A JOINT FUNCTIONAL CALCULUS FOR SECTORIAL OPERATORS
WITH COMMUTING RESOLVENTS

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1. Introduction

In this paper we are concerned with H^∞ functional calculus in the sense of the construction introduced by McIntosh [23] and developed recently by Cowling-Doust-McIntosh-Yagi [7].

Given θ in $(0, \pi]$, we will use the notation $\Sigma_\theta = \{z \in \mathbb{C} \setminus \{0\} \ : \ |\arg z| < \theta\}$ to denote the open sector of angle 2θ around the half-line $\mathbb{R}_+$. Let X be a complex Banach space. Given a linear operator A on X we denote by $D(A)$, $N(A)$ and $R(A)$ the domain, the kernel and the range of A respectively. We also denote by $\sigma(A)$ the spectrum of A and by $\rho(A)$ the resolvent set of A.

__Definition 1.1 :__ Let $\omega \in (0, \pi)$ and let A be a linear operator on X. We say that A is *pseudosectorial* of type ω if A is closed, $D(A)$ is dense in X, the spectrum $\sigma(A)$ is included in $\overline{\Sigma_\omega}$ and:

$$\forall \, \theta \in (\omega, \pi), \ \exists \, C > 0 \ \text{ such that } \ \forall \, z \in \mathbb{C} \setminus \overline{\Sigma_\theta}, \ \|z(A - z)^{-1}\| \leq C \ . \tag{1.1}$$

If moreover $N(A) = \{0\}$ and $R(A)$ is dense, we simply say that A is *sectorial* of type ω.

The purpose of this paper is the study of a joint H^∞ functional calculus for a pair of sectorial operators A, B on X with commuting resolvents, i.e. which satisfy:

$$\forall \lambda \in \rho(A), \ \forall \mu \in \rho(B), \quad (\lambda - A)^{-1}(\mu - B)^{-1} = (\mu - B)^{-1}(\lambda - A)^{-1} \ . \tag{1.2}$$

This joint functional calculus, which was first introduced by Albrecht [1], is a natural two variables analogue of McIntosh's H^∞ functional calculus. Given two sectorial operators A and B which are of type ω and ω' respectively and satisfy (1.2), and two numbers $\mu \in (\omega, \pi)$, $\mu' \in (\omega', \pi)$, it consists in defining a closed and densely defined operator $F(A, B)$ for any bounded analytic function $F \colon \Sigma_\mu \times \Sigma_{\mu'} \to \mathbb{C}$, in a way that preserves reasonnable algebraic and continuity properties. This leads to the notion of bounded H^∞ joint functional calculus for a pair of commuting sectorial operators. See Section 2 below for precise definitions and basic properties.

The first problem addressed in this paper is the following: what are the Banach spaces X for which (A, B) admits a bounded H^∞ joint functional calculus as soon as A and B each admit a bounded H^∞ functional calculus ? In [1], Albrecht proves that this is the case if X is an L^p-space, with $1 < p < +\infty$. We extend this result to a class of Banach spaces containing for example every Banach space having a local unconditional structure and every quotient of subspaces of a B-convex Banach lattice.

Later on in the paper we apply these results to the study of the sum and the product of two commuting sectorial operators A and B. As usual the operators $A + B$ and AB are here understood with their natural domains:

$$D(A + B) = D(A) \cap D(B) \tag{1.3}$$

1

$$D(AB) = \{x \in D(B) \ : \ Bx \in D(A)\} \, . \tag{1.4}$$

We recall that the assumption (1.2) ensures that $D(A+B)$ and $D(AB)$ are dense and that $A+B$ and AB are closable operators (see e.g. [30]).

Assume now that the sum of the types of A and B is strictly less than π. Then a classical problem, which goes back at least to [9], is to determine conditions under which $A+B$ is closed. This problem turns out to be strongly related to joint functional calculus. Indeed the closedness of $A+B$ is implied by the boundedness of $A(A+B)^{-1}$, which is equivalent to that of $F(A,B)$, where $F(z,z') = z(z+z')^{-1}$ (see Proposition 2.7 below). A celebrated theorem of Dore-Venni [11] (completed by Prüss-Sohr [30]) asserts that $A+B$ is closed under the following conditions: (i) The Banach space X has the UMD property (i.e. the Hilbert transform is bounded on $L^2(\mathbb{R};X)$). (ii) A and B each admit bounded imaginary powers and there exist four positive constants K, K', μ, μ', such that $\mu + \mu' < \pi$ and $\|A^{is}\| \leq Ke^{\mu|s|}$, $\|B^{is}\| \leq K'e^{\mu'|s|}$ for all s in $\mathbb{R}$. Furthermore it is proved in [12, 30] that under these conditions, $A+B$ and $\overline{AB}$ actually admit bounded imaginary powers. We will show in section 4 the following H^∞ functional calculus analogue of the Dore-Venni-Prüss-Sohr theorem. Note however that the geometric assumption on X in Theorem 1.2 is different from the UMD property.

Theorem 1.2 : Let X be a Banach space. Assume that X is either a Banach lattice or the quotient of two subspaces of a B-convex Banach lattice (for instance a quotient of subspaces of L^p, $1 < p < +\infty$). Let A, B be two sectorial operators on X, with commuting resolvents. Assume that A admits a bounded $H^\infty(\Sigma_\mu)$ functional calculus, B admits a bounded $H^\infty(\Sigma_{\mu'})$ functional calculus, and $\mu + \mu' < \pi$.

Then $A+B$ is a sectorial operator (in particular $A+B$ is closed) and:

(i) For any $\nu > \mathrm{Max}\{\mu, \mu'\}$, $A+B$ admits a bounded $H^\infty(\Sigma_\nu)$ functional calculus.

(ii) For any $\nu > \mu + \mu'$, $\overline{AB}$ admits a bounded $H^\infty(\Sigma_\nu)$ functional calculus.

In the last two sections of this paper, we study another variant of McIntosh's functional calculus which also leads to the study of the boundedness of $A(A+B)^{-1}$ (for two commuting operators A and B) and discuss some connections with the study of maximal regularity (in the L^p-sense) for the Cauchy problem.

Let A be a sectorial operator on a B-convex Banach lattice Λ (for example Λ is an L^p-space with $1 < p < +\infty$) and let $\mathcal{A}$ be its natural extension to $\Lambda(H)$, where H is a Hilbert space. Assume that $\mathcal{A}$ (or equivalently A) admits a bounded $H^\infty(\Sigma_\mu)$ functional calculus. The main result in section 5 says that for any $\nu > \mu$, this functional calculus extends in a natural way to a bounded Banach algebra homomorphism from $H^\infty(\Sigma_\nu; B(H))$ into $B(\Lambda(H))$.

Let us now recall the definition of maximal regularity, to which we apply this result. Note that with the terminology introduced in Definition 1.1, a linear operator B on a Banach space X is pseudosectorial of type strictly less than $\dfrac{\pi}{2}$ if and only if $-B$ is the infinitesimal generator of a bounded holomorphic semigroup on X, see e.g. [16].

Definition 1.3 : Let $T > 0$ and $1 < p < +\infty$ be two numbers. Let X be a Banach space and let B be a pseudosectorial operator of type strictly less than $\dfrac{\pi}{2}$ on X. Then B is said to have the *maximal regularity property* if there exists $C > 0$ such that for any f in $L^p(0,T;X)$ there exists a unique function u in $W_0^{1,p}(0,T;X) \cap L^p(0,T;D(B))$ satisfying:

$$u' + Bu = f \ \text{on} \ [0,T] \qquad \text{and} \qquad \|u\| \leq C\|f\|.$$

It is well known that this property does not depend on $1 < p < +\infty$ and $0 < T < +\infty$. Moreover if X is a Hilbert space, then any B as above has the maximal regularity property. This is due to De Simon [10]. It is an open problem to know whether any pseudosectorial operator of type $< \dfrac{\pi}{2}$ has the maximal regularity property on L^p for $1 < p \neq 2 < +\infty$. However a great progress was made by Dore and Venni [11] (see also [30]) by means of their theorem quoted above. Indeed they proved that if X is a UMD Banach space (in particular if $X = L^p$, $1 < p < +\infty$), and if B admits bounded imaginary powers which satisfy $\|B^{is}\| \leq K'e^{\mu'|s|}$ for some constants $K' > 0$ and $0 < \mu' < \dfrac{\pi}{2}$, then B has the maximal regularity property.

In the following theorem, which will be proved in Section 5, we obtain a sufficient condition for maximal regularity which combines the results of De Simon and Dore-Venni, at least to some extent.

Theorem 1.4 : Let (Ω_1, m_1) be a measure space and let $1 < q < +\infty$. Let H be a Hilbert space and B an operator on $L^q(\Omega_1, m_1)$ with a bounded $H^\infty(\Sigma_\mu)$ functional calculus for some $\mu < \dfrac{\pi}{2}$. Let C be any pseudosectorial operator of type $< \dfrac{\pi}{2}$ on H. Let $\widetilde{B}$ and $\widetilde{C}$ be the closures, which exist, of $B \otimes I_H$ and $I_{L^q} \otimes C$ respectively.

   Then $\widetilde{B} + \widetilde{C}$ is a sectorial operator of type $< \dfrac{\pi}{2}$ with the maximal regularity property.

This result can be interpreted as follows. Let $(T_t)_{t \geq 0}$ be a bounded holomorphic semigroup on $X = L^q(\Omega_1, m_1; H)$. Assume that $(T_t)_{t \geq 0}$ is obtained as a tensor product, i.e. for any $t \geq 0$, T_t is the closure of $U_t \otimes V_t$ where $(U_t)_{t \geq 0}$ and $(V_t)_{t \geq 0}$ are two bounded holomorphic semigroups defined on $L^q(\Omega_1, m_1)$ and H respectively. Then to obtain the maximal regularity property for the negative generator of $(T_t)_{t \geq 0}$, it suffices to know that, for some $\mu < \dfrac{\pi}{2}$, the negative generator of $(U_t)_{t \geq 0}$ has a bounded $H^\infty(\Sigma_\mu)$ functional calculus on $L^q(\Omega_1, m_1)$.

2. Preliminaries

First, we introduce the main notations. For θ in $(0, \pi)$, we denote by Γ_θ the oriented contour defined by:

$$\Gamma_\theta(t) = \begin{cases} -te^{i\theta} & \text{if } -\infty < t \leq 0 \\ te^{-i\theta} & \text{if } 0 \leq t < +\infty. \end{cases}$$

We will also often use the function φ defined on $\mathbb{C} \setminus \{-1\}$ by $\varphi(z) = \dfrac{z}{(1+z)^2}$.

In order to study the boundedness of special operators such as $A(A+B)^{-1}$, it will be useful to use a variant of McIntosh's one variable H^∞ functional calculus. To define it, the idea is to replace scalar holomorphic functions by operator valued holomorphic functions.

Given $\mu \in (0, \pi)$ and a Banach space E, we will denote by $H^\infty(\Sigma_\mu; E)$ the Banach space of bounded holomorphic functions $f : \Sigma_\mu \to E$, equipped with the norm:

$$\|f\| = \text{Sup}\{\|f(z)\|_E \ : \ z \in \Sigma_\mu \}.$$

Let A be a sectorial operator of type ω $(0 < \omega < \pi)$ on a complex Banach space X. The commutant of A, denoted E_A, is defined to be the closed subalgebra of $B(X)$:

$$E_A = \{T \in B(X) \ : \ T(\lambda - A)^{-1} = (\lambda - A)^{-1}T, \ \lambda \in \rho(A) \}.$$

It is easy to check that a bounded operator T on X belongs to E_A as soon as it commutes with $(\lambda - A)^{-1}$ for one element λ in $\rho(A)$.

Let $\mu > \omega$ and let E be a closed subalgebra of E_A. We consider now $H_0^\infty(\Sigma_\mu; E)$, the space of all f in $H^\infty(\Sigma_\mu; E)$ for which there exists $s > 0$ such that $\varphi^{-s}f$ belongs to $H^\infty(\Sigma_\mu; E)$. Let θ in (ω, μ). Then for any f in $H_0^\infty(\Sigma_\mu; E)$, we can define:

$$u_A(f) = \frac{1}{2\pi i} \int_{\Gamma_\theta} f(\lambda)(\lambda - A)^{-1} d\lambda \tag{2.1}$$

which is an integral converging in $B(X)$ and whose value does not depend on the choice of θ.

Now observe that the operator $u_A(\varphi) = \varphi(A) = A(I + A)^{-2}$ is one to one with range $D(A) \cap R(A)$, which is, by sectoriality of A, a dense subspace of X (see e.g. [30]). Let f be any function in $H^\infty(\Sigma_\mu; E)$. The product function $f\varphi$ belongs to $H_0^\infty(\Sigma_\mu; E)$ hence it makes sense to define:

$$u_A(f) = \varphi(A)^{-1} u_A(f\varphi) \tag{2.2}$$

with domain given, via (1.4), by:

$$D\big(u_A(f)\big) = \{x \in X \;:\; u_A(f\varphi)x \in D(A) \cap R(A)\,\}.$$

From the inclusion $E \subset E_A$ it follows easily that u_A is an algebra homomorphism from $H_0^\infty(\Sigma_\mu; E)$ into $B(X)$. This implies that for any f in $H^\infty(\Sigma_\mu; E)$, $D(u_A(f))$ contains $R(\varphi(A)) = D(A) \cap R(A)$ and is therefore dense. Since $\varphi(A)$ is bounded, $u_A(f)$ is a closed operator. Then by the closed graph theorem, $u_A(f)$ is bounded if and only if $D\big(u_A(f)\big) = X$.

We say that A admits a bounded $H^\infty(\Sigma_\mu; E)$ functional calculus if $u_A(f)$ is bounded for any $f \in H^\infty(\Sigma_\mu; E)$. These definitions coincide with those in [7, 23] in the case when $E = \mathrm{Span}\{I_X\}$. As in this case, the fundamental proposition is:

Proposition 2.1 : The following assertions are equivalent :

(i) A admits a bounded $H^\infty(\Sigma_\mu; E)$ functional calculus.

(ii) $\exists\, C > 0$ such that $\forall\, f \in H_0^\infty(\Sigma_\mu; E)$, $\|u_A(f)\| \le C\|f\|_{H^\infty(\Sigma_\mu; E)}$.

(iii) $\exists\, C > 0$ such that $\forall\, f \in H^\infty(\Sigma_\mu; E)$, $u_A(f) \in B(X)$ and $\|u_A(f)\| \le C\|f\|_{H^\infty(\Sigma_\mu; E)}$.

In this case, u_A is a (bounded) homomorphism on $H^\infty(\Sigma_\mu; E)$.

Of course when $E = \mathrm{Span}\{I_X\}$, we merely say $H^\infty(\Sigma_\mu)$ functional calculus instead of $H^\infty(\Sigma_\mu; E)$ functional calculus.

If A is pseudosectorial and if (ii) is satisfied, we will say that A admits a bounded $H_0^\infty(\Sigma_\mu; E)$ functional calculus.

We now turn to the definition of the joint H^∞ functional calculus. The following construction was first introduced by Albrecht [1] . It is entirely parallel to McIntosh's one variable construction.

Given $\mu, \mu' \in (0, \pi)$, we denote by $H^\infty(\Sigma_\mu \times \Sigma_{\mu'})$ the Banach algebra of all bounded holomorphic scalar valued functions on $\Sigma_\mu \times \Sigma_{\mu'}$ and we let Φ be defined by:

$$\Phi(z, z') = \varphi(z)\varphi(z') = \frac{zz'}{(1 + z)^2(1 + z')^2} \;.$$

Then we define:

$$H_0^\infty(\Sigma_\mu \times \Sigma_{\mu'}) = \{F \in H^\infty(\Sigma_\mu \times \Sigma_{\mu'}) \;:\; \exists\, s > 0,\; \Phi^{-s}F \in H^\infty(\Sigma_\mu \times \Sigma_{\mu'})\}.$$

Let A and B be two sectorial operators on a Banach space X, respectively of types ω and ω', with commuting resolvents. For $\mu > \omega$, $\mu' > \omega'$, and $F \in H_0^\infty(\Sigma_\mu \times \Sigma_{\mu'})$ one defines:

$$F(A, B) = -\frac{1}{4\pi^2} \int_{\Gamma_\theta \times \Gamma_{\theta'}} F(\lambda, \lambda')(\lambda - A)^{-1}(\lambda' - B)^{-1} d\lambda\; d\lambda' \tag{2.3}$$

with $(\theta, \theta') \in (\omega, \mu) \times (\omega', \mu')$. This integral converges in $B(X)$ and does not depend on the choice of (θ, θ'). To define $F(A, B)$ for general F we proceed as in the one-variable case. Namely, for any F in $H^\infty(\Sigma_\mu \times \Sigma_{\mu'})$, the function $F\Phi$ belongs to $H_0^\infty(\Sigma_\mu \times \Sigma_{\mu'})$, hence we may set:

$$F(A, B) = \Phi(A, B)^{-1}(F\Phi)(A, B). \tag{2.4}$$

4

This makes sense since $\Phi(A, B)$ is one to one.

As in the first situation, the calculus associated with (A, B) on $\Sigma_\mu \times \Sigma_{\mu'}$ by the formula (2.3) defines an algebra homomorphism from $H_0^\infty(\Sigma_\mu \times \Sigma_{\mu'})$ into $B(X)$.

To go further, we need the auxilliary functions φ_n and Φ_n defined for all $n \geq 1$ by:

$$\varphi_n(z) = \frac{n^2 z}{(n+z)(1+nz)} \quad \text{and} \quad \Phi_n(z, z') = \varphi_n(z)\varphi_n(z').$$

Lemma 2.2 : For any $k \geq 1$, we let : $R_k = R\big(\Phi^k(A, B)\big)$. Then:

(i) $\forall \, k \geq 1, \; \forall \, x \in X, \qquad \lim_{n \to \infty} \Phi_n^k(A, B)x = x.$

(ii) $\forall \, k \geq 1, \; \forall \, n \geq 1, \qquad R\big(\Phi_n^k(A, B)\big) = R_k.$

(iii) $\forall \, k \geq 1, \; R_k$ is dense in X.

(iv) $\forall \, k \geq 1, \; \Phi^k(A, B)$ is a bounded, one to one operator with dense range.

Proof : It is well known that when A and B are sectorial, then, for any x in X,
$\lim_{n \to \infty} \varphi_n(A)x = \lim_{n \to \infty} \varphi_n(B)x = x$ (see e.g. [29, Section 8.1]). Moreover the sequences $(\varphi_n(A))_{n \geq 1}$, $(\varphi_n(B))_{n \geq 1}$ ∎
are bounded, whence the result (i). The rest of the lemma is then clear. ◇

Let $F \in H^\infty(\Sigma_\mu \times \Sigma_{\mu'})$ and let $F(A, B)$ be defined by (2.4). It is a closed operator for the domain defined by: $D\big(F(A, B)\big) = \{x \in X \; : \; (F\Phi)(A, B) \in R_1 \}$. By the homomorphism property, $D\big(F(A, B)\big)$ contains R_1 hence it is dense in X by Lemma 2.2. Consequently $F(A, B)$ is bounded if and only if $D\big(F(A, B)\big) = X$. Moreover the definitions (2.3) and (2.4) are consistent for F in $H_0^\infty(\Sigma_\mu \times \Sigma_{\mu'})$.

We will now prove, for completeness, a useful convergence lemma, which is a variant of the usual convergence lemma due to McIntosh [23] :

Lemma 2.3 : Let $(F_n)_{n \geq 1}$ a bounded sequence in $H_0^\infty(\Sigma_\mu \times \Sigma_{\mu'})$ and F in $H^\infty(\Sigma_\mu \times \Sigma_{\mu'})$ such that $(F_n)_{n \geq 1}$ converges pointwise to F on $\Sigma_\mu \times \Sigma_{\mu'}$. Then:

(i) $\forall \, x \in R_1, \quad \lim_{n \to \infty} \|F_n(A, B)x - F(A, B)x\| = 0.$

(ii) If moreover $\sup_{n \geq 1} \|F_n(A, B)\|$ is finite, then $F(A, B)$ is bounded.

Proof : Let us denote $G = F\Phi$ and $G_n = F_n\Phi$. For x in R_1, there is $y \in X$ so that $x = \Phi(A, B)y$. So
$\|F_n(A, B)x - F(A, B)x\| = \|G_n(A, B)y - G(A, B)y\| =$

$$\left\| \frac{1}{4\pi^2} \int_{\Gamma_\theta \times \Gamma_{\theta'}} (G_n(\lambda, \lambda') - G(\lambda, \lambda'))(\lambda - A)^{-1}(\lambda' - B)^{-1} \, y \, d\lambda d\lambda' \right\|$$

for some $(\theta, \theta') \in (\omega, \mu) \times (\omega', \mu')$. Now, (i) follows from Lebesgue's dominated convergence theorem and then (ii) is clear. ◇

We say that (A, B) admits a bounded $H^\infty(\Sigma_\mu \times \Sigma_{\mu'})$ joint functional calculus if $F(A, B)$ is a bounded operator for any F in $H^\infty(\Sigma_\mu \times \Sigma_{\mu'})$. By combining Lemma 2.3 and the closed graph theorem as in [23], one can transfer Proposition 2.1 to the two variables setting:

Proposition 2.4 : The following assertions are equivalent:

(i) (A, B) admits a bounded $H^\infty(\Sigma_\mu \times \Sigma_{\mu'})$ joint functional calculus.

(ii) $\exists \, C > 0$ such that $\forall \, F \in H_0^\infty(\Sigma_\mu \times \Sigma_{\mu'}), \; \|F(A, B)\| \leq C\|F\|_{H^\infty(\Sigma_\mu \times \Sigma_{\mu'})}.$

(iii) $\exists \, C > 0$ such that $\forall \, F \in H^\infty(\Sigma_\mu \times \Sigma_{\mu'}), \; F(A, B) \in B(X)$ and
$\|F(A, B)\| \leq C\|F\|_{H^\infty(\Sigma_\mu \times \Sigma_{\mu'})}.$

In this case, the map $F \mapsto F(A, B)$ is a (bounded) homomorphism on $H^\infty(\Sigma_\mu \times \Sigma_{\mu'})$.

5

Next we indicate, how one can in a standard way extend this calculus to an even larger class than $H^\infty(\Sigma_\mu \times \Sigma_{\mu'})$, namely the class of all analytic functions of polynomial growth on $\Sigma_\mu \times \Sigma_{\mu'}$. Let P be an analytic function on $\Sigma_\mu \times \Sigma_{\mu'}$; we say that P is of *polynomial growth* if there is k in $\mathbb{N}$ such that $P\Phi^k \in H_0^\infty(\Sigma_\mu \times \Sigma_{\mu'})$. For such a function P, we define $P(A, B) = \Phi(A, B)^{-k}(P\Phi^k)(A, B)$. By the homomorphism property, this definition does not depend on k such that $P\Phi^k \in H_0^\infty(\Sigma_\mu \times \Sigma_{\mu'})$ and by Lemma 2.2 we immediately obtain the following:

Proposition 2.5 : Let A and B two sectorial operators on X with respective types ω and ω' and with commuting resolvents. Let (μ, μ') in $(\omega, \pi) \times (\omega', \pi)$ and P an analytic function of polynomial growth on $\Sigma_\mu \times \Sigma_{\mu'}$. Then $P(A, B)$ is a closed operator whose domain is dense in X.

We will now state two propositions, that will be very useful for our applications to maximal regularity problems.

Proposition 2.6 : Let A be a sectorial operator of type ω and B a pseudosectorial operator of type ω', whose resolvents commute. Assume $\omega + \omega' < \pi$.

Then $R(A + B)$ is dense in X and $\overline{A + B}$ is injective.

Let now $\omega < \mu < \pi - \omega'$. If we define f in $H^\infty(\Sigma_\mu, E_A)$ by $f(z) = z(z + B)^{-1}$, then the following assertions are equivalent:

(i) $u_A(f) \in B(X)$.

(ii) $A + B$ is a closed operator and:

$$\exists\, C > 0 \text{ such that } \forall\, x \in D(A) \cap D(B),\ \|Ax\| \le C\|Ax + Bx\|. \tag{2.5}$$

Proof : Let $\omega < \mu < \pi - \omega'$. Consider σ and τ in $H_0^\infty(\Sigma_\mu, E_A)$ defined by:

$$\sigma(z) = (z + B)(1 + B)^{-1}\varphi^2(z) \quad \text{and} \quad \tau(z) = (z + B)^{-1}\varphi^2(z)\,.$$

The homomorphism property yields: $(1+B)^{-1}\varphi^4(A) = u_A(\sigma)u_A(\tau)$. Since $u_A(\sigma) = (A+B)(1+B)^{-1}\varphi^2(A)$, we deduce: $R((1+B)^{-1}\varphi^4(A)) \subset R(A+B)$. Consider now $x = (1+B)^{-1}y$ in $D(B)$. Since $(1+B)^{-1}\varphi_n^4(A)y \to x$, we see that $R((1 + B)^{-1}\varphi^4(A))$ is dense in $D(B)$ and therefore dense in X. Consequently $R(A + B)$ is dense in X.

Let now x in $N(\overline{A + B})$. There exists a sequence (x_n) in $D(A) \cap D(B)$ so that $x_n \to x$ and $(A + B)x_n \to 0$. Passing to the limit in the equality $(1 + B)^{-1}\varphi^4(A)x_n = u_A(\sigma)u_A(\tau)x_n$, one gets $(1 + B)^{-1}\varphi^4(A)x = 0$, and therefore $x = 0$. This proves the first part of the proposition.

Turning to the second part, we will now show:

$$\forall\, x \in D(A) \cap D(B), \quad u_A(f\varphi)\ (A + B)x = \varphi(A)Ax. \tag{2.6}$$

So let $x \in D(A) \cap D(B)$; then given θ in (ω, μ), we have:

$u_A(f\varphi^2)\ (A + B)x =$

$$\frac{1}{2\pi i}\int_{\Gamma_\theta} \varphi^2(\lambda)\lambda(\lambda + B)^{-1}(\lambda - A)^{-1}(A + B)x\,d\lambda \quad \text{by (2.1)}$$

$$= \frac{1}{2\pi i}\int_{\Gamma_\theta} \lambda\varphi^2(\lambda)\ (\lambda - A)^{-1}x\,d\lambda - \frac{1}{2\pi i}\int_{\Gamma_\theta} \lambda\,\varphi^2(\lambda)(\lambda + B)^{-1}x\,d\lambda$$

$$= A\varphi^2(A)x \quad \text{by Cauchy's theorem.}$$

Since $x \in D(A)$, we obtain $\varphi(A)u_A(f\varphi)\ (A + B)x = \varphi^2(A)Ax$. Then (2.6) follows from the injectivity of $\varphi(A)$.

6

Let us show "(i) $\Rightarrow$ (ii)". We assume that $u_A(f)$ is bounded, then it follows again from the injectivity of $\varphi(A)$, that for any $x \in D(A) \cap D(B)$, $u_A(f)(A + B)x = Ax$. This clearly implies (2.5) and the closedness of $A + B$ can then be deduced in a usual way.

Assume now that the assertion (ii) is satisfied. From (2.6) it follows that:

$$\forall \, y \in \varphi(A)(R(A + B)), \; \|u_A(f)y\| \leq C\|y\|.$$

So, we can conclude our proof by showing that $\varphi(A)(R(A + B))$ is dense in X. Since $\overline{R(A + B)} = X$, it is enough to prove that $\varphi(A)(R(A+B))$ is dense in $R(A+B)$. This relies on the following standard arguments: let $x = (A + B)y$ in $R(A + B)$, then $x_n = \varphi_n(A)x \to x$ and $x_n \in \varphi(A)(R(A + B))$. $\diamond$

With essentially the same proof, we obtain:

Proposition 2.7 : Let A and B be two sectorial operators with commuting resolvents and respective types ω and ω'. Assume $\omega + \omega' < \pi$ and let $\omega < \mu < \pi - \omega'$ and $\omega' < \mu' < \pi - \mu$. Define F in $H^\infty(\Sigma_\mu \times \Sigma_{\mu'})$ by $F(z, z') = \dfrac{z}{z + z'}$ for any $(z, z') \in \Sigma_\mu \times \Sigma_{\mu'}$. Then the following assertions are equivalent:

(i) $F(A, B) \in B(X)$.

(ii) $A + B$ is a closed operator and:

$$\exists \, C > 0 \;\; \text{such that} \; \forall \, x \in D(A) \cap D(B), \; \|Ax\| \leq C\|Ax + Bx\|.$$

We will continue this section by fixing some notation and recalling some definitions related to the general geometric properties of the Banach spaces that we will consider in this paper.

Definition 2.8 : Let X be a Banach space and denote by $r_n(t) = \text{sign}(\sin(2^n \pi t))$, for $n \geq 1$, the Rademacher functions on the interval $I = [0, 1]$.

(i) We say that X is of type p, for $1 < p \leq 2$, if there is a constant C such that for all finite subsets $\{x_1, \ldots, x_n\}$ of X, $\left\|\sum_{i=1}^n r_i x_i\right\|_{L^1(I;X)} \leq C\left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p}$.

(ii) We say that X is of cotype q, for $2 \leq q < +\infty$, if there is a constant C such that for all finite subsets $\{x_1, \ldots, x_n\}$ of X, $\left(\sum_{i=1}^n \|x_i\|^q\right)^{1/q} \leq C\left\|\sum_{i=1}^n r_i x_i\right\|_{L^1(I;X)}$.

(iii) We say that a Banach space X is B-convex if it is of type p, for some $p > 1$.

Finally we recall an interpolation theorem of Carleson. For convenience, we will transpose to $\Sigma_{\frac{\pi}{2}}$ the statement that can be found in [15] for the upper half plane.

Theorem 2.9 (Carleson) : Let $(z_j)_{j \geq 1}$ be a sequence in $\Sigma_{\frac{\pi}{2}}$, then the following conditions are equivalent:

(i) There is $\delta > 0$ such that:

$$\forall k \geq 1, \quad \prod_{j \neq k} \left| \frac{z_k - z_j}{z_k + \bar{z}_j} \right| \geq \delta. \tag{2.7}$$

(ii) There exist a sequence $(f_j)_{j \geq 1}$ in $H^\infty(\Sigma_{\frac{\pi}{2}})$ and a constant $M > 0$ such that:

$$\forall j \geq 1 : \; f_j(z_j) = 1, \;\; \forall k \neq j : \; f_j(z_k) = 0 \;\; \text{and} \;\; \forall z \in \Sigma_{\frac{\pi}{2}} : \; \sum_{j \geq 1} |f_j(z)| \leq M. \tag{2.8}$$

Such a sequence $(z_j)_{j \geq 1}$ is called an interpolating sequence in $\Sigma_{\frac{\pi}{2}}$. Indeed, let Y be a Banach space and $(y_j)_{j \geq 1}$ a bounded sequence in Y, then the function f defined on $\Sigma_{\frac{\pi}{2}}$ by $f(z) = \sum_{j \geq 1} f_j(z)y_j$ satisfies the following properties:

$$f \in H^\infty(\Sigma_{\frac{\pi}{2}}; Y), \;\; \|f\|_{H^\infty(\Sigma_{\frac{\pi}{2}}; Y)} \leq M \sup_{j \geq 1} \|y_j\| \;\; \text{and} \;\; \forall j \geq 1 : \; f(z_j) = y_j. \tag{2.9}$$

3. On Banach spaces with the joint calculus property

In this section we describe large classes of Banach spaces having the following property:

Definition 3.1 : We say that a Banach space X has the *joint calculus property* if whenever A and B are sectorial operators on X with commuting resolvents, which are of type ω and ω' respectively and admit a bounded $H^\infty(\Sigma_\mu)$, respectively $H^\infty(\Sigma_{\mu'})$, functional calculus (where $0 < \omega < \mu$ and $0 < \omega' < \mu'$), then (A, B) admits a bounded $H^\infty(\Sigma_\beta \times \Sigma_{\beta'})$ joint functional calculus for any (β, β') in $(\mu, \pi) \times (\mu', \pi)$.

Let us first mention that there exist Banach spaces without the joint calculus property. Namely, it can be deduced from [24] that the Banach space of all compact operators on ℓ_2 fails this property. As we mentioned in our introduction, Albrecht [1] proved that L^p-spaces, with $1 < p < \infty$, have the joint calculus property. Recently, Franks and McIntosh [14] obtained a simpler proof of Albrecht's theorem based on a decomposition result for analytic functions. Since this decomposition will be an important tool in this section, we now recall it precisely.

Franks and McIntosh [14, section 3] proved the following: for any (μ, μ') in $(0, \pi) \times (0, \pi)$ and any (β, β') in $(\mu, \pi) \times (\mu', \pi)$, there are a constant $C > 0$ and sequences $(\psi_i)_{i \geq 0}, (\tilde\psi_i)_{i \geq 0}$ in $H_0^\infty(\Sigma_\mu)$, $(\varphi_j)_{j \geq 0}, (\tilde\varphi_j)_{j \geq 0}$ in $H_0^\infty(\Sigma_{\mu'})$ such that:

$$\forall p > 0, \ \sup_{\Sigma_\mu} \sum_{i \geq 0} |\psi_i|^p, \ \sup_{\Sigma_\mu} \sum_{i \geq 0} |\tilde\psi_i|^p, \ \sup_{\Sigma_{\mu'}} \sum_{j \geq 0} |\varphi_j|^p, \ \text{and} \ \sup_{\Sigma_{\mu'}} \sum_{j \geq 0} |\tilde\varphi_j|^p \ \text{are finite.} \tag{3.1}$$

$$\forall \, h \in H^\infty(\Sigma_\beta \times \Sigma_{\beta'}), \ \exists \, (\alpha_{i,j}) \in \mathbb{C}^{\mathbb{N}^2} \ \text{so that} \ \forall \, \zeta = (\zeta_1, \zeta_2) \in \Sigma_\mu \times \Sigma_{\mu'},$$

$$h(\zeta) = \sum_{i,j \geq 0} \alpha_{i,j} \psi_i(\zeta_1) \tilde\psi_i(\zeta_1) \varphi_j(\zeta_2) \tilde\varphi_j(\zeta_2), \ \text{with} \ \sup_{i,j \geq 0} |\alpha_{i,j}| \leq C \|h\|_{H^\infty(\Sigma_\beta \times \Sigma_{\beta'})}. \tag{3.2}$$

A complete reference on the notions of geometry of Banach spaces that we will use can be found in [20, 21, 26]. However, we will recall a few definitions. Let us start with the following : we say that a Banach space X has a local unconditional structure (l.u.st.) if there is a constant λ such that for any finite dimensional subspace F of X, there is a space E with an unconditional basis (e_n) and operators $A \in B(F, E)$ and $B \in B(E, X)$ such that BA is the canonical embedding of F into X and $\|A\| \, \|B\| K \leq \lambda$, where K is the unconditional basis constant of (e_n). Any Banach lattice has this property and it is even known (see [13]) that a Banach space has l.u.st. if and only if its bidual is isomorphic to a complemented subspace of a Banach lattice. We will also consider the following class of Banach spaces, introduced by Pisier [25]: a Banach space X has property (α) if there exists a constant $C > 0$ such that the inequality

$$\Big\| \sum_{1 \leq i,j \leq n} \alpha_{i,j}(r_i \otimes r_j)x_{i,j} \Big\|_{L^2(I \times I; X)} \leq C \sup_{1 \leq i,j \leq n} |\alpha_{i,j}| \, \Big\| \sum_{1 \leq i,j \leq n} (r_i \otimes r_j)x_{i,j} \Big\|_{L^2(I \times I; X)} \tag{α}$$

is satisfied for every integer n and every choice of $(\alpha_{i,j})$ in $\mathbb{C}^{n^2}$ and $(x_{i,j})$ in X^{n^2} (where (r_n) denotes the sequence of Rademacher functions on $I = [0, 1]$).

The main results of this section are the following:

Theorem 3.2 : Every Banach space with a local unconditional structure has the joint calculus property.

Theorem 3.3 : Let X be a Banach space such that X or X^* has property (α), then X has the joint calculus property.

Before to proceed with the proofs of these theorems we need to introduce a technical intermediate property : we say that a Banach space X satisfies property (A) if there is a constant $C > 0$ such that, for any n in $\mathbb{N}, (\alpha_{i,j})$ in $\mathbb{C}^{n^2}$, $(x_{i,j})$ in X^{n^2} and $(x_{i,j}^*)$ in $(X^*)^{n^2}$:

$$\Big| \sum_{1 \leq i,j \leq n} \alpha_{i,j} \langle x_{i,j}, x_{i,j}^* \rangle \Big| \leq C \sup_{1 \leq i,j \leq n} |\alpha_{i,j}|$$
$$\times \Big\| \sum_{1 \leq i,j \leq n} (r_i \otimes r_j)x_{i,j} \Big\|_{L^2(I \times I; X)} \Big\| \sum_{1 \leq i,j \leq n} (r_i \otimes r_j)x_{i,j}^* \Big\|_{L^2(I \times I; X^*)} \tag{A}$$

Then, the proofs of both Theorems 3.2 and 3.3 rely on:

Proposition 3.4 : If a Banach space X satisfies property (A), then it has the joint calculus property.

Proof : We keep the notations introduced in Definition 3.1 and apply the analytic decomposition given by (3.1) and (3.2).

To any h in $H^\infty(\Sigma_\beta \times \Sigma_{\beta'})$, we associate the partial sums of its decomposition given by (3.2), namely:
$h_n = \sum_{1 \leq i,j \leq n} \alpha_{i,j}(\psi_i \tilde{\psi}_i \otimes \varphi_j \tilde{\varphi}_j)$ for all $n \geq 1$.

By Lemma 2.3, in order to show that (A, B) admits a bounded $H^\infty(\Sigma_\beta \times \Sigma_{\beta'})$ functional calculus, it is enough to find a constant $K > 0$ such that:

$$\forall\, h \in H^\infty(\Sigma_\beta \times \Sigma_{\beta'}),\ \forall\, n \geq 1, \quad \|h_n(A,B)\| \leq K\, \|h\|_{H^\infty(\Sigma_\beta \times \Sigma_{\beta'})}. \tag{3.3}$$

So, let $x \in X$ and $x^* \in X^*$, then:

$$|\langle h_n(A,B)x, x^*\rangle| = \Big| \sum_{1 \leq i,j \leq n} \alpha_{i,j} \langle \psi_i(A)\varphi_j(B)x, \tilde{\psi}_i(A)^* \tilde{\varphi}_j(B)^* x^* \rangle \Big|$$

$$\leq C \|h\|_{H^\infty(\Sigma_\beta \times \Sigma_{\beta'})} \Big\| \sum_{1 \leq i,j \leq n} (r_i \otimes r_j)\psi_i(A)\varphi_j(B)x \Big\|_{L^2(I \times I; X)}$$

$$\times \Big\| \sum_{1 \leq i,j \leq n} (r_i \otimes r_j)\tilde{\psi}_i(A)^* \tilde{\varphi}_j(B)^* x^* \Big\|_{L^2(I \times I; X^*)}$$

by (3.2) and property (A) for X. Now, fix $(s,t) \in I \times I$. Using the fact that A and B admit a bounded H^∞ functional calculus, respectively on Σ_μ and $\Sigma_{\mu'}$, we have:

$$\Big\| \sum_{i \leq i,j \leq n} r_i(s)r_j(t)\psi_i(A)\varphi_j(B)x \Big\|_X \leq$$

$$\Big\| \sum_{1 \leq i \leq n} r_i(s)\psi_i(A) \Big\|_{B(X)} \Big\| \sum_{1 \leq j \leq n} r_j(t)\varphi_j(B) \Big\|_{B(X)} \|x\| \leq$$

$$K_1 \Big\| \sum_{1 \leq i \leq n} r_i(s)\psi_i \Big\|_{H^\infty(\Sigma_\mu)} \Big\| \sum_{1 \leq j \leq n} r_j(t)\varphi_j \Big\|_{H^\infty(\Sigma_{\mu'})} \|x\| \leq K_2 \|x\| \quad \text{by (3.1).}$$

Therefore: $\qquad \Big\| \sum_{1 \leq i,j \leq n} (r_i \otimes r_j)\psi_i(A)\varphi_j(B)x \Big\|_{L^2(I \times I; X)} \leq K_2 \|x\|.$

Similarly: $\qquad \Big\| \sum_{1 \leq i,j \leq n} (r_i \otimes r_j)\tilde{\psi}_i(A)^* \tilde{\varphi}_j(B)^* x^* \Big\|_{L^2(I \times I; X^*)} \leq K_3 \|x^*\|$

whence (3.3). $\hfill \diamond$

It follows rather easily from Goldstine's Lemma that if a Banach space X has property (A), so does its dual. Since the converse implication is obvious, we can state:

Lemma 3.5 : Let X be a Banach space. X has property (A) if and only if its dual X^* has property (A).

Proof of Theorem 3.2 : Applying Proposition 3.4, we wish to show that any Banach space with a local unconditional structure satisfies property (A). We know that a Banach space has l.u.st. if and only if its bidual is isomorphic to a complemented subspace of a Banach lattice [13]. Moreover, it is clear that if X satisfies property (A), so does any complemented subspace of X. By Lemma 3.5, it is therefore enough to show that any Banach lattice satisfies property (A).

Let X be a Banach lattice. Then for any $(\alpha_{i,j}) \in \mathbb{C}^{n^2}, (x_{i,j}) \in X^{n^2}$ and $(x_{i,j}^*) \in (X^*)^{n^2}$:

$$\Big| \sum \alpha_{i,j} \langle x_{i,j}, x_{i,j}^* \rangle \Big| \leq \sup |\alpha_{i,j}| \,\Big\| (\sum |x_{i,j}|^2)^{1/2} \Big\|_X \Big\| (\sum |x_{i,j}^*|^2)^{1/2} \Big\|_{X^*}. \tag{3.4}$$

Let us say a word about the meaning of this inequality. If the reader is not a specialist in Banach lattices, a good way to get some intuition is to imagine X (and then X^*) as a function lattice on a measure space Ω. Then, the quantity $(\sum |x_{i,j}(\omega)|^2)^{1/2}$ makes sense for almost every ω in Ω and turns out to define an element of X. The above inequality is thus a consequence of the pointwise application of the Cauchy-Schwarz inequality. A meaning can also be given to $(\sum |x_{i,j}|^2)^{1/2}$ in general Banach lattices (see [21] for complete details) and the previous inequality holds true as well.

The following two variables version of Khintchine's inequality is well known:

$$\exists C \; : \; \forall \, (a_{i,j})_{1 \leq i,j \leq n} \in \mathbb{C}^{n^2}, \quad \left(\sum |a_{i,j}|^2\right)^{1/2} \leq C \left\|\sum a_{i,j}\, r_i \otimes r_j\right\|_{L^1(I \times I)}.$$

This can be viewed as a particular case of Kahane's inequality (see e.g. [21, p. 74]). The latter inequality implies that for any $(x_{i,j})$ in X^{n^2}:

$$\left\|\left(\sum |x_{i,j}|^2\right)^{1/2}\right\|_X \leq C \; \left\|\int_{I^2} \left|\sum r_i(t)r_j(s)x_{i,j}\right| ds dt\right\|_X \quad \text{whence}$$

$$\left\|\left(\sum |x_{i,j}|^2\right)^{1/2}\right\|_X \leq C \; \left\|\sum (r_i \otimes r_j)x_{i,j}\right\|_{L^2(I^2;X)}. \tag{3.5}$$

The inequalities (3.4) and (3.5) yield the desired property (A) for X. $\diamond$

Proof of Theorem 3.3 : Assume for instance that X has property (α) and let us show property (A). With the above notation we have:

$$\left|\sum_{1 \leq i,j \leq n} \alpha_{i,j}\langle x_{i,j}, x_{i,j}^*\rangle\right| =$$

$$\left|\int_{I \times I} \left\langle \sum_{i,j} \alpha_{i,j} r_i(t)r_j(s)x_{i,j}, \sum_{k,\ell} r_k(t)r_\ell(s)x_{k,\ell}^* \right\rangle ds dt\right| \leq$$

$$\left\|\sum_{i,j} \alpha_{i,j}(r_i \otimes r_j)x_{i,j}\right\|_{L^2(I \times I;X)} \left\|\sum_{i,j} (r_i \otimes r_j)x_{i,j}^*\right\|_{L^2(I \times I;X^*)} \leq$$

$$C \sup_{i,j} |\alpha_{i,j}| \left\|\sum_{i,j} (r_i \otimes r_j)x_{i,j}\right\|_{L^2(I \times I;X)} \left\|\sum_{i,j} (r_i \otimes r_j)x_{i,j}^*\right\|_{L^2(I \times I;X^*)}$$

by the (α) property of X. When X^* has property (α), we just have to modify this proof in an obvious manner. $\diamond$

In [25], Pisier exhibited several classes of Banach spaces with property (α). From his results and Theorem 3.3, we deduce:

Corollary 3.6 : Let E be a Banach space with l.u.st. (or merely the Gordon-Lewis property).
 (i) If E has a finite cotype, then every subspace of E has the joint calculus property.
 (ii) If E^* has a finite cotype, then every quotient of E has the joint calculus property.
(iii) If E and E^* have a finite cotype, then every quotient of subspaces of E has the joint calculus property.

We refer the reader to [26] for information on the Gordon-Lewis property. From (i) it follows that every subspace of L^1 has the joint calculus property and from (iii) that every quotient of subspaces of a B-convex Banach lattice (or equivalently a Banach lattice with a finite cotype and such that its dual has a finite cotype) has the joint calculus property. We mention for completeness that Bourgain [3] constructed a B-convex Banach lattice which is not a UMD space. In order to describe the field of application of our results, let us finally notice that for any measure space (Ω, Σ, μ) and any $1 \leq p < +\infty$, if a Banach space X enjoys property (α), so does $L^p(\Omega;X)$.

We will now study the joint calculus property for a few Banach spaces that are typical for their lack of unconditional structure.

Let $\mathbb{T} = \{z \in \mathbb{C}, |z| = 1\}$ and $D = \{z \in \mathbb{C}, |z| < 1\}$. We let $L^p = L^p(\mathbb{T})$ for $1 \le p \le \infty$, where $\mathbb{T}$ is equipped with its Haar measure. Let $H^p = \{f \in L^p : \forall n < 0, \int_{\mathbb{T}} f(z)z^n dz = 0\}$ be the usual Hardy spaces over D. Lastly we denote by $A(D)$ the disk algebra which is the space of continuous functions on $\mathbb{T}$ belonging to H^∞. We denote by P the quotient mapping from L^1 onto L^1/H^1. We can now state together our positive results about these spaces.

Theorem 3.7 : L^1/H^1, H^∞, and $A(D)$ enjoy the joint calculus property.

Proof : We will show that these spaces satisfy (A). We start with L^1/H^1. In [4] Bourgain proved that L^1/H^1 is a Grothendieck space of cotype 2 or, equivalently (see [26]), that it satisfies the following lifting property:

There is a constant $C > 0$ such that, for any subset $\{x_1, \ldots, x_n\}$ of L^1/H^1, there is a subset $\{y_1, \ldots, y_n\}$ of L^1 such that:

$$\forall i \in \{1, \ldots, n\}, \ Py_i = x_i \qquad \text{and} \qquad \left\| \sum_{i=1}^n r_i y_i \right\|_{L^2(I;L^1)} \le C \left\| \sum_{i=1}^n r_i x_i \right\|_{L^2(I;L^1/H^1)}.$$

A thorough examination of the proof given in [6] (see also [26, Chapter 6]) yields the following lifting property with two variables:

There is a constant $C > 0$ such that, for any subset $\{x_{i,j}\}_{1 \le i,j \le n}$ of L^1/H^1, there is a subset $\{y_{i,j}\}_{1 \le i,j \le n}$ of L^1 such that:

$$\forall \ 1 \le i,j \le n, \ \ Py_{i,j} = x_{i,j} \ \text{ and}$$
$$\left\| \sum_{1 \le i,j \le n} (r_i \otimes r_j) y_{i,j} \right\|_{L^2(I \times I;L^1)} \le C \left\| \sum_{1 \le i,j \le n} (r_i \otimes r_j) x_{i,j} \right\|_{L^2(I \times I;L^1/H^1)}. \tag{3.6}$$

Since L^1 satisfies property (A), it follows clearly from (3.6) that L^1/H^1 enjoys also property (A) and therefore the joint calculus property by Proposition 3.4.

Notice now that H^∞ is isometric to $(L^1/H^1)^*$. So H^∞ has property (A) by Lemma 3.5. Recall finally that the dual of $A(D)$ is isometric to the ℓ_1-sum $L^1/H^1 \oplus_1 L_s^1$, where L_s^1 denotes the space of all singular measures on $\mathbb{T}$. We have already seen that L^1/H^1 has property (A). Since it is a Banach lattice, so does L_s^1. Therefore $A(D)^*$, and thus $A(D)$ have property (A). $\diamond$

Remark 3.8 : In fact it is possible to deduce from (3.6) that L^1/H^1 has property (α). Moreover, using similar arguments as in the proof of Theorem 3.7, one obtains that for any reflexive subspace R of an L^1-space $L^1(\Omega)$, the quotient space $L^1(\Omega)/R$ has the joint calculus property. More generally, if X is a Banach space with property (A) and if $Y \subset X$ is a subspace, then X/Y has property (A) (and thus the joint calculus property) as soon as Y is B-convex (see [26, 27]).

We will conclude this section with a counterexample to bounded joint functional calculus on Schatten spaces. Let S_∞ be the Banach space of all compact operators on ℓ_2. For $1 \le p < \infty$, the Schatten space S_p is defined as the space of all T in S_∞ such that $|T|^p$ has a finite trace. It can be equipped with the norm $\alpha_p(T) = (\text{tr} \, |T|^p)^{1/p}$, for which it is a Banach space.

Theorem 3.9 : For all $1 \le p \le \infty$, $p \ne 2$, the space S_p fails the joint calculus property.

Proof : Let $1 \le p \le \infty$, $p \ne 2$. We denote by $(e_j)_{j \ge 1}$ the canonical basis of ℓ_2. For any $i, j \ge 1$, we let $E_{i,j}$ be the rank one operator on ℓ_2 defined by $E_{i,j}x = \langle x, e_j \rangle e_i$ for every x in ℓ_2. Let K_{00} be the linear span of the $E_{i,j}$'s. We consider it as equipped with the S_p norm. For every family $a = (a_{i,j})_{i,j \ge 1}$ of complex numbers, we denote by $M_a : K_{00} \to K_{00}$ the so-called Schur multiplier defined by $M_a(E_{i,j}) = a_{i,j}E_{i,j}$. When M_a is bounded, we say that it is a bounded Schur multiplier on S_p.

We identify elements of S_p with their infinite matrix in the canonical basis of ℓ_2. Let $(\alpha_i)_{i=1}^\infty$ and $(\beta_i)_{i=1}^\infty$ be two sequences of positive numbers. Let D and Δ be the two diagonal matrices:

$$D = \begin{pmatrix} \alpha_1 & & 0 \\ & \cdot & \\ & & \alpha_i \\ 0 & & \cdot \end{pmatrix} \quad \text{and} \quad \Delta = \begin{pmatrix} \beta_1 & & 0 \\ & \cdot & \\ & & \beta_i \\ 0 & & \cdot \end{pmatrix}. \tag{3.7}$$

11

Then, one can define two sectorial operators A and B of type 0 (i.e. of type ω for any $\omega > 0$) on S_p as follows: let $D(A) = \{T \in S_p \ : \ DT \in S_p\}$ and for $T \in D(A)$, $A(T) = DT$. Similarly, $D(B) = \{T \in S_p \ : \ T\Delta \in S_p\}$ and for $T \in D(B)$, $B(T) = T\Delta$. For any μ in $(0, \pi)$, A and B admit a bounded $H^\infty(\Sigma_\mu)$ functional calculus. Indeed, for any f in $H^\infty(\Sigma_\mu)$, and any T in S_p, we have:

$$f(A)(T) = f(D)T \quad \text{and} \quad f(B)(T) = Tf(\Delta) \qquad \text{where}$$

$$f(D) = \begin{pmatrix} f(\alpha_1) & & 0 \\ & \cdot & \\ & & f(\alpha_i) \\ 0 & & \cdot \end{pmatrix} \text{ and } f(\Delta) = \begin{pmatrix} f(\beta_1) & & 0 \\ & \cdot & \\ & & f(\beta_i) \\ 0 & & \cdot \end{pmatrix}.$$

Let now μ and μ' in $(0, \pi)$ and $F \in H^\infty(\Sigma_\mu \times \Sigma_{\mu'})$. Then for every $i, j \geq 1$, $E_{i,j}$ belongs to $D\big(F(A, B)\big)$ and $F(A, B)(E_{i,j}) = F(\alpha_i, \beta_j)E_{i,j}$. We deduce that:

$$F(A, B) \text{ is bounded} \iff M_{(F(\alpha_i, \beta_j))} \text{ is a bounded Schur multiplier on } S_p. \tag{3.8}$$

We now fix $(\alpha_i)_{i \geq 1} = (\beta_i)_{i \geq 1} = (2^i)_{i \geq 1}$ and assume that (A, B) admits a bounded $H^\infty(\Sigma_\pi \times \Sigma_\pi)$ functional calculus. It can be easily checked that $(2^i)_{i \geq 1}$ is an interpolating sequence in Σ_π (i.e. $(\sqrt{2^i})_{i \geq 1}$ satisfies (2.7)). So by Theorem 2.9, there exist $M > 0$ and $(f_i)_{i \geq 1} \subset H^\infty(\Sigma_\pi)$ such that:

$$\forall i \geq 1 : \ f_i(2^i) = 1, \ \ \forall k \neq i : \ f_i(2^k) = 0 \ \text{ and } \ \forall z \in \Sigma_\pi, \ \sum_{i \geq 1} |f_i(z)| \leq M.$$

Let $a = (a_{i,j})_{i,j \geq 1}$ be any bounded family of complex numbers. We may define F_a in $H^\infty(\Sigma_\pi \times \Sigma_\pi)$ by $F_a(z, z') = \sum_{i,j \geq 1} a_{i,j} f_i(z) f_j(z')$. Clearly $F_a(2^i, 2^j) = a_{i,j}$ for every $i, j \geq 1$, hence, by (3.8), M_a is a bounded Schur multiplier on S_p. But the fact that this holds for all bounded a is false, as it can be seen in [26, Chapter 8]. $\diamond$

Remark 3.10 : McIntosh and Yagi used the operators considered above in [24] and showed that $A + B$ is not closed in S_∞. On the other hand, if $1 < p < \infty$, S_p is a UMD space (see [5]) and therefore, Dore-Venni's theorem insures that $A + B$ is closed. The Schatten spaces S_p, for $1 < p < \infty$ and $p \neq 2$, provide, to our knowledge, the first examples of UMD spaces that are failing the joint calculus property.

5. The $H^\infty(\Sigma_\mu; B(H))$ functional calculus on $\Lambda(H)$ - Applications to maximal regularity

Throughout this section, Λ will denote a B-convex Banach lattice, or equivalently, a Banach lattice with a finite cotype, whose dual Λ^* has a finite cotype. We recall that for $1 < p < \infty$, L^p-spaces are B-convex. We will use the following result of Maurey [22] (see also [21, p. 49-50]): If Λ is a Banach lattice with a finite cotype, then there is a constant $K > 0$ such that, for every subset $\{x_i\}_{i=1}^n$ of Λ:

$$K^{-1} \Big\| \Big(\sum_{i=1}^n |x_i|^2\Big)^{1/2} \Big\|_\Lambda \leq \Big\| \sum_{i=1}^n r_i x_i \Big\|_{L^1(I; \Lambda)} \leq K \Big\| \Big(\sum_{i=1}^n |x_i|^2\Big)^{1/2} \Big\|_\Lambda. \tag{5.1}$$

Since Λ is B-convex, it is order continuous and therefore, Λ and Λ^* can be represented as function lattices on a same measure space (Ω, Σ, μ) (see [21, 1.a, 1.b] for details). Moreover, the duality is described by:

$$\forall (x, x^*) \in \Lambda \times \Lambda^*, \ < x, x^* > = \int_\Omega x(\omega) x^*(\omega) \, d\mu(\omega).$$

Let now X be a Banach space. We introduce the classical definition:

$$\Lambda(X) = \{f : \Omega \to X \ : \ f \text{ strongly measurable and } \|f(.)\|_X \in \Lambda\}.$$

$\Lambda(X)$ equipped with $\|f\|_{\Lambda(X)} = \big\| \, \|f(.)\|_X \big\|_\Lambda$ is a Banach space and the B-convexity of Λ implies that $\Lambda \otimes X$ is dense in $\Lambda(X)$. As a typical example, notice that if $\Lambda = L^p(\Omega, \mu)$ for some $p \in (1, +\infty)$, then $\Lambda(X) = L^p(\Omega, \mu; X)$ is the classical Bochner X-valued L^p-space.

For every S in $B(X)$, $I_\Lambda \otimes S$ extends to a bounded operator $\overline{I_\Lambda \otimes S}$ on $\Lambda(X)$, with $\|\overline{I_\Lambda \otimes S}\|_{B(\Lambda(X))} = \|S\|_{B(X)}$. This allows us to consider $B(X)$ as a closed subalgebra of $B(\Lambda(X))$. Now let A be a sectorial operator on Λ. Assume that $A \otimes I_X$, defined on $D(A) \otimes X$, is closable in $\Lambda(X)$ and its closure $\mathcal{A}$ is also sectorial. Then $B(X)$ is actually a subalgebra of $E_\mathcal{A}$, thus we have:

$$B(X) \ \subset E_\mathcal{A} \ \subset \ B(\Lambda(X)).$$

It then makes sense to study the functional calculus $u_\mathcal{A}$ (see (2.1), (2.2)) on the Banach algebra $H^\infty(\Sigma_\mu; B(X))$. It will be shown in Section 6 that this functional calculus is not always bounded when A has a bounded H^∞ functional calculus. The purpose of this section is to establish positive results in the Hilbert space case and to apply them to maximal regularity. The following introductory proposition deals with the extension to $\Lambda(H)$, where H is a Hilbert space, of a bounded or of a sectorial operator on Λ.

Proposition 5.1 : Let H be a Hilbert space and $0 \leq \omega < \pi$.

 (i) There exists a constant $C > 0$ such that for every T in $B(\Lambda)$, $T \otimes I_H$ is bounded on $\Lambda \otimes H$ and satisfies $\|T \otimes I_H\| \leq C\|T\|$. Moreover, if T is injective then its extension $\mathcal{T} = \overline{T \otimes I_H}$ to $\Lambda(H)$ is injective.

 (ii) If A is a sectorial operator of type ω on Λ, then $A \otimes I_H$, defined on $D(A) \otimes H$, is a closable operator, whose closure $\mathcal{A}$ is sectorial of type ω on $\Lambda(H)$.
 If moreover A admits a bounded $H^\infty(\Sigma_\mu)$ functional calculus, so does $\mathcal{A}$.

Proof : We start with the proof of (i). Let $T \in B(\Lambda)$ and let y in $\Lambda \otimes H$. Then $y = \sum_{i=1}^n x_i \otimes e_i$, where $\{x_i\}_{i=1}^n \subset \Lambda$ and $(e_i)_{i=1}^n$ is an orthonormal system. We have $(T \otimes I_H)y = \sum_{i=1}^n Tx_i \otimes e_i$ hence to obtain $\|T \otimes I_H\| \leq C\|T\|$, it suffices to have:

$$\Big\|\big(\sum_{i=1}^n |Tx_i|^2\big)^{1/2}\Big\|_\Lambda \ \leq \ C\|T\| \ \Big\|\big(\sum_{i=1}^n |x_i|^2\big)^{1/2}\Big\|_\Lambda.$$

Such an inequality is actually true for any Banach lattice Λ by a result of Krivine [17] (see also [21, p. 93]). Note however that under our assumption of B-convexity, it can be proved directly as a consequence of (5.1). Indeed we clearly have:

$$\Big\|\sum_{i=1}^n r_i Tx_i\Big\|_{L^1(I;\Lambda)} \leq \|T\| \ \Big\|\sum_{i=1}^n r_i x_i\Big\|_{L^1(I;\Lambda)}.$$

For any $h \in H$ and any $y \in \Lambda(H)$, we may define the element of Λ $\tilde{h}(y) = \langle h, y(.) \rangle$. Suppose now that T is injective and let y in $\Lambda(H)$ such that $\mathcal{T}y = 0$. Then for any $h \in H$, $T(\tilde{h}(y)) = \tilde{h}(\mathcal{T}y) = 0$. Hence $\tilde{h}(y) = 0$, i.e. $\langle h, y(\omega) \rangle = 0$ for almost every ω in Ω. Since Λ is B-convex, the essential image of y is separable (see [21, Proposition 1.a.9]) and therefore we get $y = 0$.

Let us now prove (ii). We give ourselves a sectorial operator A of type ω on Λ. Let us consider $(y_n)_{n=1}^\infty$ a sequence in $D(A) \otimes H$ such that $y_n \to 0$ and $(A \otimes I_H)y_n \to y$ in $\Lambda(H)$. By point (i) proved above, we can set $\mathcal{R} = \overline{(I + A)^{-1} \otimes I_H}$ in $B(\Lambda(H))$. We let $z = \mathcal{R}y$. Clearly, $\mathcal{R}(A \otimes I_H)y_n = ((I - (I + A)^{-1}) \otimes I_H)y_n \to z$. Since $(I - (I + A)^{-1}) \otimes I_H$ is bounded, $z = 0$. But (i) implies that $\mathcal{R}$ is injective, so $y = 0$. The operator $A \otimes I_H$ is therefore closable. Moreover $\mathcal{A} = \overline{A \otimes I_H}$ clearly has dense range and domain and is injective, for the same reasons as in (i).

In order to conclude this proof we show that $\rho(A) = \rho(\mathcal{A})$ and:

$$\forall \, \lambda \in \rho(A), \quad (\lambda - \mathcal{A})^{-1} = \overline{(\lambda - A)^{-1} \otimes I_H}. \tag{5.2}$$

13

The inclusion $\rho(\mathcal{A}) \subset \rho(A)$ is obvious, so consider λ in $\rho(A)$. Then $\lambda - \mathcal{A}$ is injective, so it is enough to prove that:

$$\forall\, z \in \Lambda(H), \quad y = (\overline{(\lambda - A)^{-1} \otimes I_H})z \in D(\mathcal{A}) \;\; \text{and} \;\; (\lambda - \mathcal{A})y = z.$$

Let $z \in \Lambda(H)$ and (z_n) in $\Lambda \otimes H$ so that $z_n \to z$. Let $y_n = ((\lambda - A)^{-1} \otimes I_H)z_n$. Since $(\lambda - A)^{-1} \otimes I_H$ is bounded, (y_n) is a Cauchy sequence included in $D(A) \otimes I_H$. So y_n converges to y, for some y in $\Lambda(H)$. Moreover $\mathcal{A}y_n = \lambda y_n - z_n \to \lambda y - z$. Thus $y \in D(\mathcal{A}), (\lambda - \mathcal{A})y = z$ and $y = (\overline{(\lambda - A)^{-1} \otimes I_H})z$.

As an obvious consequence of (5.2) and (i), we obtain that $\mathcal{A}$ is sectorial of type ω.

Assume finally that A admits a bounded $H^\infty(\Sigma_\mu)$ functional calculus. Then it follows clearly from (5.2) and (2.1) that for any f in $H_0^\infty(\Sigma_\mu)$, $f(\mathcal{A}) = \overline{f(A) \otimes I_H}$. Thus point (i) and Proposition 2.1 imply that $\mathcal{A}$ admits a bounded $H^\infty(\Sigma_\mu)$ functional calculus. $\diamond$

Theorem 5.2 : Let $0 \leq \omega < \mu < \pi$. If A is a sectorial operator of type ω on Λ and admits a bounded $H^\infty(\Sigma_\mu)$ functional calculus, then, for any ν in (μ, π), $\mathcal{A}$ admits a bounded $H^\infty(\Sigma_\nu; B(H))$ functional calculus.

Let us start with a few preliminary lemmas. The first one is an extended version of the quadratic estimates of [7].

Lemma 5.3 : Let x in Λ and ψ in $H_0^\infty(\Sigma_\mu)$. Then for almost all ω in Ω, the application $h(\omega) : t \mapsto (\psi(tA)x)(\omega)$ belongs to the Hilbert space $H_0 = L^2\big((0, \infty), \frac{dt}{t}\big)$ and h is an element of $\Lambda(H_0)$.

Moreover, for a given ψ in $H_0^\infty(\Sigma_\mu)$, there is a constant $C > 0$, such that, for any x in Λ, $\|h\|_{\Lambda(H_0)} \leq C\|x\|_\Lambda$.

Proof : Let us fix ψ in $H_0^\infty(\Sigma_\mu)$. Since $D(A) \cap R(A)$ is dense in Λ, it suffices to show that this lemma holds for any x in $D(A) \cap R(A)$. So let x belong to $D(A) \cap R(A)$, we can write $x = \varphi(A)y$ with y in Λ. Then, by Fubini's theorem, the map $f : t \mapsto \psi(e^t A)x$ belongs to $L^1(\mathbb{R}; \Lambda)$ and, for all t in $\mathbb{R}$, $\hat{f}(t) = \hat{\psi}_e(t)A^{it}x$, where $\psi_e \in L^1(\mathbb{R})$ is defined by $\psi_e(t) = \psi(e^t)$. Then, for almost every ω in Ω, $f_\omega : t \mapsto (\psi(e^t A)x)(\omega)$ is in $L^1(\mathbb{R})$ and, for all s in $\mathbb{R}$, $\hat{f}_\omega(s) = \hat{\psi}_e(s)(A^{is}x)(\omega)$. Now, by Plancherel's theorem, we get that $\omega-$almost everywhere, $h(\omega) \in H_0$ and

$$\|h(\omega)\|_{H_0} = \frac{1}{2\pi}\Big(\int_{-\infty}^{+\infty} |\hat{\psi}_e(s)(A^{is}x)(\omega)|^2 ds\Big)^{1/2}. \tag{5.3}$$

Finally, following the proof of Theorem 6.6 in [7], we get that the righthand side term in (5.3) defines an element of Λ whose norm is less than $C\|x\|_\Lambda$, where C is a constant independent of x. $\diamond$

Next, we generalize Lemma 5.3 to the case of finite sequences in Λ.

Lemma 5.4 : Let ψ in $H_0^\infty(\Sigma_\mu)$. There exists a constant $C > 0$ such that for any finite subset $\{x_i\}_{i=1}^n$ of Λ:

$$\Big\|\Big(\int_0^\infty \sum_{i=1}^n |\psi(tA)x_i|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_\Lambda \leq C\Big\|\Big(\sum_{i=1}^n |x_i|^2\Big)^{1/2}\Big\|_\Lambda. \tag{5.4}$$

Proof : By Lemma 5.3 and by the properties of the Banach lattices, both quantities in the inequality (5.4) make sense. So let $\{x_i\}_{i=1}^n \subset \Lambda$ and define $h_i \in \Lambda(H_0)$ as above by $h_i(\omega) : t \mapsto (\psi(tA)x_i)(\omega)$. Then for any fixed $\omega \in \Omega$:

$$\Big(\int_0^\infty \sum_{i=1}^n |\psi(tA)x_i|^2(\omega)\frac{dt}{t}\Big)^{1/2} = \Big(\sum_{i=1}^n \|h_i(\omega)\|_{H_0}^2\Big)^{1/2}$$

$$= \Big\|\sum_{i=1}^n r_i h_i(\omega)\Big\|_{L^2(I;H_0)} \leq K_1 \Big\|\sum_{i=1}^n r_i h_i(\omega)\Big\|_{L^1(I;H_0)}$$

by Kahane's inequality (see e.g. [21, p. 74]). By convexity we then have:

$$\Big\|\Big(\int_0^\infty \sum_{i=1}^n |\psi(tA)x_i|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_\Lambda \leq K_1 \Big\|\sum_{i=1}^n r_i h_i\Big\|_{L^1(I;\Lambda(H_0))}.$$

14

But we can write:

$$\Big\|\sum_{i=1}^{n} r_i h_i\Big\|_{L^1(I;\Lambda(H_0))} = \int_I \Big\| \Big(\int_0^\infty |\psi(tA).\sum_{i=1}^{n} r_i(s)x_i|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_\Lambda ds.$$

Hence applying Lemma 5.3 to $\sum r_i(s)x_i$ for all s in I, we obtain:

$$\Big\|\sum_{i=1}^{n} r_i h_i\Big\|_{L^1(I;\Lambda(H_0))} \leq C \int_I\Big\|\sum_{i=1}^{n} r_i(s)x_i\Big\|_\Lambda ds = C \Big\|\sum_{i=1}^{n} r_i x_i\Big\|_{L^1(I;\Lambda)}$$

whence the result by (5.1). $\diamond$

Proof of Theorem 5.2 : Let α such that $\omega < \mu < \alpha < 2\alpha - \mu < \nu < \pi$. Consider γ in $H_0^\infty(\Sigma_\mu)$ such that $\gamma = \psi^2$ with ψ in $H_0^\infty(\Sigma_\mu)$ and satisfying:

$$\exists\, K > 0 \text{ s.t. } \forall\, s \in \mathbb{R},\ \hat{\gamma}_e(s) \geq K e^{-\alpha|s|}$$

where $\gamma_e \in L^1(\mathbb{R})$ is defined by $\gamma_e(t) = \gamma(e^t)$. Such a γ exists (see [7], example 4.7). We then obtain a constant $C > 0$ such that:

$$\forall\, b \in H_0^\infty(\Sigma_\nu; B(H)),\ \exists\, \beta \in L^\infty(\mathbb{R}_+\,;\, B(H)) \cap L^1(\mathbb{R}_+, \frac{dt}{t}\,;\, B(H)):$$

$$\|\beta\|_{L^\infty(\mathbb{R}_+;B(H))} \leq C\, \|b\|_{H^\infty(\Sigma_\mu;B(H))}\ \text{ and }\ \forall\, z \in \Sigma_\mu,\ b(z) = \int_0^\infty \beta(t)\gamma(tz)\frac{dt}{t}\,. \tag{5.5}$$

This result is proved in the scalar case (i.e. for $b \in H_0^\infty(\Sigma_\nu)$) in the course of the proof of Theorem 4.4 in [7]. It is easy to check that the same proof works as well for vector valued functions.

In the sequel, we simply denote by u, the functional calculus associated with $\mathcal{A}$ and defined on $H_0^\infty(\Sigma_\nu; B(H))$ by (2.1). For $b \in H_0^\infty(\Sigma_\nu; B(H)), \theta \in (\omega, \mu)$ and $x \in \Lambda \otimes H$, we have:

$$u(b)x = \frac{1}{2\pi i} \int_{\Gamma_\theta} \Big((\lambda - A)^{-1} \otimes \int_0^\infty \beta(t)\gamma(t\lambda)\frac{dt}{t}\Big)x\, d\lambda\,,$$

where β is given by (5.5). So, by Fubini's theorem:

$$u(b)x = \int_0^\infty (\gamma(tA) \otimes \beta(t))\, x\, \frac{dt}{t}\,.$$

Let us consider $(x, x^*) \in (\Lambda \otimes H) \times (\Lambda^* \otimes H)$. We can write $x = \sum_{i=1}^{n} x_i \otimes e_i$ and $x^* = \sum_{i=1}^{n} x_i^* \otimes e_i$, where $\{x_i\}_{i=1}^n \subset \Lambda$, $\{x_i^*\}_{i=1}^n \subset \Lambda^*$ and $\{e_i\}_{i=1}^n$ is an orthonormal system of H. Then letting $\beta_{i,j}(t) = \langle \beta(t)e_j, e_i\rangle_H$, we have:

$$\langle u(b)x, x^*\rangle = \int_0^\infty \sum_{1\leq i,j\leq n} \beta_{i,j}(t)\, \langle \gamma(tA)x_j, x_i^*\rangle_{\Lambda,\Lambda^*}\, \frac{dt}{t}$$

$$= \int_0^\infty \sum_{1\leq i,j\leq n} \beta_{i,j}(t)\, \langle \psi(tA)x_j, \psi(tA)^*x_i^*\rangle_{\Lambda,\Lambda^*}\, \frac{dt}{t}$$

$$= \int_\Omega \int_0^\infty \sum_{1\leq i,j\leq n} \beta_{i,j}(t)(\psi(tA)x_j)(\omega)(\psi(tA)^*x_i^*)(\omega)\, \frac{dt}{t}\, d\omega$$

15

by Fubini's theorem. Hence $|\langle u(b)x, x^* \rangle|$ is less than

$$\int_\Omega \int_0^\infty \|\beta(t)\|_{B(H)} \Big(\sum_{i=1}^n |(\psi(tA)x_i)(\omega)|^2\Big)^{1/2} \Big(\sum_{i=1}^n |(\psi(tA)^*x_i^*)(\omega)|^2\Big)^{1/2} \frac{dt}{t}\, d\omega$$

$$\leq \|\beta\|_{L^\infty} \int_\Omega \Big(\int_0^\infty \sum_{i=1}^n |(\psi(tA)x_i)(\omega)|^2 \frac{dt}{t}\Big)^{1/2} \Big(\int_0^\infty \sum_{i=1}^n |(\psi(tA)^*x_i^*)(\omega)|^2 \frac{dt}{t}\Big)^{1/2} d\omega$$

by the Cauchy-Schwarz inequality

$$\leq K \|b\| \Big\| \Big(\int_0^\infty \sum_{i=1}^n |\psi(tA)x_i|^2 \frac{dt}{t}\Big)^{1/2} \Big\|_\Lambda \Big\| \Big(\int_0^\infty \sum_{i=1}^n |\psi(tA)^*x_i^*|^2 \frac{dt}{t}\Big)^{1/2} \Big\|_{\Lambda^*}.$$

The Banach lattice Λ is reflexive because it is B-convex [21, Theorem 1.c.5]. Therefore A^* admits a bounded $H^\infty(\Sigma_\mu)$ functional calculus on Λ^*. Moreover, $\psi(tA)^* = \widetilde{\psi}(tA^*)$, where $\widetilde{\psi} \in H_0^\infty(\Sigma_\mu)$ is defined by $\widetilde{\psi}(z) = \overline{\psi(\overline{z})}$ for any $z \in \Sigma_\mu$. Hence applying Lemma 5.4 to A and A^*, we obtain that:

$$|\langle u(b)x, x^* \rangle| \leq KC^2 \|b\| \Big\| \Big(\sum_{i=1}^n |x_i|^2\Big)^{1/2} \Big\|_\Lambda \Big\| \Big(\sum_{i=1}^n |x_i^*|^2\Big)^{1/2} \Big\|_{\Lambda^*}$$

$$= KC^2 \|b\| \|x\|_\Lambda \|x^*\|_{\Lambda^*}.$$

From the B-convexity of Λ, it follows that $\Lambda(H)^*$ is canonically identified with $\Lambda^*(H)$. Then by density of $\Lambda \otimes H$ in $\Lambda(H)$ and $\Lambda^* \otimes H$ in $\Lambda^*(H)$, we get from the above estimate that $\|u(b)\| \leq KC^2 \|b\|$. $\diamond$

In order to apply Theorem 5.2, we need to extend to $\Lambda(X)$ an operator B defined on some Banach space X. The situation is simpler than in Proposition 5.1.

Lemma 5.5 : Let B a pseudosectorial operator of type ω' on X and let μ' in (ω', π).

(i) $I_\Lambda \otimes B$, defined on $\Lambda \otimes D(B)$, is closable and its closure $\mathcal{B}$ is pseudosectorial of type ω'. Moreover, $D(\mathcal{B}) = \Lambda(D(B))$, where $D(B)$ is equipped with its graph norm: $\|x\|_{D(B)} = \|x\|_X + \|Bx\|_X$.

(ii) If B is sectorial of type ω', so is $\mathcal{B}$.

(iii) If B admits a bounded $H^\infty(\Sigma_{\mu'})$ functional calculus, so does $\mathcal{B}$.

Proof : Let $(x_n)_{n=1}^\infty$ be a sequence in $\Lambda \otimes D(B)$ so that $x_n \to 0$ and $(I_\Lambda \otimes B)x_n \to y$. Then there is a subsequence $(x_n')_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that for almost all ω in Ω, $x_n'(\omega) \to 0$ and $B(x_n'(\omega)) \to y(\omega)$. Since B is closed, we have that $y = 0$ and therefore $I_\Lambda \otimes B$ is closable. Then our description of $D(\mathcal{B})$ is a straightforward consequence of the definition of $\Lambda(D(B))$. The operator $\mathcal{B}$ is pseudosectorial on account of the following fact :

$$\rho(B) \subset \rho(\mathcal{B}) \quad \text{and} \quad \forall \lambda \in \rho(B),\ (\lambda - \mathcal{B})^{-1} = \overline{I_\Lambda \otimes (\lambda - B)^{-1}}. \tag{5.6}$$

The assertion (ii) is then clear. From (5.6), it follows that for any f in $H_0^\infty(\Sigma_{\mu'})$, $f(\mathcal{B}) = \overline{I_\Lambda \otimes f(B)}$, which yields (iii). $\diamond$

Then, as an application of Theorem 5.2 and Proposition 2.6, we have the following:

Corollary 5.6 : Let A be an operator on a B-convex Banach lattice Λ admitting a bounded $H^\infty(\Sigma_\mu)$ functional calculus and let B a pseudosectorial operator of type ω' on a Hilbert space H. Denote by $\mathcal{A}$ and $\mathcal{B}$ the extensions described above of A and B to $\Lambda(H)$ and assume that $\mu + \omega' < \pi$.

Then $\mathcal{A} + \mathcal{B}$ is a closed one to one operator on $\Lambda(H)$ with dense range and:

$$\exists C > 0\ :\ \forall\, x \in D(\mathcal{A}) \cap D(\mathcal{B}), \quad \|\mathcal{A}x\| \leq C\|(\mathcal{A} + \mathcal{B})x\|. \tag{5.7}$$

Proof of Theorem 1.4 : Under the assumptions of Theorem 1.4, the fact that $B \otimes I_H$ and $I_{L^q} \otimes C$ are closable follows from Proposition 5.1 and Lemma 5.5. We let $\widehat{B} = \overline{B \otimes I_H}$ and $\widehat{C} = \overline{I_{L^q} \otimes C}$. Then by

16

Corollary 5.6, $\widetilde{B} + \widetilde{C}$ is a closed one to one operator with a dense range. The fact that $\widetilde{B} + \widetilde{C}$ is actually sectorial of type $< \dfrac{\pi}{2}$ can be easily deduced from Theorem 5.2 but also follows from the estimates given in [9, Section 3].

Now let $T > 0$, $1 < p < \infty$. Consider $A = \dfrac{d}{dt}$ on $L^p(0,T)$ with domain $W_0^{1,p}(0,T)$ and let

$$\Lambda = L^p(0,T;L^q(\Omega_1)).$$

Then $A \otimes I_{L^q(\Omega_1)}$ is closable in Λ and $\overline{A \otimes I_{L^q(\Omega_1)}}$ admits a bounded $H^\infty(\Sigma_\eta)$ functional calculus for any $\eta > \dfrac{\pi}{2}$ (see [11]). The space Λ is a B-convex Banach lattice and it is easy to check that $\Lambda(H)$ can be identified with $L^p(0,T;L^q(\Omega_1;H))$ and that the closure of $\overline{A \otimes I_{L^q(\Omega_1)}} \otimes I_H$, provided by Proposition 5.1, is $\mathcal{A} = \dfrac{d}{dt}$ on $L^p(0,T;L^q(\Omega_1;H))$ with domain $W_0^{1,p}(0,T;L^q(\Omega_1;H))$.

We call $\mathcal{B}$ and $\mathcal{C}$ respectively, the closures of $I_{L^p(0,T)} \otimes \widetilde{B}$ and $I_{L^p(0,T)} \otimes \widetilde{C}$ in $\Lambda(H)$. On the other hand, by Lemma 5.5, $I_{L^p(0,T)} \otimes B$ is closable on Λ and its closure admits a bounded $H^\infty(\Sigma_\mu)$ functional calculus. It can easily be checked that $\mathcal{B}$ is the closure of $\overline{I_{L^p(0,T)} \otimes B} \otimes I_H$. We also have that $\mathcal{C}$ is the closure of $I_\Lambda \otimes C$ and is a sectorial operator of type ω' on $\Lambda(H)$.

By Theorem 3.2, $(\overline{A \otimes I_{L^q(\Omega_1)}}, \overline{I_{L^p(0,T)} \otimes B})$ admits on Λ a bounded $H^\infty(\Sigma_\alpha \times \Sigma_\beta)$ joint functional calculus, for any (α,β) in $(\dfrac{\pi}{2},\pi) \times (\mu,\pi)$ and, by Proposition 5.1, so does $(\mathcal{A},\mathcal{B})$. By Proposition 2.7, this implies that there exists $C_1 > 0$ such that:

$$\forall\, x \in D(\mathcal{A}) \cap D(\mathcal{B}), \quad \|\mathcal{A}x\| \le C_1\|(\mathcal{A}+\mathcal{B})x\|. \tag{5.8}$$

Now let $S = \overline{A \otimes I_L^q(\Omega_1)} + \overline{I_{L^p(0,T)} \otimes B}$, then by Corollary 4.3, S admits a bounded $H^\infty(\Sigma_\nu)$ functional calculus, for any $\nu > \dfrac{\pi}{2}$. It is standard to verify that $\overline{S \otimes I_H} = \mathcal{A} + \mathcal{B}$. Then, in view of Corollary 5.6, there exists $C_2 > 0$ such that:

$$\forall\, x \in D(\mathcal{A}) \cap D(\mathcal{B}) \cap D(\mathcal{C}), \quad \|(\mathcal{A}+\mathcal{B})x\| \le C_2\|(\mathcal{A}+\mathcal{B}+\mathcal{C})x\|. \tag{5.9}$$

From (5.8) and (5.9), it follows of course that $\mathcal{A}(\mathcal{A}+\mathcal{B}+\mathcal{C})^{-1}$ is bounded. As is well known this is equivalent to the maximal regularity of $\widetilde{B} + \widetilde{C}$. $\diamond$

6. A characterization of Hilbert spaces in terms of functional calculi

Let X be a Banach space, p in $(1,+\infty)$ and let $\mathbb{T} = \{z \in \mathbb{C} \,:\, |z| = 1\}$. We will start with the study of the first derivation operator on $L^p(\mathbb{T};X)$. So consider $A = iz\dfrac{d}{dz}$ on $L^p(\mathbb{T})$ with domain $D(A) = W^{1,p}(\mathbb{T})$ and $\mathcal{A} = iz\dfrac{d}{dz}$ on $L^p(\mathbb{T};X)$ with domain $D(\mathcal{A}) = W^{1,p}(\mathbb{T};X)$ which is also the closure of $A \otimes I_X$ in $L^p(\mathbb{T};X)$. It is known that $\mathcal{A}$ is pseudosectorial of type $\dfrac{\pi}{2}$ and that, for any μ in $(\dfrac{\pi}{2},\pi)$, $\mathcal{A}$ admits a bounded $H_0^\infty(\Sigma_\mu)$ functional calculus if and only if X is a UMD space [11, 29].

We recall (see Section 5), that $B(X)$ is canonically identified with a closed subalgebra of $E_\mathcal{A}$. Then we have the following characterization:

Theorem 6.1 : Let $\dfrac{\pi}{2} < \mu < \pi$.

$\mathcal{A}$ admits a bounded $H_0^\infty(\Sigma_\mu;B(X))$ functional calculus if and only if X is isomorphic to a Hilbert space.

Proof : The "if" part is a consequence of Theorem 5.2. So let us assume that $\mathcal{A}$ admits a bounded $H_0^\infty(\Sigma_\mu;B(X))$ functional calculus. We denote as usual by:

$$u_\mathcal{A} \colon H_0^\infty(\Sigma_\mu;B(X)) \to B(L^p(\mathbb{T};X))$$

17

the induced bounded homomorphism. Then we will show that X is of type 2 and of cotype 2 (see Definition 2.8). This, by Kwapien's theorem [18] will yield the conclusion.

Let us prove first that X is of type 2. So let $\{x_k\}_{k=1}^n$ be a finite subset of X and fix (e, e^*) in $X \times X^*$ with $\langle e, e^* \rangle = \|e\|_X = \|e^*\|_{X^*} = 1$.

The sequence $(\sqrt{2^k} e^{i\frac{\pi}{4}})_{k \geq 1}$ satisfies the condition (i) of Theorem 2.9 (the verification is left to the reader). So there exists a sequence $(f_k)_{k \geq 1}$ in $H^\infty(\Sigma_\pi)$ and a constant $M > 0$ such that:

$$\forall k \geq 1: \ f_k(i2^k) = 1, \ \ \forall j \neq k: \ f_j(i2^k) = 0 \ \text{ and } \ \forall z \in \Sigma_\pi: \ \sum_{k \geq 1} |f_k(z)| \leq M$$

Now, consider, for $1 \leq k \leq n$, $T_k \in B(X)$ defined by $T_k x = e^*(x) \dfrac{x_k}{\|x_k\|}$ for all $x \in X$ (we may assume $x_k \neq 0$). For z in Σ_π, define $f(z) = \displaystyle\sum_{k \geq 1} f_k(z) T_k$. Then $f \in H^\infty(\Sigma_\pi; B(X))$, $\|f\|_{H^\infty(\Sigma_\pi; B(X))} \leq M$ and $\forall \, k \in \{1, \ldots, n\}, \ f(i2^k) = T_k$.

We also consider P in $L^p(\mathbb{T}; X)$ defined by: $\forall \, z \in \mathbb{T}, \quad P(z) = \Big(\displaystyle\sum_{k=1}^n \|x_k\| \, z^{2^k}\Big) e$.

Let ε_k be defined on $\mathbb{T}$ by $\varepsilon_k(z) = z^{2^k}$. It is known (see [28]) that there is a constant $C > 0$ such that for any $y_1, \ldots, y_n$ in a Banach space Y:

$$C^{-1} \Big\| \sum_{k=1}^n \varepsilon_k y_k \Big\|_{L^p(\mathbb{T}; Y)} \leq \Big\| \sum_{k=1}^n r_k y_k \Big\|_{L^1(I; Y)} \leq C \Big\| \sum_{k=1}^n \varepsilon_k y_k \Big\|_{L^p(\mathbb{T}; Y)}. \tag{6.1}$$

In particular, this yields (with $Y = \mathbb{C}$):

$$\|P\|_{L^p(\mathbb{T}; X)} \leq C \Big(\sum_{k=1}^n \|x_k\|^2 \Big)^{1/2}. \tag{6.2}$$

Now, we need to introduce the auxilliary functions defined, for N in $\mathbb{N}$ and $\lambda \in \Sigma_\mu$, by, $\xi_N(\lambda) = \dfrac{1}{1+\lambda/N} - \dfrac{1}{1+\lambda N}$. For any $N > 0$, $\xi_N \in H_0^\infty(\Sigma_\mu)$ and $\|\xi_N\|_{H^\infty(\Sigma_\mu)} \leq C_\mu$, where C_μ does not depend on N. Using the Cauchy Residue Theorem, one gets:

$$\forall \, z \in \mathbb{T}, \quad \big(u_{\mathcal{A}}(\xi_N f) P\big)(z) = \sum_{k=1}^n \xi_N(i2^k) z^{2^k} x_k.$$

This, combined with (6.1) and (6.2) implies:

$$\Big\| \sum_{k=1}^n r_k \xi_N(i2^k) x_k \Big\|_{L^1(I; X)} \leq C^2 C_\mu M \|u_{\mathcal{A}}\| \Big(\sum_{k=1}^n \|x_k\|^2 \Big)^{1/2}. \tag{6.3}$$

Finally, for any z in Σ_μ, $\displaystyle\lim_{N \to +\infty} \xi_N(z) = 1$. Thus, passing to the limit in (6.3), we get that X is of type 2.

In order to prove that X is of cotype 2, we follow the same steps, after exchanging the T_k's with the operators defined by: $\forall x \in X$, $S_k x = x_k^*(x) e$; where $x_k^* \in X^*$, $\|x_k^*\|_{X^*} = 1$ and $x_k^*(x_k) = \|x_k\|$. $\diamond$

We will now show how to use transference methods in order to obtain the same characterization if we replace $\mathbb{T}$ by $\mathbb{R}$. So let X be a Banach space, $1 < p < +\infty$, $A_0 = \dfrac{d}{dt}$ on $L^p(\mathbb{R})$ with domain $W^{1,p}(\mathbb{R})$ and $\mathcal{A}_0 = \dfrac{d}{dt}$ on $L^p(\mathbb{R}; X)$ with domain $W^{1,p}(\mathbb{R}; X)$. We have similarly:

<u>Theorem 6.2 :</u> Let $\frac{\pi}{2} < \mu < \pi$.

$\mathcal{A}_0$ admits a bounded $H^\infty(\Sigma_\mu; B(X))$ functional calculus if and only if X is isomorphic to a Hilbert space.

Proof : Notice first that $-\mathcal{A}_0$ and $-\mathcal{A}$ generate the groups of translations in $L^p(\mathbb{R}; X)$ and $L^p(\mathbb{T}; X)$, that we denote by $\left(e^{-t\mathcal{A}_0}\right)_{t\in\mathbb{R}}$ and $\left(e^{-t\mathcal{A}}\right)_{t\in\mathbb{R}}$.

Lemma 6.3 : For any k in $L^1(\mathbb{R}; B(X))$:

$$\left\|\int_{\mathbb{R}} e^{-t\mathcal{A}} k(t)\, dt\right\|_{B(L^p(\mathbb{T};X))} \leq \left\|\int_{\mathbb{R}} e^{-t\mathcal{A}_0} k(t)\, dt\right\|_{B(L^p(\mathbb{R};X))}.$$

In the case when k belongs to $L^1(\mathbb{R})$, this result follows from some well known work of Calderon on transference, using the group action of $\mathbb{R}$ on $\mathbb{T}$ by translation (see e.g. [8, Chapters 1 and 2]). This extension to the $B(X)$-valued case is straightforward and left to the reader.

Suppose now that $u_{\mathcal{A}_0}\colon H_0^\infty(\Sigma_\mu; B(X)) \to B(L^p(\mathbb{R}; X))$ is bounded. Let $\mathcal{R}_0$ be the algebra of all rational functions, belonging to $H_0^\infty(\Sigma_\mu)$, with poles outside $\overline{\Sigma_\mu}$. Clearly $\mathcal{R}_0$ is a subalgebra of $H_0^\infty(\Sigma_\mu)$. Any f in $\mathcal{R}_0$ is the Laplace transform of some k in $L^1(\mathbb{R}_+)$. More precisely:

$$\forall\, z \in \Sigma_{\frac{\pi}{2}}\ :\quad f(z) = \int_0^\infty e^{-tz} k(t)\, dt\,. \tag{6.4}$$

This inspired the so-called Phillips functional calculus and it is known that for any f in $\mathcal{R}_0$, $u_{\mathcal{A}_0}(f)$ is equal to $\displaystyle\int_0^\infty e^{-t\mathcal{A}_0} k(t)\, dt$, where k is given by (6.4). Then, this can be extended in the following way: for any f in $\mathcal{R}_0 \otimes B(X)$, there exists k in $L^1(\mathbb{R}_+; B(X))$ such that:

$$\forall z \in \Sigma_{\frac{\pi}{2}}\ :\quad f(z) = \int_0^\infty e^{-tz} k(t)\, dt\ \ \text{and}\ \ u_{\mathcal{A}_0}(f) = \int_0^\infty e^{-t\mathcal{A}_0} k(t)\, dt\,.$$

This is as well true for $u_{\mathcal{A}}$. So, by Lemma 6.3, we have that $u_{\mathcal{A}}$ is bounded on $\mathcal{R}_0 \otimes B(X)$:

$$\exists\, K > 0\ :\ \forall\, f \in \mathcal{R}_0 \otimes B(X),\ \|u_{\mathcal{A}}(f)\| \leq K\|f\|_{H^\infty(\Sigma_\mu; B(X))}\,. \tag{6.5}$$

Now, consider f in $H_0^\infty(\Sigma_\mu) \otimes B(X)$. For $\varepsilon > 0$ and z in $\mathbb{C} \setminus \{-\frac{1}{\varepsilon}\}$, we denote $g_\varepsilon(z) = \dfrac{\varepsilon + z}{1 + \varepsilon z}$. Notice that $g_\varepsilon(\overline{\Sigma_\mu})$ is a compact subset of Σ_μ. Hence it is not hard to deduce from Runge's theorem that there is a sequence $(f_n)_{n\geq 1}$ in $\mathcal{R}_0 \otimes B(X)$ such that $f_n \circ g_\varepsilon$ converges to $f \circ g_\varepsilon$ uniformly on Σ_μ. Fix now m in $\mathbb{N}$. It follows from (6.5) that

$$\forall\varepsilon > 0,\ \forall m > 0,\ \forall n > 0\ :\ \|u_{\mathcal{A}}(\varphi_m (f_n \circ g_\varepsilon))\| \leq K\|\varphi_m\|_{H^\infty(\Sigma_\mu)}\|f_n \circ g_\varepsilon\|_{H^\infty(\Sigma_\mu; B(X))}.$$

Since $(\varphi_m)_{m\geq 1}$ is uniformly bounded in $H^\infty(\Sigma_\mu)$, there is a constant $K' > 0$ such that:

$$\forall\varepsilon > 0,\ \forall m > 0\ :\ \|u_{\mathcal{A}}(\varphi_m (f \circ g_\varepsilon))\| \leq K'\|f\|_{H^\infty(\Sigma_\mu; B(X))}.$$

By Lebesgue's dominated convergence theorem, we have:

$$\forall x \in X,\ \lim_{\varepsilon \to 0} u_{\mathcal{A}}(\varphi_m (f \circ g_\varepsilon))x = u_{\mathcal{A}}(f)\varphi_m(A)x\,.$$

Therefore, letting m tend to $+\infty$ we get: $\|u_{\mathcal{A}}(f)\| \leq K'\|f\|_{H^\infty(\Sigma_\mu; B(X))}$. Finally, notice that it follows from the proof of Theorem 6.1 that the boundedness of $u_{\mathcal{A}}$ restricted to $H_0^\infty(\Sigma_\mu) \otimes B(X)$ actually implies that X is isomorphic to a Hilbert space. $\diamond$

Remark 6.4 : Let $1 < p < \infty$ and let $A_0 = \dfrac{d}{dt}$ on $L^p(\mathbb{R})$ as above. The classical proof of the fact that for any $\mu > \dfrac{\pi}{2}$, A_0 admits a bounded $H^\infty(\Sigma_\mu)$ functional calculus relies on Mihlin's multiplier theorem on $L^p(\mathbb{R})$. The arguments involved in this proof can be adapted to the operator valued framework hence it

is possible to derive a direct proof of Theorem 5.2 for A_0 from Mihlin's theorem on $L^p(\mathbb{R}; H)$ for $B(H)$-valued multipliers [2, Theorem 6.1.6.]. In the same manner, if we assume that for a given Banach space X, Mihlin's theorem holds on $L^p(\mathbb{R}; X)$ for $B(X)$-valued multipliers, then one can show that $\mathcal{A}_0$ admits a bounded $H^\infty(\Sigma_\mu; B(X))$ functional calculus on $L^p(\mathbb{R}; X)$ for any $\mu > \dfrac{\pi}{2}$. Thus as a consequence of Theorem 6.2, we obtain that X is necessarily isomorphic to Hilbert space. The fact that the operator valued Mihlin multiplier theorem only holds on Hilbert spaces has been known for a long time and goes back to G. Pisier (unpublished).

Remark 6.5 : Given a sectorial operator A which admits a bounded $H^\infty(\Sigma_\mu)$ functional calculus on a Banach space X, a natural question is: does A automatically admit a bounded $H^\infty(\Sigma_\mu, E_A)$ functional calculus ? This holds to be true when X is a Hilbert space. This result is implicit in [19] and its proof relies upon techniques from [23]. We were informed by Albrecht and McIntosh that they also obtained this result (paper in preparation). When we leave Hilbert spaces, the situation turns out to be much more complicated. Indeed, Theorem 6.2 shows that on L^p-spaces $(1 < p \neq 2 < \infty)$, the boundedness of $H^\infty(\Sigma_\mu)$ functional calculus for an operator A does not imply the boundedness of $H^\infty(\Sigma_\mu, E_A)$ functional calculus.

References

[1] D. Albrecht, Functional calculi of commuting unbounded operators, PhD Thesis, Monash University, Australia (1994).

[2] J. Bergh and A.J. Löfström, Interpolation spaces, Springer Verlag, New-York (1976).

[3] J. Bourgain, Some remarks on Banach spaces in which martingale difference sequences are unconditional, Arkiv Math. 21 (1983), 163-168.

[4] J. Bourgain, New Banach space properties of the disk algebra and H^∞, Acta Math. 152 (1984), 1-48.

[5] J. Bourgain, Vector valued singular integrals and the $H^1 - BMO$ duality, Probability Theory and Harmonic Analysis (J.-A. Chao and W.A. Woyczynski, Eds.), Dekker, New York and Basel (1986), 1-19.

[6] J. Bourgain and W. Davis, Martingale transforms and complex uniform convexity, Trans. Amer. Math. Soc. 294 (1986), 501-515.

[7] M. Cowling, I. Doust, A. McIntosh and A. Yagi, Banach space operators with a bounded H^∞ functional calculus, Journal of Aust. Math. Society (Series A) 60 (1996), 51-89.

[8] R.R. Coifman and G. Weiss, Transference methods in analysis, CBMS regional conference series in mathematics, A.M.S., number 31 (1977).

[9] G. Da Prato and P. Grisvard, Sommes d'opérateurs linéaires et équations différentielles opérationnelles, J. Math. pures et appl. 54 (1975), 305-387.

[10] L. De Simon, Un' applicazione della theoria degli integrali singolari allo studio delle equazioni differenziali lineare astratte del primo ordine, Rend. Sem. Mat., Univ. Padova (1964), 205-223.

[11] G. Dore and A. Venni, On the closedness of the sum of two closed operators, Math. Z. 196 (1987), 189-201.

[12] G. Dore and A. Venni, Some results about complex powers of closed operators, J. Math. Anal. Appl. 149 (1990), 124-136.

[13] T. Figiel, W. Johnson and L. Tzafriri, On Banach lattices and spaces having local unconditional structure, with applications to Lorentz function spaces, J. Approx. Theory 13 (1975), 395-412.

[14] E. Franks and A. McIntosh, Discrete quadratic estimates and holomorphic functional calculi of operators in Banach spaces, preprint (1995).

[15] J. B. Garnett, Bounded analytic functions, Academic Press, New-York (1981).

[16] J.A. Goldstein, Semigroups of linear operators and applications, Oxford University Press, New York (1985).

[17] J. L. Krivine, Théorèmes de factorisation dans les espaces de Banach réticulés, Sémi- naire Maurey-Schwarz (1973-74), Exposés 22-23, Ecole Polytechnique, Paris.

[18] S. Kwapien, Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients, Studia Math. 44 (1972), 583-595.

[19] C. Le Merdy, On dilation theory for C_0-semigroups on Hilbert space, Indiana Univ. Math. J., to appear.

[20] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces I, Springer Verlag, Berlin (1977).

[21] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces II, Springer Verlag, Berlin (1979).

[22] B. Maurey, Type et cotype dans les espaces munis de structures locales inconditionnelles, Séminaire Maurey-Schwartz (1973-74), Exposés 24-25, Ecole Polytechnique, Paris.

[23] A. McIntosh, Operators which have an H^∞ functional calculus, Miniconference on Operator Theory and Partial Differential Equations, Proc. Cent. Math. Analysis, A.N.U., Canberra 14 (1986), 210-231.

[24] A. McIntosh and A. Yagi, Operators of type ω without a bounded H^∞ functional calculus, Miniconference on Operators in Analysis, Proc. Cent. Math. Analysis, A.N.U., Canberra 24 (1989), 159-172.

[25] G. Pisier, Some results on Banach spaces without local unconditional structure, Compositio Math. 37 (1978), 3-19.

[26] G. Pisier, Factorization of linear operators and geometry of Banach spaces, CBMS regional conference series in mathematics, A.M.S., number 60 (1986).

[27] G. Pisier, Quotients of Banach spaces of cotype q, Proc. Amer. Math. Soc. 85 (1982), 32-36.

[28] G. Pisier, Les inégalités de Khintchine-Kahane d'après C. Borell, Séminaire sur la géométrie des espaces de Banach (1977-78), Exposé n° 7, Ecole Polytechnique.

[29] J. Prüss, Evolutionary Integral Equations and Applications, Birkhäuser Verlag, Monographs in Mathematics vol. 87 (1993).

[30] J. Prüss and H. Sohr, On operators with bounded imaginary powers in Banach spaces, Math. Z. 203 (1990), 429-452.