the physical motivations of studying (1.1). In [12, 8], however, the main existence results are obtained, through a constrained minimization argument, only up to an unknown Lagrange multiplier.

Subsequently a general existence result for (1.1) was derived in [7]. To overcome the undefiniteness of J the idea in [7] is to introduce a change of variable and to rewrite the functional J with this new variable. Then critical points are search in an associated Orlicz space (see [7] for details).

The aim of the present paper is to give a simple and shorter proof of the results of [7], which do not use Orlicz spaces, but rather is developed in the usual $H^1(\mathbb{R}^N)$ space. The fact that we work in $H^1(\mathbb{R}^N)$ also permit to cover a different class of nonlinearities. In particular we give full treatment of the autonomous case and for non autonomous problems we do not assume that,

$$s \to \frac{g(x,s)}{s} :]0,\infty[\to \mathbb{R} \text{ is non decreasing in } s.$$

Following the strategy developed in [4] on a related problem we also make use of a change of unknown $v = f^{-1}(u)$ and define an associated equation that we shall call dual. If $v \in H^1(\mathbb{R}^N)$ is classical solution of

$$-\Delta v = \frac{1}{\sqrt{1+2f^2(v)}}\, g(x, f(v)), \tag{1.2}$$

$u = f(v)$ is a classical solution of (1.1).

Equations of the form (1.2) are of semilinear elliptic type and one can try to solve them by a variational approach. In particular we shall see that, under very general conditions on g, the "natural" functional associated to (1.2), $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \, dx - \int_{\mathbb{R}^N} G(x, f(v)) \, dx$$

is well defined and of class C^1 on $H^1(\mathbb{R}^N)$.

The dual approach is introduced in Section 2. In Section 3 we deal with autonomous problems, when (1.1) is of the form,

$$-\Delta u - \Delta(u^2)u = g(u), \qquad u \in H^1(\mathbb{R}^N). \tag{1.3}$$

Autonomous problems seems to play an important role in physical phenomena (see [3] for example) and we obtain here an existence result under assumptions we believe to be nearly optimal. We assume that the nonlinear term g satisfies :

(g0) $g(s)$ is locally Hölder continuous on $[0, \infty[$.

(g1) $-\infty < \liminf_{s\to 0} \frac{g(s)}{s} \leq \limsup_{s\to 0} \frac{g(s)}{s} = -\nu < 0$ for $N \geq 3$,
$\lim_{s\to 0} \frac{g(s)}{s} = -\nu \in (-\infty, 0)$ for $N = 1, 2$.

(g2) When $N \geq 3$, $\lim_{s\to\infty} \frac{|g(s)|}{s^{\frac{3N+2}{N-2}}} = 0$.
When $N = 2$, for any $\alpha > 0$ there exists $C_\alpha > 0$ such that

$$|g(s)| \leq C_\alpha e^{\alpha s^2} \quad \text{for all } s \geq 0.$$

(g3) When $N \geq 2$, there exists $\xi_0 > 0$ such that $G(\xi_0) > 0$,
When N=1, there exists $\xi_0 > 0$ such that

$$G(\xi) < 0 \text{ for all } \xi \in]0, \xi_0[, \; G(\xi_0) = 0 \text{ and } g(\xi_0) > 0.$$

Remark 1.1 *An easy calculation shows that (g0)-(g3) are satisfied in the model case* $g(s) = |s|^2 s - \nu s$.

Theorem 1.2 *Assume that (g0)-(g3) hold. Then (1.3) admits a solution* $u_0 \in H^1(\mathbb{R}^N)$ *having the following properties :*

(i) $u_0 > 0$ *on* $\mathbb{R}^N$.

(ii) u_0 *is spherically symmetric :* $u_0(x) = u_0(r)$ *with* $r = |x|$ *and* u_0 *decreases with respect to* r.

(iii) $u_0 \in C^2(\mathbb{R}^N)$.

(iv) u_0 *together with its derivatives up to order 2 have exponential decay at infinity*

$$|D^\alpha u_0(x)| \leq Ce^{-\delta|x|},\ x \in \mathbb{R}^N,$$

for some $C, \delta > 0$ *and for* $|\alpha| \leq 2$.

We prove Theorem 1.2 searching for a critical point of the functional I, which is here autonomous. As we shall see the existence of a critical point follows almost directly, from classical results on scalar field equations due to Berestycki-Lions [1] when $N = 1$ or $N \geq 3$ and Berestycki-Gallouët-Kavian [2] when $N = 2$.

In Section 4 we assume that (1.1) is of the form,

$$-\Delta u - \Delta(u^2)u + V(x)u = h(u). \tag{1.4}$$

We require $V \in C(\mathbb{R}^N, \mathbb{R})$ and $h \in C(\mathbb{R}^+, \mathbb{R})$, to be Hölder continuous and to satisfy

(V0) There exists $V_0 > 0$ such that $V(x) \geq V_0 > 0$ on $\mathbb{R}^N$.

(V1) $\lim_{|x|\to\infty} V(x) = V(\infty)$ and $V(x) \leq V(\infty)$ on $\mathbb{R}^N$.

(h0) $\displaystyle\lim_{s\to 0} \frac{h(s)}{s} = 0.$

(h1) There exists $p < \infty$ if $N = 1, 2$ and $p < \frac{3N+2}{N-2}$ if $N \geq 3$ such that $|h(s)| \leq C(1 + |s|^p)$, $\forall s \in \mathbb{R}$, for a $C > 0$.

(h2) There exists $\mu \geq 4$ such that, $\forall s > 0$,

$$0 < \mu H(s) \leq h(s)s \ \text{ with } \ H(s) = \int_0^s h(t)\, dt.$$

Our main result is the following :

Theorem 1.3 *Assume that (V0)-(V1) and (h0)-(h1) hold. Then (1.4) has a positive non trivial solution if one of the following conditions hold :*

1) (h2) hold with a $\mu > 4$.

2) (h2) hold with $\mu = 4$ *with* $p \leq 5$ *if* $N = 3$ *and* $p < \frac{3N+4}{N}$ *if* $N \geq 4$ *in (h1).*

The proof of Theorem 1.3 also relies on the study of the functional I. We first show that I possess a mountain pass geometry and denote by $c > 0$ the mountain pass level (see Lemma 4.2). To find a critical point the main difficulties to overcome are the possible unboundedness of the Palais-Smale (or Cerami) sequences and a lack of compactness since (1.4) is set on all $\mathbb{R}^N$.

For the second difficulty we use some recent results presented in [9] and [10] which imply that, under conditions (V0)-(V1), the mountain pass level $c > 0$ is below (if $V \not\equiv V(\infty)$) the first level of possible loss of compactness (see Theorem 3.4 and Lemma 4.3).

For the first difficulty we distinguish the cases $\mu > 4$ and $\mu = 4$ in (h2). In the case $\mu > 4$, it is direct to prove that all Cerami sequences of I are bounded. To show it in the case $\mu = 4$ is more involved and for this we make use of an idea introduced in [7].

Notation : Throughout the article the letter C will denote various positive constants whose exact value may change from line to line but are not essential to the analysis of the problem. Also if we take a subsequence of a sequence $\{v_n\}$ we shall denote it again $\{v_n\}$.

2 The dual formulation

We start with some preliminary results. Let f be defined by

$$f'(t) = \frac{1}{\sqrt{1+2f^2(t)}} \quad \text{and} \quad f(0) = 0$$

on $[0, +\infty[$ and by $f(t) = -f(-t)$ on $]-\infty, 0]$.

Lemma 2.1 *1) f is uniquely defined, C^∞ and invertible.*

2) $|f'(t)| \leq 1$, for all $t \in \mathbb{R}$.

3) $\dfrac{f(t)}{t} \to 1$ as $t \to 0$.

4) $\dfrac{f(t)}{\sqrt{t}} \to 2^{\frac{1}{4}}$ as $t \to +\infty$.

Proof. Points 1)-3) are immediate. To see 4) we integrate

$$\int_0^t f'(s)\sqrt{1+2f^2(s)}\, ds = t.$$

Using the changes of variables $x = f(s)$ and $x = \frac{1}{\sqrt{2}}\, Sh(y)$ we obtain that

$$\frac{1}{2\sqrt{2}}\,[sinh^{-1}(\sqrt{2}f(t))] + \frac{1}{4\sqrt{2}}\, sinh2[Sh^{-1}(\sqrt{2}f(t))] = t.$$

Thus, $sinh2[Sh^{-1}(\sqrt{2}f(t))] \sim 4\sqrt{2}t$ in the sense that, as $t \to +\infty$,

$$\frac{sinh2[sinh^{-1}(\sqrt{2}f(t))]}{4\sqrt{2}t} \to 1.$$

We set $a(t) = sinh^{-1}(\sqrt{2}f(t))$. Then $a(t)$ satisfies $sinh[2a(t)] \sim 4\sqrt{2}t$ and we deduce that

$$a(t) \sim \frac{1}{2}ln(4\sqrt{2}t + \sqrt{32t^2+1}).$$

Finally since $2sinh(t) \sim e^t$ it follows that

$$2\sqrt{2}f(t) \sim e^{\frac{1}{2}ln(4\sqrt{2}t + \sqrt{32t^2+1})} \sim 2\sqrt{2}\, 2^{\frac{1}{4}}\sqrt{t}$$

and the lemma is proved. ♠

Lemma 2.2 *For all* $t \in \mathbb{R}$,

$$\frac{1}{2}f(t) \leq \frac{t}{\sqrt{1+2f^2(t)}} \leq f(t).$$

Proof. To establish the first inequality we need to show that, for all $t \geq 0$,

$$\sqrt{1+2f^2(t)}\, f(t) \leq 2t.$$

In this aim we study the function $g : \mathbb{R}^+ \to \mathbb{R}$, defined by

$$g(t) = 2t - \sqrt{1+2f^2(t)}\, f(t).$$

We have $g(0) = 0$ and, since $f'(t)\sqrt{1+2f^2(t)} = 1$, $\forall t \in \mathbb{R}$, that $g'(t) = 1 - 2f'^2(t)f^2(t)$. It follows that $g'(t) \geq 0$ since $1 - 2f'^2(t)f^2(t) = f'^2(t)$ and the first inequality is proved. The second one is derived in a similar way. ♠

We now present our dual approach. For simplicity we set $H = H^1(\mathbb{R}^N)$ and denote by $||\cdot||$ its standard norm. We assume that $g(x,s)$ is such that $I : H \to \mathbb{R}$ given by

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\, dx - \int_{\mathbb{R}^N} G(x, f(v))\, dx$$

with $G(x,s) = \int_0^s g(x,t)\, dt$, is well defined and of class C^1 ($f : \mathbb{R} \to \mathbb{R}$ is the function previously introduced).

Let $v \in H \cap C^2(\mathbb{R}^N)$ be a critical point of I. Since $f'^2(t)(1+2f^2(t)) \equiv 1$, it satisfies

$$-\Delta v = \frac{1}{\sqrt{1+2f^2(v)}}\, g(x, f(v)). \tag{2.1}$$

We set $u = f(v)$ (i.e. $v = f^{-1}(u)$). Clearly $u \in C^2(\mathbb{R}^N)$ and $u \in H$. Indeed $\nabla u = f'(v)\nabla v$ and $|f'(t)| \leq 1$, $\forall t \in \mathbb{R}$.

We have $\nabla v = (f^{-1})'(u)\nabla u$ and

$$\Delta v = (f^{-1})''(u)\, |\nabla u|^2 + (f^{-1})'(u)\, \Delta u. \tag{2.2}$$

Since $(f^{-1})'(t) = \dfrac{1}{f'[f^{-1}(t)]}$, it follows that

$$(f^{-1})'(t) = \sqrt{1 + 2f^2(f^{-1}(t))} = \sqrt{1+2t^2} \ \text{ and } \ (f^{-1})''(t) = \frac{2t}{\sqrt{1+2t^2}}.$$

Thus, from (2.2), we deduce that

$$\Delta v = \frac{2u}{\sqrt{1+2u^2}}\,|\nabla u|^2 + \sqrt{1+2u^2}\,\Delta u$$

and consequently, from (2.1), that

$$-\frac{2u}{\sqrt{1+2u^2}}\,|\nabla u|^2 - \sqrt{1+2u^2}\,\Delta u - \frac{1}{\sqrt{1+2u^2}}\,g(x,u) = 0.$$

This can be rewrite as

$$\frac{1}{\sqrt{1+2u^2}}\,\big[(-1-2u^2)\Delta u - 2u|\nabla u|^2 - g(x,u)\big] = 0.$$

Since $\Delta(u^2)u = 2u|\nabla u|^2 + 2u^2\Delta u$ it shows that $u \in H \cap C^2(\mathbb{R}^N)$ satisfies (1.1).

At this point it is clear that to obtain a classical solution of (1.1) it suffices to obtain a critical point of I of class C^2.

3 Autonomous cases

In this section (1.1) is of the form

$$-\Delta u - \Delta(u^2)u = g(u), \qquad u \in H. \tag{3.1}$$

with the nonlinearity g satisfying (g0)-(g3). Because we look for positive solutions we may assume without restriction that $g(s) = 0$, $\forall s \le 0$. Following our dual approach we shall obtain the existence of solutions for (3.1) studying the associated dual equation

$$-\Delta v = \frac{1}{\sqrt{1+2f^2(v)}}\,g(f(v)), \qquad v \in H. \tag{3.2}$$

In this aim, we now recall some classical results due to Berestycki-Lions [1] and Berestycki-Gallouët-Kavian [2] on equations of the form

$$-\Delta v = k(v), \qquad v \in H. \tag{3.3}$$

These authors show that the natural functional corresponding to (3.3), $J : H \to \mathbb{R}$ given by

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx - \int_{\mathbb{R}^N} K(v)\,dx$$

where $K(s) = \int_0^s k(t)\,dt$ is of class C^1, if k satisfies the conditions :

(k0) $k(s) \in C(\mathbb{R}^+, \mathbb{R})$ (and $k(s) = 0$, $\forall s \leq 0$).

(k1) $-\infty < \liminf_{s\to 0} \frac{k(s)}{s} \leq \limsup_{s\to 0} \frac{k(s)}{s} = -\nu < 0$ for $N \geq 3$,

$\lim_{s\to 0} \frac{k(s)}{s} = -\nu \in (-\infty, 0)$ for $N = 1, 2$.

(k2) When $N \geq 3$, $\lim_{s\to\infty} \frac{|k(s)|}{s^{\frac{N+2}{N-2}}} = 0$.
When $N = 2$, for any $\alpha > 0$ there exists $C_\alpha > 0$ such that

$$|k(s)| \leq C_\alpha e^{\alpha s^2} \quad \text{for all } s \geq 0.$$

We recall that a solution $v \in H$ of (3.3) is said to be a least energy solution if and only if

$$J(v) = m \ \text{ where } \ m = inf\{J(v), v \in H\backslash\{0\} \text{ is a solution of (3.3)}\}.$$

The following result is given in [1] when $N = 1$ or $N \geq 3$ and in [2] when $N = 2$.

Theorem 3.1 *Assume that (k0)-(k2) and (k3) hold with*

*(k3) When $N \geq 2$, there exists $\xi_0 > 0$ such that $K(\xi_0) > 0$.
When N=1, there exists $\xi_0 > 0$ such that*

$$K(\xi) < 0 \text{ for all } \xi \in]0, \xi_0[,\ K(\xi_0) = 0 \ \text{ and } k(\xi_0) > 0.$$

Then $m > 0$ and there exists a least energy solution $\omega(x)$ of (3.3) which satisfies :

(i) $\omega > 0$ on $\mathbb{R}^N$.

(ii) ω is spherically symmetric : $\omega(x) = \omega(r)$ with $r = |x|$ and ω decreases with respect to r.

(iii) $\omega \in C^2(\mathbb{R}^N)$.

(iv) ω together with its derivatives up to order 2 have exponential decay at infinity

$$|D^\alpha \omega(x)| \leq Ce^{-\delta|x|},\ x \in \mathbb{R}^N,$$

for some $C, \delta > 0$ and for $|\alpha| \leq 2$.

Now observe that equation (3.2) is of the form $-\Delta v = k(v)$ with

$$k(s) = \frac{1}{\sqrt{1 + 2f^2(s)}}\, g(f(s)). \tag{3.4}$$

We claim that if $g(s)$ satisfies (g0)-(g3) then $k(s)$ given by (3.4) satisfies (k0)-(k3). Indeed the fact that (k0) holds is trivial. The conditions (k1),(k2) follow, respectively, from Lemma 2.1 (ii) and (iii). To check (k3) when $N \geq 2$ it suffices to notice that

$$G(\xi_0) > 0 \text{ for a } \xi_0 > 0 \Longleftrightarrow \exists s_0 > 0 \text{ such that } G(f(s_0)) > 0.$$

Clearly (k3) also holds when $N = 1$. Having proved our claim we directly obtain from Theorem 3.1 :

Theorem 3.2 *Assume that (g0)-(g2) hold. Then the functional $I : H \to \mathbb{R}$ given by*

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \, dx - \int_{\mathbb{R}^N} G(f(v)) \, dx$$

is well defined and of class C^1. If in addition g satisfies (g3) then (3.2) has a least energy solution $\omega(x)$ which possesses the properties (i)-(iv) of Theorem 3.1.

At this point turning back to equation (3.1), Theorem 1.2 follows directly from Theorem 3.2 and the properties of f (see Lemma 2.1).

Remark 3.3 *In [1] the authors justify the growth restriction (k2) considering the special nonlinearities $k(s) = \lambda|s|^{p-1}s - ms$ where $\lambda, m > 0$. They show that in this case (3.3) has no solution when $p \geq \frac{N+2}{N-2}$. In contrast, Theorem 1.2 says that solutions of (3.1) do exist for all $1 < p < \frac{3N+2}{N-2}$.*

In the next section we shall use the fact that the least energy solution $\omega(x)$ given in Theorem 3.2 has a mountain pass characterization. Indeed, in [9] for $N \geq 2$ and in [10] for $N = 1$, Theorem 3.1 is complemented in the following way :

Theorem 3.4 *Assume that (k0)-(k3) hold. Then setting*

$$\Gamma = \{\gamma \in C([0,1], H), \gamma(0) = 0 \text{ and } J(\gamma(1)) < 0\},$$

we have $\Gamma \neq \emptyset$ and $b = m$ with

$$b \equiv \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)).$$

Moreover for any least energy solution $\omega(x)$ as given in Theorem 3.1, there exists a path $\gamma \in \Gamma$ such that $\gamma(t)(x) > 0$ for all $x \in \mathbb{R}^N$ and $t \in (0,1]$ satisfying $\omega \in \gamma([0,1])$ and

$$\max_{t \in [0,1]} J(\gamma(t)) = b.$$

Remark 3.5 *In [9], [10] it is also proved that under (k0)-(k2) there exists $\alpha_0 > 0$, $\delta_0 > 0$ such that*

$$J(v) \geq \alpha_0 \, ||v||^2 \text{ when } ||v|| \leq \delta_0.$$

4 Non autonomous cases

In this section we assume that (1.1) is of the form

$$-\Delta u - \Delta(u^2)u + V(x)u = h(u), \qquad u \in H. \tag{4.1}$$

with the potential $V(x)$ satisfying (V0)-(V1) and the nonlinearity $h(s)$, (h0)-(h2). Here again we use our dual approach and first look to critical points of $I : H \to \mathbb{R}$ given by

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + V(x)f^2(v)\,dx - \int_{\mathbb{R}^N} H(f(v))\,dx.$$

Namely for solutions $v \in H$ of

$$-\Delta v = \frac{1}{\sqrt{1+2f^2(v)}}\,[-V(x)f(v) + h(f(v))]. \tag{4.2}$$

From Section 3 we readily deduce that I is well defined and of class C^1 under conditions (V0)-(V1) and (h0)-(h1). Let us show that I has a mountain pass geometry, in the sense that,

$$\Gamma = \{\gamma \in C([0,1], H), \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\} \neq \emptyset,$$

and

$$c \equiv \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) > 0.$$

For this we first mention a direct consequence of (h2).

Remark 4.1 *The function $t \to H(st)t^{-4}$ is increasing on $\mathbb{R}^+$, for all $s > 0$. In particular there is $C > 0$ such that $H(s) \geq Cs^4$ for $s \geq 1$ and $\lim_{s\to+\infty} h(s)s^{-1} = \infty$.*

Lemma 4.2 *Under (V0)-(V1) and (h0)-(h2) I has a mountain pass geometry.*

Proof. From the assumptions (V0)-(V1) we have

$$k_1(s) \;\leq\; \frac{1}{\sqrt{1+2f^2(v)}}\,[-V(x)f(v) + h(f(v))] \;\leq\; k_2(s)$$

where

$$k_1(s) = \frac{1}{\sqrt{1+2f^2(v)}}\,[-V(\infty)f(v) + h(f(v))] \quad \text{and}$$

$$k_2(s) = \frac{1}{\sqrt{1+2f^2(v)}}[-V_0 f(v) + h(f(v))].$$

The nonlinearities $k_1(s)$ and $k_2(s)$ both satisfy assumptions (k0)-(k3). Thus, from Remark 3.5, we deduce (considering $k_2(s)$) that there exists $\alpha_0 > 0$, $\delta_0 > 0$ such that

$$I(v) \geq \alpha_0\,||v||^2 \text{ when } ||v|| \leq \delta_0. \tag{4.3}$$

Namely the origin is a strict local minimum. Also since the functional corresponding to $k_1(s)$ is negative at some point we deduce that $\Gamma \neq \emptyset$. ♠

Lemma 4.3 *Assume that (V0)-(V1) and (h0)-(h2) hold. Let* $\{v_n\} \subset H$ *be a bounded Palais-Smale sequence for* I *at level* $c > 0$*. Then, up to a subsequence,* $v_n \rightharpoonup v \neq 0$ *with* $I'(v) = 0$*.*

Proof. Since $\{v_n\}$ is bounded, we can assume that, up to a subsequence, $v_n \rightharpoonup v$. Let us prove that $I'(v) = 0$. Noting that $C_0^\infty(\mathbb{R}^N)$ is dense in H, it suffices to check that $I'(v)\varphi = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. But we readily have, using Lebesgue's Theorem, that

$$\begin{aligned} I'(v_n)\varphi - I'(v)\varphi &= \int_{\mathbb{R}^N} \nabla(v_n - v)\nabla\varphi \, dx \\ &+ \int_{\mathbb{R}^N} \left(\frac{-f(v_n)}{\sqrt{1+2f^2(v_n)}} + \frac{f(v)}{\sqrt{1+2f^2(v)}} \right) V(x)\varphi \, dx \\ &+ \int_{\mathbb{R}^N} \left(\frac{h(f(v_n))}{\sqrt{1+2f^2(v_n)}} - \frac{h(f(v))}{\sqrt{1+2f^2(v)}} \right) \varphi \, dx \to 0, \end{aligned}$$

since $v_n \rightharpoonup v$ weakly in H and strongly in $L^q_{loc}(\mathbb{R}^N)$ for $q \in [2, \frac{2N}{N-2}[$ if $N \geq 3, q \geq 2$ if $N = 1, 2$. Thus recalling that $I'(v_n) \to 0$ we indeed have $I'(v) = 0$. At this point if $v \neq 0$ the lemma is proved. Thus we assume that $v = 0$. We claim that in this case $\{v_n\}$ is also a Palais-Smale sequence for the functional $\tilde{I} : H \to \mathbb{R}$ defined by

$$\tilde{I}(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\infty) f^2(v) \, dx - \int_{\mathbb{R}^N} H(f(v)) \, dx$$

at the level $c > 0$. Indeed, as $n \to \infty$,

$$\tilde{I}(v_n) - I(v_n) = \int_{\mathbb{R}^N} [V(\infty) - V(x)] f^2(v_n) \, dx \to 0$$

since $V(x) \to V(\infty)$ as $|x| \to \infty$, $|f(s)| \leq |s|, \forall s \in \mathbb{R}$ and $v_n \to 0$ in $L^2_{loc}(\mathbb{R}^N)$. Also, for the same reasons, we have

$$\sup_{||u|| \leq 1} |(\tilde{I}'(v_n) - I'(v_n), u)| = \sup_{||u|| \leq 1} |\int_{\mathbb{R}^N} \frac{f(v_n)u}{\sqrt{1+2f^2(v_n)}} [V(\infty) - V(x)] \, dx| \to 0.$$

Next we claim that the situation : For all $R > 0$

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{y+B_R} v_n^2 \, dx = 0,$$

which we will refer to as the vanishing case cannot occurs. From (h0)-(h1) and Lemma 2.1, $\forall \varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that

$$h(f(s))f(s) \leq \varepsilon s^2 + C_\varepsilon |s|^{\frac{p+1}{2}} \quad \text{for all } s \in \mathbb{R}. \tag{4.4}$$

Thus, for any $v \in H$,

$$\int_{\mathbb{R}^N} h(f(v))f(v)\,dx \le \varepsilon \int_{\mathbb{R}^N} v^2\,dx + C_\varepsilon \int_{\mathbb{R}^N} |v|^{\frac{p+1}{2}}\,dx \tag{4.5}$$

and using Lemma 2.2 we see that $\forall \varepsilon > 0$,

$$\begin{aligned}\lim_{n\to\infty} \int_{\mathbb{R}^N} h(f(v_n))\,\frac{v_n}{\sqrt{1+2f^2(v_n)}}\,dx &\le \lim_{n\to\infty} \int_{\mathbb{R}^N} h(f(v_n))f(v_n)\,dx \\ &\le \lim_{n\to\infty}\left[\varepsilon \int_{\mathbb{R}^N} v_n^2\,dx + C_\varepsilon \int_{\mathbb{R}^N} |v_n|^{\frac{p+1}{2}}\,dx\right] \\ &\le \varepsilon \lim_{n\to\infty} \int_{\mathbb{R}^N} v_n^2\,dx,\end{aligned}$$

because, if $\{v_n\}$ vanish, $v_n \to 0$ strongly in $L^q(\mathbb{R}^N)$ for any $q \in]2, \frac{2N}{N-2}[$ (a proof of this result is given in Lemma 2.18 of [5] and is a special case of Lemma I.1 of [11]). We then deduce that,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} h(f(v_n))\,\frac{v_n}{\sqrt{1+2f^2(v_n)}}\,dx = 0.$$

This implies, since $I'(v_n)v_n \to 0$, that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 + V(x)f^2(v_n)\,dx \to 0$$

in contradiction with the fact that $I(v_n) \to c > 0$. Thus $\{v_n\}$ does not vanish and there exists $\alpha > 0$, $R < \infty$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n\to\infty} \int_{y_n + B_R} v_n^2\,dx \ge \alpha > 0.$$

Let $\tilde{v_n}(x) = v_n(x + y_n)$. Since $\{v_n\}$ is a Palais-Smale sequence for $\tilde{I}$, $\{\tilde{v}_n\}$ also. Arguing as in the case of $\{v_n\}$ we get that $\tilde{v}_n \rightharpoonup \tilde{v}$, up to a subsequence, with $\tilde{I}'(\tilde{v}) = 0$. Since $\{\tilde{v}_n\}$ is non-vanishing we also have that $\tilde{v} \neq 0$.

Now observe that, because of Lemma 2.2, for all $x \in \mathbb{R}^N$, $n \in \mathbb{N}$,

$$f^2(\tilde{v_n}) - \frac{f(\tilde{v_n})\tilde{v_n}}{\sqrt{1+2f^2(\tilde{v_n})}} \ge 0,$$

also, because of condition (h2), for all $x \in \mathbb{R}^N$, $n \in \mathbb{N}$,

$$\frac{1}{2}\,\frac{h(f(\tilde{v_n}))\tilde{v_n}}{\sqrt{1+2f^2(\tilde{v_n})}} - H(f(\tilde{v_n})) \ge 0.$$

Indeed, for all $x \in \mathbb{R}^N$, $n \in \mathbb{N}$,

$$\frac{1}{2}\,\frac{h(f(\tilde{v_n}))\tilde{v_n}}{\sqrt{1+2f^2(\tilde{v_n})}} \;\ge\; \frac{1}{2}\,h(f(\tilde{v_n}))f(\tilde{v_n}) \;\ge\; \frac{\mu}{4}\,H(f(\tilde{v_n})) \;\ge\; H(f(\tilde{v_n})).$$

Thus, from Fatou's lemma, we get

$$
\begin{aligned}
c &= \limsup_{n\to\infty} \left[\tilde{I}(\tilde{v}_n) - \frac{1}{2}\tilde{I}'(\tilde{v}_n)\tilde{v}_n \right] \\
&= \limsup_{n\to\infty} \frac{1}{2}\int_{\mathbb{R}^N} \left[f^2(\tilde{v_n}) - \frac{f(\tilde{v_n})\tilde{v_n}}{\sqrt{1+2f^2(\tilde{v_n})}} \right] V(\infty)\, dx \\
&+ \limsup_{n\to\infty} \int_{\mathbb{R}^N} \left[\frac{1}{2}\frac{h(f(\tilde{v_n}))\tilde{v_n}}{\sqrt{1+2f^2(\tilde{v_n})}} - H(f(\tilde{v_n})) \right] dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^N} \left[f^2(\tilde{v}) - \frac{f(\tilde{v})\tilde{v}}{\sqrt{1+2f^2(\tilde{v})}} \right] V(\infty)\, dx + \int_{\mathbb{R}^N} \left[\frac{1}{2}\frac{h(f(\tilde{v}))\tilde{v}}{\sqrt{1+2f^2(\tilde{v})}} - H(f(\tilde{v})) \right] dx \\
&= \tilde{I}(\tilde{v}) - \frac{1}{2}\tilde{I}'(\tilde{v})\tilde{v} = \tilde{I}(\tilde{v}).
\end{aligned}
$$

Namely $\tilde{v} \neq 0$ is a critical point of $\tilde{I}$ satisfying $\tilde{I}(\tilde{v}) \leq c$. We deduce that the least energy level $\tilde{m}$ for $\tilde{I}$ satisfies $\tilde{m} \leq c$. We denote by $\tilde{\omega}$ a least energy solution as provided by Theorem 3.1. Now applying Theorem 3.4 to the functional $\tilde{I}$ we can find a path $\gamma(t) \in C([0,1], H)$ such that $\gamma(t)(x) > 0$, $\forall\, x \in \mathbb{R}^N$, $\forall\, t \in (0,1]$, $\gamma(0) = 0$, $\tilde{I}(\gamma(1)) < 0$, $\tilde{\omega} \in \gamma([0,1])$ and

$$\max_{t\in[0,1]} \tilde{I}(\gamma(t)) = \tilde{I}(\tilde{\omega}).$$

Without restriction we can assume that $V(x) \leq V(\infty)$ but $V \not\equiv V(\infty)$ in (V1) (otherwise there is nothing to prove). Thus

$$I(\gamma(t)) < \tilde{I}(\gamma(t)) \quad \text{for all } t \in (0,1]$$

and it follows that

$$c \leq \max_{t\in[0,1]} I(\gamma(t)) < \max_{t\in[0,1]} \tilde{I}(\gamma(t)) \leq c.$$

This is a contradiction and the lemma is proved. ♠

At this point to end the proof of Theorem 1.3 we just need to show that there exists a Palais-Smale sequence for I as in Lemma 4.3. From Lemma 4.2 we know (see [6]) that I possesses a Cerami sequence at the level $c > 0$. Namely a sequence $\{v_n\} \subset H$ such that

$$I(v_n) \to c \ \text{ and } \ ||I'(v_n)||_{H^{-1}}(1 + ||v_n||) \to 0 \quad \text{as } n \to \infty.$$

Lemma 4.4 *Assume that (V0)-(V1) and (h0)-(h2) hold. Then all Cerami sequences for I at the level $c > 0$ are bounded in H.*

Proof. First we observe that if a sequence $\{v_n\} \subset H$ satisfies

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx + \int_{\mathbb{R}^N} V(x) f^2(v_n) \, dx \qquad \text{is bounded} \tag{4.6}$$

then it is bounded in H. To see this we just need to show that $\int_{\mathbb{R}^N} v_n^2 dx$ is bounded. We write

$$\int_{\mathbb{R}^N} v_n^2 \, dx = \int_{\{x \,:\, |v_n(x)| \leq 1\}} v_n^2 \, dx + \int_{\{x \,:\, |v_n(x)| > 1\}} v_n^2 \, dx.$$

By Remark 4.1, there exists a $C > 0$ such that $H(s) \geq Cs^4$ for all $s \geq 1$ and thus, because of the behavior of f at infinity, for a $C > 0$, $H(f(s)) \geq Cs^2$, for all $s \geq 1$. It follows that

$$\int_{\{x \,:\, |v_n(x)| > 1\}} v_n^2 \, dx \leq \frac{1}{C} \int_{\{x \,:\, |v_n(x)| > 1\}} H(f(v_n)) \, dx \leq \frac{1}{C} \int_{\mathbb{R}^N} H(f(v_n)) \, dx.$$

Also, for a $C > 0$, since $f(s) \geq Cs$ for all $s \in [0, 1]$, (see Lemma 2.1) we also have

$$\int_{\{x \,:\, |v_n(x)| \leq 1\}} v_n^2 \, dx \leq \frac{1}{C} \int_{\{x \,:\, |v_n(x)| \leq 1\}} f^2(v_n) \, dx \leq \frac{1}{C} \int_{\mathbb{R}^N} f^2(v_n) \, dx.$$

At this point the boundedness of $\{v_n\} \subset H$ is clear.

Now let $\{v_n\} \subset H$ be an arbitrary Cerami sequence for I at the level $c > 0$. We have for any $\phi \in H$

$$\frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(v_n) \, dx - \int_{\mathbb{R}^N} H(f(v_n)) \, dx = c + o(1), \tag{4.7}$$

$$\begin{aligned} I'(v_n)\phi &= \int_{\mathbb{R}^N} \nabla v_n \nabla \phi \, dx + \int_{\mathbb{R}^N} V(x) \frac{f(v_n)\phi}{\sqrt{1 + 2f^2(v_n)}} \, dx \\ &- \int_{\mathbb{R}^N} \frac{h(f(v_n))\phi}{\sqrt{1 + 2f^2(v_n)}} \, dx. \end{aligned} \tag{4.8}$$

Choosing $\phi = \phi_n = \sqrt{1 + 2f^2(v_n)} f(v_n)$ we have, from Lemma 2.1, $||\phi_n||_2 \leq C||v_n||_2$ and

$$|\nabla \phi_n| = (1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)}) |\nabla v_n| \leq 2 |\nabla v_n|.$$

Thus $||\phi_n|| \leq C||v_n||$ and, in particular, recording that $\{v_n\} \subset H$ is a Cerami sequence

$$\begin{aligned} I'(v_n)\phi_n &= \int_{\mathbb{R}^N} (1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)}) |\nabla v_n|^2 \, dx + \int_{\mathbb{R}^N} V(x) f^2(v_n) \, dx \\ &- \int_{\mathbb{R}^N} h(f(v_n)) f(v_n) \, dx = o(1). \end{aligned} \tag{4.9}$$

Now using (h2) it follows computing (4.7) $-\frac{1}{\mu}$(4.9) that

$$\begin{aligned}\int_{\mathbb{R}^N}(\frac{1}{2}-\frac{1}{\mu}(1+\frac{2f^2(v_n)}{1+2f^2(v_n)}))|\nabla v_n|^2\,dx \quad &+ \quad \frac{1}{4}\int_{\mathbb{R}^N}V(x)f^2(v_n)\,dx \\ &\leq \quad c+o(1). \end{aligned} \tag{4.10}$$

Since $1+\dfrac{2f^2(v_n)}{1+2f^2(v_n)} \leq 2$, if $\mu > 4$ we immediately deduce that (4.6) hold and thus $\{v_n\} \subset H$ is bounded. If $\mu = 4$ we obtain from (4.10)

$$\frac{1}{4}\int_{\mathbb{R}^N}\frac{|\nabla v_n|^2}{1+2f^2(v_n)}\,dx + \frac{1}{4}\int_{\mathbb{R}^N}V(x)f^2(v_n)\,dx \leq c+o(1). \tag{4.11}$$

Denoting $u_n = f(v_n)$, we have $|\nabla v_n|^2 = (1+2f^2(v_n))|\nabla u_n|^2$ and (4.7), (4.11) give

$$\int_{\mathbb{R}^N}(1+2u_n^2)|\nabla u_n|^2\,dx + \int_{\mathbb{R}^N}V(x)u_n^2\,dx - 2\int_{\mathbb{R}^N}H(u_n)\,dx = 2c+o(1). \tag{4.12}$$

$$\frac{1}{4}\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^N}V(x)u_n^2\,dx \leq c+o(1). \tag{4.13}$$

From (4.13) we see that $\{u_n\} \subset H$ is bounded. Thus since, by (h0)-(h1),

$$H(s) \leq |s|^2 + C|s|^{p+1} \tag{4.14}$$

we see, from the Sobolev embedding, that if $p \leq \frac{N+2}{N-2}$ then $\int_{\mathbb{R}^N} H(u_n)\,dx$ is bounded and from (4.12) we get (4.6). When $N = 3$ the condition corresponds to $p \leq 5$. In the case where we assume $p < \frac{3N+4}{N}$ let us show that

$$\int_{\mathbb{R}^N}H(u_n)\,dx = o\left(\int_{\mathbb{R}^N}u_n^2|\nabla u_n|^2\,dx\right) \text{ if } \int_{\mathbb{R}^N}u_n^2|\nabla u_n|^2\,dx \to \infty. \tag{4.15}$$

Using Holder inequality, we have for $\theta = \frac{(N-2)(p-1)}{2N+4}$

$$\int_{\mathbb{R}^N}|u_n|^{p+1}\,dx \leq C\left(\int_{\mathbb{R}^N}|u_n|^2\,dx\right)^{1-\theta}\left(\int_{\mathbb{R}^N}|u_n|^{\frac{4N}{N-2}}\,dx\right)^{\theta}.$$

Also

$$\begin{aligned}\left(\int_{\mathbb{R}^N}|u_n^2|^{\frac{2N}{N-2}}\,dx\right)^{\theta} \quad &\leq \quad C\left(\int_{\mathbb{R}^N}|\nabla(u_n^2)|^2\,dx\right)^{\frac{\theta N}{N-2}} \\ &= \quad C\left(\int_{\mathbb{R}^N}u_n^2|\nabla u_n|^2\,dx\right)^{\frac{\theta N}{N-2}}\end{aligned}$$

where $\frac{\theta N}{N-2} < 1$ since $p < \frac{3N+4}{N}$. Recalling (4.14) and the boundedness of $\{u_n\}$ in $L^2(\mathbb{R}^N)$ this proves (4.15). Thus from (4.12) we see that $\int_{R^N} H(u_n)\,dx$ is bounded and thus (4.6) hold. At this point the lemma is proved. ♠

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