Estimation risk for the VaR of portfolios driven by semi-parametric multivariate models

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Objectives

- **Setup:**
  - A portfolio of assets with time-varying composition,
  - the vector of individual returns follows a general dynamic model.
- **Aims:**
  - Estimate the conditional risk of the portfolio (market risk).
  - Evaluate the accuracy of the estimation (model risk):
    $\Rightarrow$ quantify simultaneously the market and estimation risks.
  - Compare univariate and multivariate approaches.
    - Crystallized portfolios;
    - Optimal (conditional) mean-variance portfolios;
    - Minimal VaR porfolios.

Risk factors

- $\boldsymbol{p}_t = (p_{1t}, \dots, p_{mt})'$ vector of prices of m assets
- $\boldsymbol{y}_t = (y_{1t}, \dots, y_{mt})'$ vector of log-returns, $y_{it} = \log(p_{it}/p_{i,t-1})$
- V_t value of a portfolio composed of $\mu_{i,t-1}$ units of asset i, for $i = 1, \dots, m$:

$$V_t = \sum_{i=1}^{m} \mu_{i,t-1} p_{it}$$

- Self-financing constraint: At date t, the investor may rebalance his portfolio in such a way that

SF $\sum_{i=1}^{m} \mu_{i,t-1} p_{it} = \sum_{i=1}^{m} \mu_{i,t} p_{it}$

Return of the portfolio

Under SF, the return of the portfolio over the period $[t-1,t]$, assuming $V_{t-1} \neq 0$, is

$$\frac{V_t}{V_{t-1}} - 1 = \sum_{i=1}^{m} a_{i,t-1} \exp(y_{it}) - 1 \approx r_t$$

where

$$r_t = \sum_{i=1}^{m} a_{i,t-1} y_{it} = \mathbf{a}'_{t-1} \boldsymbol{y}_t,$$

with

$$a_{i,t-1} = \frac{\mu_{i,t-1} p_{i,t-1}}{\sum_{j=1}^{m} \mu_{j,t-1} p_{j,t-1}}, \quad i = 1, \ldots, m,$$

and $\mathbf{a}_{t-1} = (a_{1,t-1}, \ldots, a_{m,t-1})', \quad \boldsymbol{y}_t = (y_{1t}, \ldots, y_{mt})'$.

Conditional VaR of the portfolio's return

The *conditional* VaR of the portfolio's return r_t at risk level $\alpha \in (0,1)$ is defined by

$$P_{t-1}\left[r_t < -\mathrm{VaR}_{t-1}^{(\alpha)}(r_t)\right] = \alpha$$

where P_{t-1} denotes the historical distribution conditional on $\{\boldsymbol{p}_u, u < t\}$.

Consequence

Evaluation of the conditional VaR can be achieved by a

- Multivariate approach:
  dynamic model for the vector of risk factors $\boldsymbol{y}_t$
- Univariate approach:
  dynamic model for the portfolio's return r_t

Dynamic model for the vector of log-returns

Multivariate model with GARCH-type errors:

$$\boldsymbol{y}_t = \boldsymbol{m}_t(\boldsymbol{\theta}_0) + \boldsymbol{\epsilon}_t, \qquad \boldsymbol{\epsilon}_t = \boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0)\boldsymbol{\eta}_t$$

where $\boldsymbol{\eta}_t \overset{iid}{\sim} (\boldsymbol{0}, \boldsymbol{I}_m)$, $\quad \boldsymbol{\theta}_0 \in \mathbb{R}^d$

$$\boldsymbol{m}_t(\boldsymbol{\theta}_0) = \boldsymbol{m}(\boldsymbol{y}_{t-1}, \boldsymbol{y}_{t-2}, \dots, \boldsymbol{\theta}_0), \qquad \boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0) = \boldsymbol{\Sigma}(\boldsymbol{y}_{t-1}, \boldsymbol{y}_{t-2} \dots, \boldsymbol{\theta}_0).$$

Examples of MGARCH

Thus

$$r_t = \mathbf{a}'_{t-1}\boldsymbol{m}_t(\boldsymbol{\theta}_0) + \mathbf{a}'_{t-1}\boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0)\boldsymbol{\eta}_t,$$

and

$$\text{VaR}_{t-1}^{(\alpha)}(r_t) = -\mathbf{a}'_{t-1}\boldsymbol{m}_t(\boldsymbol{\theta}_0) + \text{VaR}_{t-1}^{(\alpha)}\left(\mathbf{a}'_{t-1}\boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0)\boldsymbol{\eta}_t\right).$$

A simplification for elliptic conditional distributions

$$\boldsymbol{\epsilon}_t = \boldsymbol{m}_t(\boldsymbol{\theta}_0) + \boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0)\boldsymbol{\eta}_t, \qquad (\boldsymbol{\eta}_t) \text{ iid } (\mathbf{0}, \boldsymbol{I}_m),$$

Assume that the errors $\boldsymbol{\eta}_t$ have a spherical distribution:

A1: for any non-random vector $\boldsymbol{\lambda} \in \mathbb{R}^m$, $\boldsymbol{\lambda}'\boldsymbol{\eta}_t \stackrel{d}{=} \|\boldsymbol{\lambda}\|\eta_{1t}$

where $\|\cdot\|$ is the euclidean norm on $\mathbb{R}^m$.

Remark: means that the conditional law of $\boldsymbol{\epsilon}_t$ is elliptic.

Under **A1**

$$\text{VaR}_{t-1}^{(\alpha)}(r_t) = -\mathbf{a}_{t-1}'\boldsymbol{m}_t(\boldsymbol{\theta}_0) + \left\|\mathbf{a}_{t-1}'\boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0)\right\| \text{VaR}^{(\alpha)}(\eta),$$

where $\text{VaR}^{(\alpha)}(\eta)$ is the (marginal) VaR of η_{1t}.

Example of spherical distributions

Assumption on the conditional variance model

B1: There exists a continuously differentiable function $\boldsymbol{G}:\mathbb{R}^d \mapsto \mathbb{R}^d$ such that for any $\boldsymbol{\theta} \in \Theta$, any $K > 0$, and any sequence $(\mathbf{x}_i)_i$ on $\mathbb{R}^m$,

$$K\boldsymbol{\Sigma}(\mathbf{x}_1,\mathbf{x}_2,\ldots;\boldsymbol{\theta}) = \boldsymbol{\Sigma}(\mathbf{x}_1,\mathbf{x}_2,\ldots;\boldsymbol{\theta}^*), \quad \text{and}$$

$$m(\mathbf{x}_1,\mathbf{x}_2,\ldots;\boldsymbol{\theta}) = m(\mathbf{x}_1,\mathbf{x}_2,\ldots;\boldsymbol{\theta}^*)$$

where $\boldsymbol{\theta}^* = \boldsymbol{G}(\boldsymbol{\theta},K)$.

▸ Examples of the CCC and DCC-GARCH

VaR parameter for an elliptic conditional distribution

At the risk level $\alpha \in (0, 0.5)$, the conditional VaR of the portfolio's return is

$$\begin{aligned} \text{VaR}_{t-1}^{(\alpha)}(r_t) &= -\mathbf{a}_{t-1}' \boldsymbol{m}_t(\boldsymbol{\theta}_0) + \text{VaR}_{t-1}^{(\alpha)}\left(\mathbf{a}_{t-1}' \boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0) \boldsymbol{\eta}_t\right) \\ &= -\mathbf{a}_{t-1}' \boldsymbol{m}_t(\boldsymbol{\theta}_0) + \left\|\mathbf{a}_{t-1}' \boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0)\right\| \text{VaR}^{(\alpha)}(\eta) \\ &= -\mathbf{a}_{t-1}' \boldsymbol{m}_t(\boldsymbol{\theta}_0^*) + \|\mathbf{a}_{t-1}' \boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0^*)\|, \end{aligned}$$

where, under **B1**,

$$\boldsymbol{\theta}_0^* = G\left(\boldsymbol{\theta}_0, \text{VaR}^{(\alpha)}(\eta)\right).$$

The parameter $\boldsymbol{\theta}_0^*$ can be called conditional VaR parameter.

Remark: The conditional VaR parameter

- does not depend on the portfolio composition
- summarizes the risk at a given level

Estimating the conditional VaR parameter

- Observations: $\boldsymbol{y}_1,\dots,\boldsymbol{y}_n$ (+ initial values $\tilde{\boldsymbol{y}}_0,\tilde{\boldsymbol{y}}_{-1},\dots$).
- $\widehat{\boldsymbol{\theta}}_n$: estimator of $\boldsymbol{\theta}_0$
- $\tilde{\boldsymbol{m}}_t(\boldsymbol{\theta}) = \boldsymbol{m}(\boldsymbol{y}_{t-1},\dots,\boldsymbol{y}_1,\tilde{\boldsymbol{y}}_0,\tilde{\boldsymbol{y}}_{-1},\dots,\boldsymbol{\theta})$
  $\tilde{\boldsymbol{\Sigma}}_t(\boldsymbol{\theta}) = \boldsymbol{\Sigma}(\boldsymbol{y}_{t-1},\dots,\boldsymbol{y}_1,\tilde{\boldsymbol{y}}_0,\tilde{\boldsymbol{y}}_{-1},\dots,\boldsymbol{\theta})$
- Residuals: $\widehat{\boldsymbol{\eta}}_t = \tilde{\boldsymbol{\Sigma}}_t^{-1}(\widehat{\boldsymbol{\theta}}_n)\{\boldsymbol{y}_t - \tilde{\boldsymbol{m}}_t(\widehat{\boldsymbol{\theta}}_n)\} = (\widehat{\eta}_{1t},\dots,\widehat{\eta}_{mt})'$.

Under the sphericity assumption,

$$\widehat{\mathrm{VaR}}^{(\alpha)}_{S,t-1}(r) = -\mathbf{a}'_{t-1}\tilde{\boldsymbol{m}}_t(\widehat{\boldsymbol{\theta}}_n) + \|\mathbf{a}'_{t-1}\tilde{\boldsymbol{\Sigma}}_t(\widehat{\boldsymbol{\theta}}_n)\|\widehat{\mathrm{VaR}}^{(\alpha)}_n(\eta)$$

where $\widehat{\mathrm{VaR}}^{(\alpha)}_n(\eta) = \xi_{n,1-2\alpha}$
is the $(1-2\alpha)$-quantile of $\{|\widehat{\eta}_{it}|, 1 \le i \le m, 1 \le t \le n\}$.

Assumptions

A2: $(\boldsymbol{y}_t)$ is a strictly stationary and nonanticipative solution.

A3: We have $\widehat{\boldsymbol{\theta}}_n \to \boldsymbol{\theta}_0$, a.s. and the following expansion

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0\right) \stackrel{o_P(1)}{=} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \boldsymbol{\Delta}_{t-1} \boldsymbol{V}(\boldsymbol{\eta}_t),$$

where $\boldsymbol{\Delta}_{t-1} \in \mathscr{F}_{t-1}$, $\boldsymbol{V} : \mathbb{R}^m \mapsto \mathbb{R}^K$ for some $K \geq 1$, $E\boldsymbol{V}(\boldsymbol{\eta}_t) = 0$, var$\{\boldsymbol{V}(\boldsymbol{\eta}_t)\} = \boldsymbol{\Upsilon}$ is nonsingular and $E\boldsymbol{\Delta}_t = \boldsymbol{\Lambda}$ is full row rank.

Example of the Gaussian QML

A4: The functions $\boldsymbol{\theta} \mapsto \boldsymbol{m}(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots; \boldsymbol{\theta})$ and $\boldsymbol{\theta} \mapsto \boldsymbol{\Sigma}(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots; \boldsymbol{\theta})$ are $\mathscr{C}^1$.

A5: $|\eta_{1t}|$ has a density f which is continuous and strictly positive in a neighborhood of $\xi_{1-2\alpha}$ (the $(1-2\alpha)$-quantile of $|\eta_{1t}|$).

Asymptotic distribution

Asymptotic normality

$$\sqrt{n}\begin{pmatrix} \hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0 \\ \xi_{n,1-2\alpha} - \xi_{1-2\alpha} \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Xi} := \begin{pmatrix} \boldsymbol{\Psi} & \boldsymbol{\Xi}_{\boldsymbol{\theta}\xi} \\ \boldsymbol{\Xi}'_{\boldsymbol{\theta}\xi} & \zeta_{1-2\alpha} \end{pmatrix}\right),$$

where $\boldsymbol{\Omega}' = E\left[\{\text{vec}(\boldsymbol{\Sigma}_t^{-1})\}'\left\{\frac{\partial}{\partial \boldsymbol{\vartheta}'}\text{vec}(\boldsymbol{\Sigma}_t)\right\}\right]$, $\boldsymbol{W}_\alpha = \text{Cov}(\boldsymbol{V}(\boldsymbol{\eta}_t), N_t)$, $\gamma_\alpha = \text{var}(N_t)$, with $N_t = \sum_{j=1}^m \mathbf{1}_{\{|\eta_{jt}|<\xi_{1-2\alpha}\}} - 1 + 2\alpha$, and

$$\boldsymbol{\Xi}_{\boldsymbol{\theta}\xi} = \frac{-1}{m}\left\{\xi_{1-2\alpha}\boldsymbol{\Psi}\boldsymbol{\Omega} + \frac{1}{f(\xi_{1-2\alpha})}\boldsymbol{\Lambda}\boldsymbol{W}_\alpha\right\}, \quad \boldsymbol{\Psi} = E(\boldsymbol{\Delta}_t\boldsymbol{\Upsilon}\boldsymbol{\Delta}_t')$$

$$\zeta_{1-2\alpha} = \frac{1}{m^2}\left\{\xi_{1-2\alpha}^2\boldsymbol{\Omega}'\boldsymbol{\Psi}\boldsymbol{\Omega} + \frac{2\xi_{1-2\alpha}}{f(\xi_{1-2\alpha})}\boldsymbol{\Omega}'\boldsymbol{\Lambda}\boldsymbol{W}_\alpha + \frac{\gamma_\alpha}{f^2(\xi_{1-2\alpha})}\right\}.$$

Estimation of the asymptotic variance

- Most quantities involved in the asymptotic covariance matrix Ξ can be estimated by empirical means.
- The estimation of

$$\mathbf{\Omega}' = E\left[\left\{\text{vec}\left(\mathbf{\Sigma}_t^{-1}\right)\right\}'\left\{\frac{\partial}{\partial \boldsymbol{\vartheta}'}\text{vec}\left(\mathbf{\Sigma}_t\right)\right\}\right]$$

can be delicate due to the presence of the derivatives of $\mathbf{\Sigma}_t$.

▸ Example: linear SRE on the derivatives of H_t

Asymptotic normality of the VaR-parameter estimator

VaR-parameter: $\boldsymbol{\theta}_0^* = G\left(\boldsymbol{\theta}_0, \mathrm{VaR}^{(\alpha)}(\eta)\right)$

A simple application of the delta method gives the asymptotic distribution of the estimator

$$\widehat{\boldsymbol{\theta}}_n^* = G\left\{\widehat{\boldsymbol{\theta}}_n, \widehat{\mathrm{VaR}}_n^{(\alpha)}(\eta)\right\}.$$

VaR parameter

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}}_n^* - \boldsymbol{\theta}_0^*\right) \xrightarrow{\mathscr{L}} \mathscr{N}\left(\mathbf{0}, \Xi^* := \dot{\boldsymbol{G}}\Xi\dot{\boldsymbol{G}}'\right)$$

with

$$\dot{\boldsymbol{G}} = \left[\frac{\partial G(\boldsymbol{\theta}, \xi)}{\partial(\boldsymbol{\theta}', \xi)}\right]_{(\boldsymbol{\theta}_0, \xi_{1-2\alpha})}.$$

Evaluation of the estimation risk

$$\widehat{\mathrm{VaR}}^{(\alpha)}_{S,t-1}(r) = -\mathbf{a}'_{t-1}\widetilde{\boldsymbol{m}}_t(\widehat{\boldsymbol{\theta}}_n) + \|\mathbf{a}'_{t-1}\widetilde{\boldsymbol{\Sigma}}_t(\widehat{\boldsymbol{\theta}}_n)\|\widehat{\mathrm{VaR}}^{(\alpha)}_n(\eta)$$

An asymptotic $(1-\alpha_0)\%$ confidence interval for $\mathrm{VaR}_t(\alpha)$ has bounds given by

$$\widehat{\mathrm{VaR}}^{(\alpha)}_{S,t-1}(r_t) \pm \frac{1}{\sqrt{n}}\Phi^{-1}_{1-\alpha_0/2}\left\{\boldsymbol{\delta}'_{t-1}\widehat{\boldsymbol{\Xi}}\boldsymbol{\delta}_{t-1}\right\}^{1/2},$$

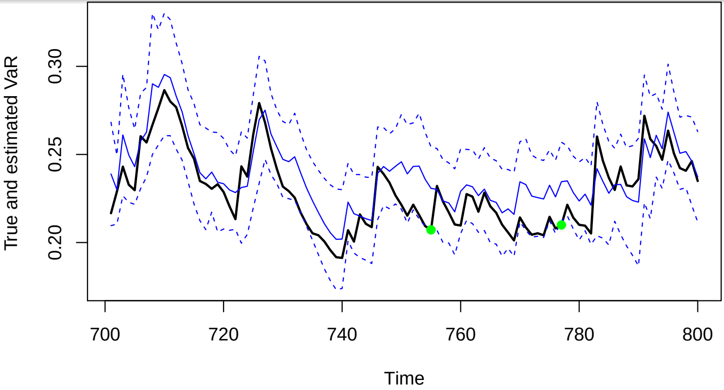
where

$$\boldsymbol{\delta}'_{t-1} = \left[\mathbf{a}'_{t-1}\frac{\partial\widetilde{\boldsymbol{m}}(\widehat{\boldsymbol{\theta}}_n)}{\partial\boldsymbol{\theta}'} + \frac{(\mathbf{a}_{t-1}\otimes\mathbf{a}_{t-1})'}{2\|\mathbf{a}'_{t-1}\widetilde{\boldsymbol{\Sigma}}_t(\widehat{\boldsymbol{\theta}}_n)\|}\frac{\partial \mathrm{vec}\widetilde{H}_t(\widehat{\boldsymbol{\theta}}_n)}{\partial\boldsymbol{\theta}'} \qquad \|\mathbf{a}'_{t-1}\widetilde{\boldsymbol{\Sigma}}_t(\widehat{\boldsymbol{\theta}}_n)\|\right],$$

with $\widetilde{\boldsymbol{H}}_t(\cdot) = \widetilde{\boldsymbol{\Sigma}}_t(\cdot)\widetilde{\boldsymbol{\Sigma}}'_t(\cdot)$.

Remark: The statistical estimation risk α_0 is not related to the financial risk α.

Accuracy intervals for the estimated conditional VaR



1%-VaR (**true** in full black line, estimated in full blue line) and estimated 95%-confidence intervals (dotted blue line) on a simulation of a fixed portfolio of a bivariate BEKK (700 values for the estimation of the VaR parameter).

Filtered Historical Simulation (FHS) approach

Barone-Adesi et al. (J. of Future Markets, 1999), Mancini and Trojani (JFE, 2011)

Relies on

i) interpreting the conditional VaR as the α-quantile of a linear combination (depending on t) of the components of $\boldsymbol{\eta}_t$:

$$\mathrm{VaR}_{t-1}^{(\alpha)}(r_t) = \mathrm{VaR}_{t-1}^{(\alpha)}\left\{b_t(\boldsymbol{\theta}_0) + \boldsymbol{c}_t'(\boldsymbol{\theta}_0)\boldsymbol{\eta}_t\right\}$$

where $b_t(\boldsymbol{\theta}) = \mathbf{a}_{t-1}'\boldsymbol{m}_t(\boldsymbol{\theta})$ and $\boldsymbol{c}_t'(\boldsymbol{\theta}) = \mathbf{a}_{t-1}'\boldsymbol{\Sigma}_t(\boldsymbol{\theta})$.

ii) replacing $\boldsymbol{\eta}_t$ by the GARCH residuals $\widehat{\boldsymbol{\eta}}_s$ and computing the empirical α-quantile of the estimated linear combination.

$$\widehat{\mathrm{VaR}}_{FHS,t-1}^{(\alpha)}(r) = -q_\alpha\left(\{b_t(\widehat{\boldsymbol{\theta}}_n) + \boldsymbol{c}_t'(\widehat{\boldsymbol{\theta}}_n)\widehat{\boldsymbol{\eta}}_s, \quad 1 \le s \le n\}\right).$$

Remark: for each value of s, $b_t(\widehat{\boldsymbol{\theta}}_n) + \boldsymbol{c}_t'(\widehat{\boldsymbol{\theta}}_n)\widehat{\boldsymbol{\eta}}_s$ is a simulated value of the return r_t conditional on the past prices.

Notations and assumptions

Let $\boldsymbol{c}:\boldsymbol{\Theta}\mapsto\mathbb{R}^m$ and $b:\boldsymbol{\Theta}\mapsto\mathbb{R}$ be $\mathscr{C}^1$ functions.

$\xi_\alpha(\boldsymbol{\theta})$: α-quantile of $b(\boldsymbol{\theta})+\boldsymbol{c}'(\boldsymbol{\theta})\boldsymbol{\eta}_t(\boldsymbol{\theta})$,

$\xi_{n,\alpha}(\boldsymbol{\theta})$: empirical α-quantile of $\{b(\boldsymbol{\theta})+\boldsymbol{c}'(\boldsymbol{\theta})\boldsymbol{\eta}_t(\boldsymbol{\theta}),1\leq t\leq n\}$.

Suppose $\xi_\alpha(\boldsymbol{\theta}_0)>0$ and $\boldsymbol{c}'(\boldsymbol{\theta}_0)\boldsymbol{\eta}_t$ admits a density f_c which is continuous and strictly positive in a neighborhood of $x_0=-b(\boldsymbol{\theta}_0)+\xi_\alpha(\boldsymbol{\theta}_0)$.

Asymptotic distribution

Estimator of the quantile of a linear combination of η_t

Under the previous assumptions (but without the sphericity assumption **A1**),

$$\sqrt{n}\{\xi_{n,\alpha}(\widehat{\boldsymbol{\theta}}_n)-\xi_\alpha(\boldsymbol{\theta}_0)\} \stackrel{\mathscr{L}}{\to} \mathscr{N}\left(0,\sigma^2:=\boldsymbol{\omega}'\boldsymbol{\Psi}\boldsymbol{\omega}+2\boldsymbol{\omega}'\boldsymbol{\Lambda}\boldsymbol{A}_\alpha+\frac{\alpha(1-\alpha)}{f_{\boldsymbol{c}}^2(x_0)}\right),$$

where $\boldsymbol{A}_\alpha=\mathrm{Cov}(\boldsymbol{V}(\boldsymbol{\eta}_t),\mathbf{1}_{\{b(\boldsymbol{\theta}_0)-\boldsymbol{c}'(\boldsymbol{\theta}_0)\boldsymbol{\eta}_t<\xi_\alpha(\boldsymbol{\theta}_0)\}})$,

$$\boldsymbol{\omega}'=\left[\boldsymbol{c}'(\boldsymbol{\theta}_0)E(\boldsymbol{C}_t)-\frac{\partial b}{\partial\boldsymbol{\theta}'}(\boldsymbol{\theta}_0)\quad \boldsymbol{d}_\alpha'\left\{(\boldsymbol{c}'(\boldsymbol{\theta}_0)\otimes\boldsymbol{I}_m)E(\boldsymbol{\Omega}_t^*)-\frac{\partial\boldsymbol{c}}{\partial\boldsymbol{\theta}'}(\boldsymbol{\theta}_0)\right\}\right],$$

$\boldsymbol{d}_\alpha=E(\boldsymbol{\eta}_t\mid b(\boldsymbol{\theta}_0)+\boldsymbol{c}'(\boldsymbol{\theta}_0)\boldsymbol{\eta}_t=\xi_\alpha(\boldsymbol{\theta}_0))$,
$\boldsymbol{\Omega}_t^*$ and $\boldsymbol{C}_t$ are matrices involving the derivatives of $\boldsymbol{\Sigma}_t$ and $\boldsymbol{m}_t$.

1. General framework
2. Estimating the conditional VaR
   - Multivariate estimation under ellipticity
   - Relaxing the ellipticity assumption
   - Univariate approaches
3. Numerical comparison of the different VaR estimators

Two univariate approaches

- Naive approach: estimate a univariate GARCH model on the series of portfolio returns.
  Generally invalid due to the time-varying combination of the individual returns.
- Virtual Historical Simulation (VHS): reconstitute a "virtual portfolio" whose returns are built using the current composition of the portfolio.

Invalidity of the naive univariate approach

- For crystallized portfolios ($\mu_{i,t-1} = \mu_i, \forall i, \forall t$), in general

$$P(\boldsymbol{a}_{t-1} \in \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}) \to 1 \quad \text{as } t \to \infty.$$

  The composition tends to be totally undiversified, but is not always close to the same single-asset composition $\boldsymbol{e}_i$.

  ▸ Illustration of the nonstationarity

  In general, the naive method based on a fixed stationary model for r_t will produce poor results.

- For static portfolios ($a_{i,t-1} = a_i$ for all i and t) the non stationarity issue vanishes.

  However, on simulated series, multivariate models outperform univariate models for estimating the VaR's of static portfolios.

Virtual Historical Simulation

Given the current portfolio composition $\boldsymbol{a}_{t-1} = \boldsymbol{x}$, we construct a (stationary) series of virtual returns mimicking the current return

$$r_s^*(\boldsymbol{x}) = \boldsymbol{x}'\boldsymbol{y}_s \qquad s \in \mathbb{Z}.$$

We have a model of the form

$$r_s^*(\boldsymbol{x}) = \mu_s(\boldsymbol{x}) + \sigma_s(\boldsymbol{x})u_s, \qquad E_{s-1}(u_s) = 0, \quad \text{var}_{s-1}(u_s) = 1.$$

The conditional VaR thus satisfies

$$\text{VaR}_{t-1}^{(\alpha)}(r_t) = -\mu_t(\boldsymbol{a}_{t-1}) + \sigma_t(\boldsymbol{a}_{t-1})\text{VaR}_{t-1}^{(\alpha)}(u_t)$$

STEP 1: Compute the virtual returns $r_s^*(\boldsymbol{x})$ for $s = 1, \dots, n$.
STEP 2: Estimate $\mu_s(\boldsymbol{x})$ and $\sigma_s(\boldsymbol{x})$. Let $\hat{u}_s = \{r_s^*(\boldsymbol{x}) - \hat{\mu}_s(\boldsymbol{x})\}/\hat{\sigma}_s(\boldsymbol{x})$.
STEP 3: Compute the α-quantile $\xi_{n,\alpha}^u(\boldsymbol{x})$ of $\{\hat{u}_s, 1 \le s \le n\}$ and let

$$\widehat{\text{VaR}}_{VHS,t-1}^{(\alpha)}(r) = -\hat{\mu}_t(\boldsymbol{x}) - \hat{\sigma}_t(\boldsymbol{x})\xi_{n,\alpha}^u(\boldsymbol{x}).$$

Remarks on Step 2: estimation of a univariate model for the virtual returns

- To obtain asymptotic properties of the procedure, we make parametric assumptions on the univariate model:

$$\sigma_s(\boldsymbol{x};\boldsymbol{\varrho}) = \sigma(r^*_{s-1}(\boldsymbol{x}), r^*_{s-2}(\boldsymbol{x}), \ldots; \boldsymbol{\varrho}),$$

- In general, a multivariate GARCH-type model for $\boldsymbol{y}_t$ is not compatible with a univariate GARCH for $r^*_s(\boldsymbol{x}) = \boldsymbol{x}'\boldsymbol{y}_s$.
  - Due to the fact that the conditional distribution of $r^*_s(\boldsymbol{x})$ is not only a function of the past virtual returns.
  - If a GARCH(1,1) is used in Step 2, it will generally be an approximation.
- Under the sphericity assumption **A1**, (u_t) is i.i.d.

1. General framework
2. Estimating the conditional VaR
3. Numerical comparison of the different VaR estimators
   - On dynamic portfolios
   - On portfolios of exchange rates
   - Appendix

Simulation designs

- Different cDCC-GARCH(1,1) models for $m=2$ assets. Designs
- For the Minimum variance portfolio Illustration

$$r_t^* = \boldsymbol{\epsilon}_t' \boldsymbol{a}_{t-1}^*, \quad \boldsymbol{a}_{t-1}^* = \frac{\boldsymbol{\Sigma}_t^{-2}(\boldsymbol{\theta}_0)\boldsymbol{e}}{\boldsymbol{e}'\boldsymbol{\Sigma}_t^{-2}(\boldsymbol{\theta}_0)\boldsymbol{e}},$$

  the true conditional VaR is explicit under sphericity, and is evaluated by means of simulations otherwise.
- $N=100$ independent simulations of the cDCC-GARCH(1,1) model.
  - First $n_1 = 1000$ observations: estimation of $\boldsymbol{\theta}_0$ + empirical quantiles of the residuals.
  - Last $n - n_1 = 1000$ simulations: comparison of the theoretical conditional VaR's of the portfolio with the three estimates (spherical, FHS and VHS methods).

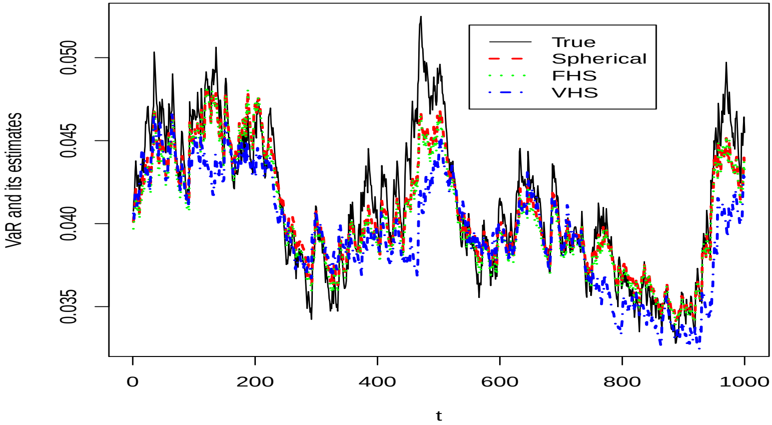
Empirical Relative Efficiency

Table: Relative efficiency of the Spherical method with respect to the FHS method (S/F) and with respect to the VHS method (S/V).

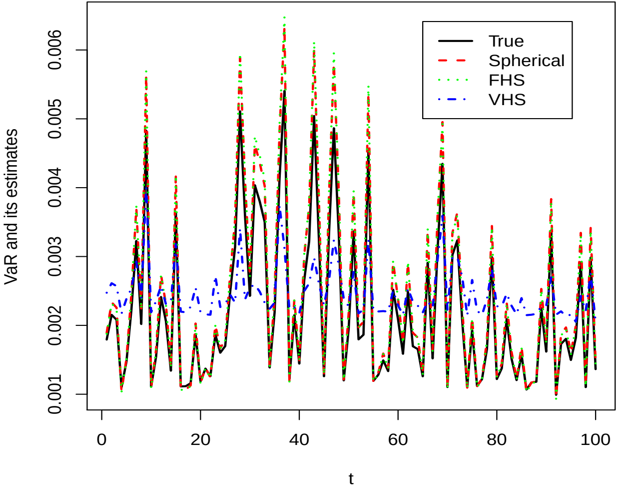
n_1	α		A	B	C	D	E	F	G	H	BEKK
1000	1%	S/F	1.30	1.11	2.35	1.62	1.53	1.51	1.57	1.36	1.41
		S/V	91.6	23.4	303.	79.8	1.93	2.53	4.43	2.23	8.27
	5%	S/F	1.14	1.03	2.07	1.00	1.25	1.08	1.33	1.01	1.13
		S/V	55.4	15.7	267.	82.5	1.75	2.44	4.14	2.01	8.23
			A*	B*	C*	D*	E*	F*	G*	H*	BEKK*
1000	1%	S/F	0.08	0.03	0.02	0.02	0.06	0.03	0.03	0.04	0.05
		S/V	2.20	2.43	2.31	1.67	0.05	0.04	0.07	0.06	0.50
	5%	S/F	0.34	0.19	0.09	0.11	0.30	0.24	0.21	0.29	0.34
		S/V	3.78	6.68	10.2	8.72	0.26	0.35	0.59	0.44	2.65

A-H: Spherical innovations; A*-H*: Non spherical innovations

The two components follow persistent volatility models



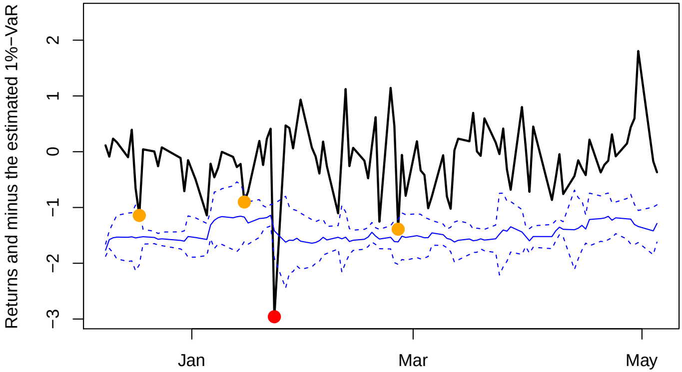
Two very different volatility models for the two components (design A)



Daily returns of exchange rates against the Euro

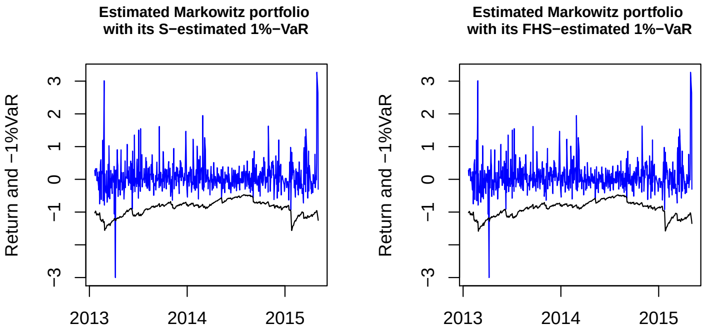
- Canadian Dollar (CAD), Chinese Yuan (CNY), British Pound (GBP), Japanese Yen (JPY) and US Dollar (USD).
- January 14, 2000 to May 5, 2015 ($n = 2582$).
- 2 settings
  - A BEKK model estimated over the whole sample except the last 100 returns. Equally-weighted crystalized portfolio ($\mu_i = 1$ for $i = 1, \ldots, 5$). VaR estimates based on sphericity.
  - DCCC GARCH(1,1) model on the first 2000 observations with estimated minimum-variance portfolio. Backtesting (unconditional coverage, independence of violations, conditional coverage).

Equally-weighted portfolio of 5 exchange rates



Returns for the period 09/12/2014 to 05/05/2015, estimated 1%- VaR and 95%-confidence interval based on the estimation of a BEKK model.

Minimum-variance portfolio of 5 exchange rates



Returns of estimated minimum-variance portfolios of 5 exchange rates and their estimated VaR's.

Backtests

Christoffersen (2003), Escanciano and Olmo (2010, 2011)

Table: p-values of three backtests for minimum-variance portfolios

Method	α	% of Viol	UC	IND	CC
Spherical	1%	2/582	0.065	0.906	0.182
FHS	1%	2/582	0.065	0.906	0.182
Spherical	5%	20/582	0.067	0.232	0.092
FHS	5%	18/582	0.023	0.283	0.043

Conclusions: univariate approaches

- Not always a good idea to fit a stationary univariate GARCH model on portfolios returns:
  - does not exploit the multivariate dynamics of the risk factors;
  - the naive approach (based on a fixed stationary model) is generally inconsistent when the composition of the portfolio is time-varying;
  - The VHS approach circumvents the non stationarity problem but
    - is generally found inefficient in simulations compared to the multivariate approaches,
    - is not necessarily simpler to implement (GARCH models have to be re-estimated at any date and for any portfolio composition),
    - does not allow to choose optimally the weights of the portfolio.

Conclusions: multivariate approaches

- For both approaches, asymptotic CIs for the conditional VaR can be built.
  $\Rightarrow$ allows to visualize on the same graph both market and estimation risks.
- Exploiting the sphericity simplifies estimation and also gives more accurate VaRs when this assumption holds.
- The method based on sphericity may yield inconsistent VaR estimators when this assumption is in failure.
- The FHS method performs well in both cases and outperforms the first approach in the absence of sphericity.

Conclusions: multivariate approaches

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Thanks for your attention!

Vector GARCH model

$$\begin{aligned} \boldsymbol{\epsilon}_t &= \boldsymbol{H}_t^{1/2}\boldsymbol{\eta}_t, \quad \boldsymbol{H}_t \text{ positive definite, } (\boldsymbol{\eta}_t) \text{ iid } (\boldsymbol{0},\boldsymbol{I}) \\ \text{vech}(\boldsymbol{H}_t) &= \boldsymbol{\omega} + \sum_{i=1}^{q} \boldsymbol{A}^{(i)}\text{vech}(\boldsymbol{\epsilon}_{t-i}\boldsymbol{\epsilon}'_{t-i}) + \sum_{j=1}^{p} \boldsymbol{B}^{(j)}\text{vech}(\boldsymbol{H}_{t-j}) \end{aligned}$$

- The most direct generalization of univariate GARCH
- Positivity conditions are difficult to obtain
- No explicit stationarity conditions

BEKK-GARCH model

Engle and Kroner (1995), Comte and Lieberman (2003)

$$\begin{cases} \boldsymbol{\epsilon}_t = \boldsymbol{H}_t^{1/2}\boldsymbol{\eta}_t, & (\boldsymbol{\eta}_t) \text{ iid } (\boldsymbol{0}, \boldsymbol{I}) \\ \boldsymbol{H}_t = \boldsymbol{\Omega} + \sum_{i=1}^{q}\sum_{k=1}^{K} \boldsymbol{A}_{ik}\boldsymbol{\epsilon}_{t-i}\boldsymbol{\epsilon}'_{t-i}\boldsymbol{A}'_{ik} + \sum_{j=1}^{p}\sum_{k=1}^{K} \boldsymbol{B}_{jk}\boldsymbol{H}_{t-j}\boldsymbol{B}'_{jk} \end{cases}$$

- Coefficients of a BEKK representation are difficult to interpret
- Positivity conditions are simple. Identifiability of a BEKK representation requires additional constraints.
- Stationarity conditions exist (Boussama, Fuchs, Stelzer, 2011) but no explicit solution can be exhibited

Constant Conditional Correlation (CCC) model

Bollerslev (1990); Extended CCC by Jeantheau (1998)

$$\underline{\boldsymbol{h}}_t = \begin{pmatrix} h_{11,t} \\ \vdots \\ h_{mm,t} \end{pmatrix}, \quad \boldsymbol{D}_t = \text{diag}\left(h_{11,t}^{1/2}, \dots, h_{mm,t}^{1/2}\right), \quad \underline{\boldsymbol{\epsilon}}_t = \begin{pmatrix} \epsilon_{1t}^2 \\ \vdots \\ \epsilon_{mt}^2 \end{pmatrix}.$$

$$\begin{cases} \boldsymbol{\epsilon}_t = \boldsymbol{H}_t^{1/2}\boldsymbol{\eta}_t, \qquad \boldsymbol{H}_t = \boldsymbol{D}_t\boldsymbol{R}\boldsymbol{D}_t, \quad \boldsymbol{R}: \text{correlation matrix} \\ \underline{\boldsymbol{h}}_t = \boldsymbol{\omega} + \sum_{i=1}^{q} \mathbf{A}_i \underline{\boldsymbol{\epsilon}}_{t-i} + \sum_{j=1}^{p} \mathbf{B}_j \underline{\boldsymbol{h}}_{t-j} \end{cases}$$

- Simple conditions ensuring the positive definiteness of H_t.
- Explicit stationarity condition (of the form $\gamma < 0$...)
- The assumption of CCC can be too restrictive

Dynamic Conditional Correlation (DCC) model

Engle (2002)

$$H_t = D_t R_t D_t, \qquad R_t = (\operatorname{diag} Q_t)^{-1/2} Q_t (\operatorname{diag} Q_t)^{-1/2},$$

where $\eta_t^* = D_t^{-1} \epsilon_t$ and

$$Q_t = (1 - \alpha - \beta) S + \alpha \eta_{t-1}^* \eta_{t-1}^{*'} + \beta Q_{t-1},$$

where $\alpha, \beta \geq 0, \alpha + \beta < 1$, S is a correlation matrix

- The existence of strictly stationary solution is a complex issue (recent PhD thesis by Malongo, 2014)
- No asymptotic theory of estimation exists
- Incorrect interpretation of S as $\operatorname{Var}(\eta_t^*)$ and Q_t as $\operatorname{Var}_{t-1}(\eta_t^*)$.

Dynamic Conditional Correlation (DCC) model

Corrected DCC (Aielli (2013)

$$\boldsymbol{Q}_t = (1-\alpha-\beta)\boldsymbol{S} + \alpha \boldsymbol{Q}_{t-1}^{*1/2}\boldsymbol{\eta}_{t-1}^{*}\boldsymbol{\eta}_{t-1}^{*'}\boldsymbol{Q}_{t-1}^{*1/2} + \beta \boldsymbol{Q}_{t-1},$$

where $\boldsymbol{Q}_t^* = \mathrm{diag}(\boldsymbol{Q}_t)$.

- Identifiability constraint: diag$(\boldsymbol{S}) = \boldsymbol{I}_m$.
- Parcimony but the $m(m-1)/2$ conditional correlations have the same dynamic structure.

Return

Example: Linear SRE on the derivatives of $\boldsymbol{H}_t$

BEKK-GARCH(1,1) model:

$$\boldsymbol{\epsilon}_t = \boldsymbol{H}_t^{1/2}\boldsymbol{\eta}_t, \qquad \boldsymbol{H}_t = \boldsymbol{C}_0 + \boldsymbol{A}_0\boldsymbol{\epsilon}_{t-1}\boldsymbol{\epsilon}'_{t-1}\boldsymbol{A}'_0 + \boldsymbol{B}_0\boldsymbol{H}_{t-1}\boldsymbol{B}'_0$$

Let $\boldsymbol{\theta} = (\text{vec}(\boldsymbol{A})', \text{vec}(\boldsymbol{B})', \text{vec}(\boldsymbol{C})')'$. For $j = 1, \dots, 3d$,

$$\begin{aligned}\frac{\partial \text{vec}(\boldsymbol{H}_t)}{\partial \theta_j} &= \frac{\partial \text{vec}(\boldsymbol{C})}{\partial \theta_j} + \frac{\partial (\boldsymbol{A}\otimes\boldsymbol{A})}{\partial \theta_j}\text{vec}(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}'_t) \\ &+ \frac{\partial (\boldsymbol{B}\otimes\boldsymbol{B})}{\partial \theta_j}\text{vec}(\boldsymbol{H}_{t-1}) + (\boldsymbol{B}\otimes\boldsymbol{B})\frac{\partial \text{vec}(\boldsymbol{H}_{t-1})}{\partial \theta_j},\end{aligned}$$

allows to compute recursively the derivatives of $\boldsymbol{H}_t$ (for some initial values).
We note that $\boldsymbol{\Sigma}_t \frac{\partial \boldsymbol{\Sigma}_t}{\partial \theta_i} + \frac{\partial \boldsymbol{\Sigma}_t}{\partial \theta_i}\boldsymbol{\Sigma}_t = \frac{\partial \boldsymbol{H}_t}{\partial \theta_i}$. Thus

$$(\boldsymbol{I}_m \otimes \boldsymbol{\Sigma}_t + \boldsymbol{\Sigma}_t \otimes \boldsymbol{I}_m)\,\text{vec}\left(\frac{\partial \boldsymbol{\Sigma}_t}{\partial \theta_i}\right) = \text{vec}\left(\frac{\partial \boldsymbol{H}_t}{\partial \theta_i}\right).$$

Return

Steps of the proof (I)

1. We have

$$\sqrt{n}(\xi_{n,1-2\alpha}-\xi_{1-2\alpha}) = \arg\min_{z\in\mathbb{R}} Q_n(z)$$

where

$$Q_n(z) = \sum_{k=1}^{m}\sum_{t=1}^{n}\left\{\rho_{1-2\alpha}\left(|\widehat{\eta}_{kt}|-\xi_{1-2\alpha}-\frac{z}{\sqrt{n}}\right)-\rho_{1-2\alpha}(|\eta_{kt}|-\xi_{1-2\alpha})\right\}.$$

2. We show that

$$|\widehat{\eta}_{kt}| = |\eta_{kt}| - u_{kt}\boldsymbol{M}'_{kt}(\widehat{\boldsymbol{\theta}}_n-\boldsymbol{\theta}_0)+o_P(n^{-1/2}),$$

where $u_{kt}=\pm 1$, and $\boldsymbol{M}_{kt}$ is a matrix depending on the derivatives of $\boldsymbol{m}_t$ and $\boldsymbol{\Sigma}_t$.

Steps of the proof (II)

3. We use the identity, for $u \neq 0$,

$$\rho_\tau(u-v) - \rho_\tau(u) = -v(\tau - \mathbf{1}_{\{u<0\}}) + \int_0^v \left\{\mathbf{1}_{\{u\le s\}} - \mathbf{1}_{\{u<0\}}\right\} ds$$

4. $Q_n(z) = \sum_{k=1}^m zX_{n,k} + Y_{n,k} + I_{n,k}(z) + J_{n,k}(z)$, where

$$\begin{aligned} X_{n,k} &= \frac{1}{\sqrt{n}} \sum_{t=1}^n (\mathbf{1}_{\{|\eta_{kt}|<\xi_{1-2\alpha}\}} - 1 + 2\alpha), \\ I_{n,k}(z) &= \sum_{t=1}^n \int_0^{z/\sqrt{n}} (\mathbf{1}_{\{|\eta_{kt}|\le \xi_{1-2\alpha}+s\}} - \mathbf{1}_{\{|\eta_{kt}|<\xi_{1-2\alpha}\}}) ds, \\ J_{n,k}(z) &= \sum_{t=1}^n \int_{z/\sqrt{n}}^{(z+R_{t,n,k})/\sqrt{n}} (\mathbf{1}_{\{|\eta_{kt}|\le \xi_{1-2\alpha}+s\}} - \mathbf{1}_{\{|\eta_{kt}|<\xi_{1-2\alpha}\}}) ds, \end{aligned}$$

with $R_{t,n,k} \stackrel{o_P(1)}{=} u_{kt} \boldsymbol{M}'_{kt} \sqrt{n}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0)$.

Steps of the proof (III)

5. We have $I_{n,k}(z) \to \frac{z^2}{2} f(\xi_{1-2\alpha})$ in probability as $n \to \infty$, and

$$\sum_{k=1}^{m} J_{n,k}(z) \overset{o_P(1)}{=} z\xi_{1-2\alpha} f(\xi_{1-2\alpha}) \boldsymbol{\Omega}' \sqrt{n}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) + A$$

6. We have

$$\sqrt{n}(\xi_{n,1-2\alpha} - \xi_{1-2\alpha}) \overset{o_P(1)}{=} -\frac{\xi_{1-2\alpha}}{m} \boldsymbol{\Omega}' \sqrt{n}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) - \frac{1}{f(\xi_{1-2\alpha})} \frac{1}{m\sqrt{n}} \sum_{t=1}^{n} N_t$$

and the conclusion follows.

Return

Example of spherical distribution

If $V \sim \chi^2_\nu$ independent of $\boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_m)$, then

$$\frac{\boldsymbol{Z}}{\sqrt{V/\nu}} \sim t_m(\nu)$$

follows the spherical multivariate Student with ν degrees of freedom. Since

$$\boldsymbol{Z} = \|\boldsymbol{Z}\| \frac{\boldsymbol{Z}}{\|\boldsymbol{Z}\|} \text{ with } R^2 := \|\boldsymbol{Z}\|^2 \sim \chi^2_m \text{ independent of } \boldsymbol{S} := \frac{\boldsymbol{Z}}{\|\boldsymbol{Z}\|}$$

uniformly distributed on the Sphere of $\mathbb{R}^d$,

$$t_m(\nu) \sim \varrho \boldsymbol{S}, \quad \varrho = \sqrt{\frac{V}{\nu}} R \sim \sqrt{\frac{\nu}{\chi^2_\nu}} \sqrt{\chi^2_m}, \quad V, R, \boldsymbol{S} \text{ independent.}$$

Return

Example: Gaussian QML

For the pure GARCH model $\boldsymbol{\epsilon}_t = \boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0)\boldsymbol{\eta}_t$, let the Gaussian QMLE

$$\widehat{\boldsymbol{\theta}}_n = \arg\min_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} n^{-1}\sum_{t=1}^{n}\widetilde{\ell}_t(\boldsymbol{\theta}) \qquad \text{where} \quad \widetilde{\ell}_t(\boldsymbol{\theta}) = \boldsymbol{\epsilon}_t'\widetilde{\boldsymbol{H}}_t^{-1}(\boldsymbol{\theta})\boldsymbol{\epsilon}_t + \log|\widetilde{\boldsymbol{H}}_t(\boldsymbol{\theta})|,$$

with $\widetilde{\boldsymbol{H}}_t(\boldsymbol{\theta}) = \widetilde{\boldsymbol{\Sigma}}_t(\boldsymbol{\theta})\widetilde{\boldsymbol{\Sigma}}_t'(\boldsymbol{\theta})$. Under some regularity conditions we have

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0\right) \overset{o_P(1)}{=} \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\boldsymbol{\Delta}_{t-1}V(\boldsymbol{\eta}_t)$$

with

$$\boldsymbol{\Delta}_{t-1} = J^{-1}\frac{\partial \mathrm{vec}'\boldsymbol{H}_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}}\left\{\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta}_0)\otimes\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta}_0)\right\}$$

and

$$V(\boldsymbol{\eta}_t) = \mathrm{vec}\left\{\boldsymbol{I}_m - \boldsymbol{\eta}_t\boldsymbol{\eta}_t'\right\}.$$

◂ Return

Some references on QML estimation for GARCH:

- **ARCH(q) or GARCH(1,1):** Weiss (Econ. Theory, 1986), Lee and Hansen (Econ. Theory, 1994), Lumsdaine (Econometrica, 1996),
- **GARCH(p,q):** Berkes, Horváth and Kokoszka (Bernoulli, 2003), Francq and Zakoïan (Bernoulli, 2004), Hall and Yao (Econometrica, 2003), Mikosch and Straumann (Ann. Statist., 2006).
- **More general stationary GARCH models:** Straumann and Mikosch (Ann. Statist., 2006), Robinson and Zaffaroni (Ann. Statist., 2006), Bardet and Wintenberger (Ann. Statist., 2009), Meitz and Saikkonen (Econ. Theory, 2011).

Return

Example: **B1** for CCC and DCC-GARCH models

$$\begin{cases} \boldsymbol{\epsilon}_t = \boldsymbol{\Sigma}_t \boldsymbol{\eta}_t, \qquad \boldsymbol{\Sigma}_t^2 = \boldsymbol{D}_t \boldsymbol{R}_t \boldsymbol{D}_t, \quad \boldsymbol{D}_t^2 = \mathrm{diag}(\underline{\boldsymbol{h}}_t), \\ \underline{\boldsymbol{h}}_t = \boldsymbol{\omega} + \sum_{i=1}^{q} \mathbf{A}_i \underline{\boldsymbol{\epsilon}}_{t-i} + \sum_{j=1}^{p} \mathbf{B}_j \underline{\boldsymbol{h}}_{t-j}, \quad \underline{\boldsymbol{\epsilon}}_t = \begin{pmatrix} \epsilon_{1t}^2 \\ \vdots \\ \epsilon_{mt}^2 \end{pmatrix} \end{cases}$$

where $\boldsymbol{R}_t$ is a correlation matrix:

$\boldsymbol{R}_t = \boldsymbol{R}(\boldsymbol{\rho})$ for CCC and $\boldsymbol{R}_t = \boldsymbol{R}(\boldsymbol{\epsilon}_u, u < t; \boldsymbol{\rho})$ for DCC.

With

$$\boldsymbol{\vartheta} = (\boldsymbol{\omega}', \mathrm{vec}'(\boldsymbol{A}_1), \dots, \mathrm{vec}'(\boldsymbol{B}_p), \boldsymbol{\rho}')',$$

we have

$$G(\boldsymbol{\vartheta}, K) = \left(K^2 \boldsymbol{\omega}', K^2 \mathrm{vec}'(\boldsymbol{A}_1), \dots, K^2 \mathrm{vec}'(\boldsymbol{A}_q), \mathrm{vec}'(\boldsymbol{B}_1), \dots, \mathrm{vec}'(\boldsymbol{B}_p), \boldsymbol{\rho}'\right)'.$$

Return to B1

Example

An equally weighted portfolio of 3 assets:

$$V_t = \sum_{i=1}^{3} p_{it}.$$

The vector of the log-returns

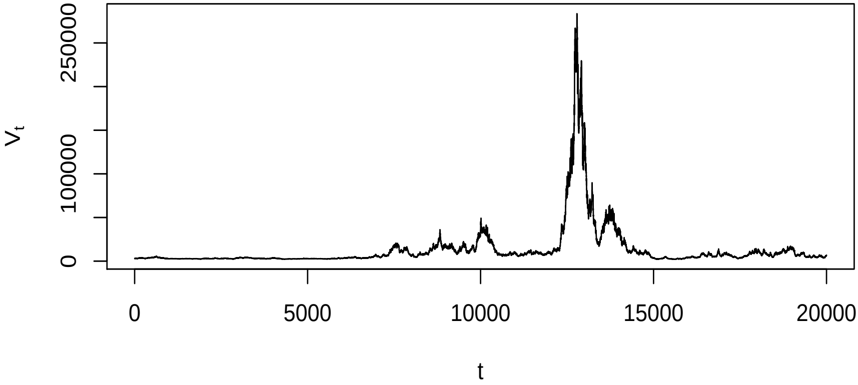
$$\boldsymbol{y}_t \sim \text{iid } \mathcal{N}(\boldsymbol{0}, \boldsymbol{DRD}),$$

with

$$\boldsymbol{D} = \begin{pmatrix} 0.01 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0.04 \end{pmatrix}, \quad \boldsymbol{R} = \begin{pmatrix} 1 & -0.855 & 0.855 \\ -0.855 & 1 & -0.810 \\ 0.855 & -0.810 & 1 \end{pmatrix}.$$

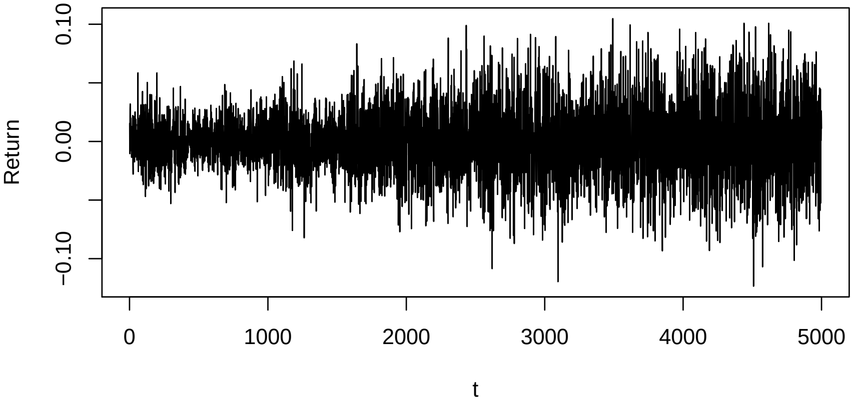
The composition of the log-return portfolio is not constant: $a_{i,t-1} = \frac{p_{i,t-1}}{\sum_{j=1}^{3} p_{j,t-1}}$.

A trajectory of (V_t)



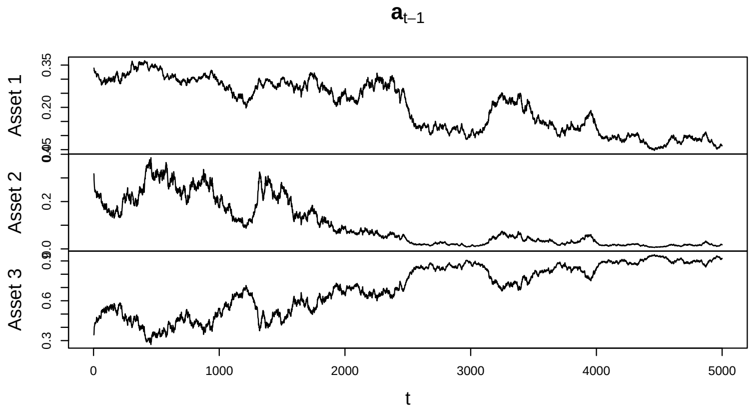
The process (V_t) is non stationary.

A trajectory of (r_t)



The return process (r_t) (also non stationary)

Time-varying composition of the portfolio



Time-varying composition of the portfolio

Return

DCC-GARCH model for the individual returns

$$\begin{cases} \boldsymbol{\epsilon}_t = \boldsymbol{\Sigma}_t \boldsymbol{\eta}_t, \quad \boldsymbol{\Sigma}_t^2 = \boldsymbol{D}_t \boldsymbol{R}_t \boldsymbol{D}_t, \quad \boldsymbol{D}_t^2 = \mathrm{diag}(\underline{\boldsymbol{h}}_t), \\ \underline{\boldsymbol{h}}_t = \boldsymbol{\omega}_0 + \mathbf{A}_0 \underline{\boldsymbol{\epsilon}}_{t-1} + \mathbf{B}_0, \underline{\boldsymbol{h}}_{t-1}, \quad \underline{\boldsymbol{\epsilon}}_t = \begin{pmatrix} \epsilon_{1t}^2 \\ \vdots \\ \epsilon_{mt}^2 \end{pmatrix} \end{cases}$$

where $\mathbf{B}_0$ is diagonal, and the correlation $\boldsymbol{R}_t$ follows the cDCC model (Engle (2002), Aielli (2013))

$$\boldsymbol{R}_t = \boldsymbol{Q}_t^{*-1/2} \boldsymbol{Q}_t \boldsymbol{Q}_t^{*-1/2},$$

$$\boldsymbol{Q}_t = (1-\alpha_0-\beta_0)\boldsymbol{S}_0 + \alpha_0 \boldsymbol{Q}_{t-1}^{*1/2} \boldsymbol{\eta}_{t-1}^{*} \boldsymbol{\eta}_{t-1}^{*'} \boldsymbol{Q}_{t-1}^{*1/2} + \beta_0 \boldsymbol{Q}_{t-1},$$

where $\alpha_0, \beta_0 \geq 0, \alpha_0+\beta_0 < 1$, $\boldsymbol{S}_0$ is a correlation matrix, $\boldsymbol{Q}_t^*$ is the diagonal matrix with the same diagonal elements as $\boldsymbol{Q}_t$, and $\boldsymbol{\eta}_t^* = \boldsymbol{D}_t^{-1} \boldsymbol{\epsilon}_t$.

Designs of the numerical experiments

Table: Design of Monte Carlo experiments.

	ω_0'	$(\text{vec}A_0)'$	$\text{diag}B_0$	$S_0(1,2)$	α	β	P_η
A	$(10^{-6}, 4\times 10^{-6})$	$(0.01, 0.01, 0.01, 0.07)$	$(0, 0.92)$	0.7	0.04	0.95	$\mathcal{N}(0, I_2)$
B	$(10^{-6}, 4\times 10^{-6})$	$(0.01, 0.01, 0.01, 0.07)$	$(0, 0.92)$	0.7	0.04	0.95	$\mathcal{S}t_7$
C	$(10^{-6}, 4\times 10^{-6})$	$(0.01, 0.01, 0.01, 0.07)$	$(0, 0.92)$	0	0	0	$\mathcal{N}(0, I_2)$
D	$(10^{-6}, 4\times 10^{-6})$	$(0.01, 0.01, 0.01, 0.07)$	$(0, 0.92)$	0	0	0	$\mathcal{S}t_7$
E	$(10^{-5}, 10^{-5})$	$(0.07, 0.00, 0.00, 0.07)$	$(0.92, 0.92)$	0.7	0.04	0.95	$\mathcal{N}(0, I_2)$
F	$(10^{-5}, 10^{-5})$	$(0.07, 0.00, 0.00, 0.07)$	$(0.92, 0.92)$	0.7	0.04	0.95	$\mathcal{S}t_7$
G	$(10^{-5}, 10^{-5})$	$(0.07, 0.00, 0.00, 0.07)$	$(0.92, 0.92)$	0	0	0	$\mathcal{N}(0, I_2)$
H	$(10^{-5}, 10^{-5})$	$(0.07, 0.00, 0.00, 0.07)$	$(0.92, 0.92)$	0	0	0	$\mathcal{S}t_7$

Designs A*-H* are the same as Designs A-H, except that P_η follows an asymmetric AEPD (introduced by Zhu and Zinde-Walsh (2009)).

◂ Numerical experiments

More details on the estimators

- Conditional VaR of the minimum-variance portfolio:

$$\mathrm{VaR}_{t-1}^{(\alpha)}\left(r_t^*\right) = \left\| \boldsymbol{a}_{t-1}^{*'} \boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0) \right\| F_{|\eta_1|}^{-1}(1-2\alpha) = \frac{1}{\sqrt{\boldsymbol{e}'\boldsymbol{\Sigma}_t^{-2}(\boldsymbol{\theta}_0)\boldsymbol{e}}} F_{|\eta_1|}^{-1}(1-2\alpha)$$

- Estimates obtained from the spherical and FHS methods:

$$\widehat{\mathrm{VaR}}_{S,t-1}^{(\alpha)}(r^*) = \frac{\xi_{n_1,1-2\alpha}}{\sqrt{\boldsymbol{e}'\widetilde{\boldsymbol{\Sigma}}_t^{-2}(\widehat{\boldsymbol{\theta}}_{n_1})\boldsymbol{e}}},$$

$$\widehat{\mathrm{VaR}}_{FHS,t-1}^{(\alpha)}(r^*) = -q_\alpha\left(\left\{\frac{\boldsymbol{e}'\widetilde{\boldsymbol{\Sigma}}_t^{-1}(\widehat{\boldsymbol{\theta}}_{n_1})\widehat{\boldsymbol{\eta}}_u}{\boldsymbol{e}'\widetilde{\boldsymbol{\Sigma}}_t^{-2}(\widehat{\boldsymbol{\theta}}_{n_1})\boldsymbol{e}}, u=1,\dots,n_1\right\}\right),$$

For the VHS method, the estimator is baised on GARCH(1,1).

Numerical experiments

Empirical Relative Efficiency

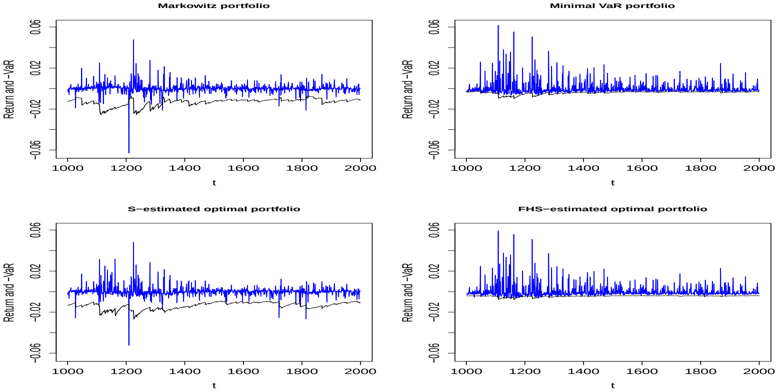
Table: Relative efficiency of the spherical method with respect to the FHS method.

n_1	α	A	B	C	D	E	F	G	H
500	1%	1.181	1.109	2.567	2.350	1.076	1.174	1.232	1.424
	5%	1.209	1.029	1.813	1.403	1.181	1.115	1.122	1.186
1000	1%	1.301	1.105	2.354	1.623	1.533	1.511	1.572	1.549
	5%	1.144	1.025	2.070	0.999	1.249	1.077	1.332	1.011
		A*	B*	C*	D*	E*	F*	G*	H*
500	1%	1.366	0.509	1.562	0.388	1.303	0.865	1.664	0.918
	5%	1.256	0.477	1.741	0.216	1.112	0.589	1.158	0.337
1000	1%	1.045	0.381	0.957	0.211	1.598	0.507	1.852	0.526
	5%	1.356	0.289	1.225	0.129	1.203	0.339	1.303	0.337

A-H: Spherical innovations; A*-H*: Non spherical innovations

Numerical experiments

Minimum VaR portfolios



Numerical experiments

Three competing VaR estimators (assuming $\boldsymbol{\mu}_t = 0$)

- $\widehat{\mathrm{VaR}}^{(\alpha)}_{t-1}(\epsilon^{(P)}) = \|\mathbf{a}'_{t-1}\widetilde{\boldsymbol{\Sigma}}_t(\widehat{\boldsymbol{\vartheta}}_n)\|\xi_{n,1-2\alpha}$

  based on an elliptic distribution for the conditional distribution of the risk factor returns.

- $\widehat{\mathrm{VaR}}^{(\alpha)}_{FHS,t-1}(\epsilon^{(P)}) = -\xi_{n,\alpha}(t,\widehat{\boldsymbol{\vartheta}}_n)$

  the filtered historical simulation VaR based on a multivariate GARCH-type model.

- $\widehat{\mathrm{VaR}}^{(\alpha)}_{U,t-1}(\epsilon^{(P)}) = -\widetilde{\sigma}_t(\widehat{\boldsymbol{\zeta}}_n)\widehat{F}_\nu(\alpha)$

  based on a univariate volatility model for the return r_t of the portfolio: $r_t = \sigma_t(\boldsymbol{\zeta})\nu_t$ where $\sigma_t(\boldsymbol{\zeta}) = \sigma(\epsilon^{(P)}_{t-1},\ldots;\boldsymbol{\zeta})$.

Advantages and drawbacks

Static model

Consider the static model $r_t = \boldsymbol{a}'\boldsymbol{\epsilon}_t = \boldsymbol{a}'\boldsymbol{\Sigma}_t(\boldsymbol{\vartheta}_0)\boldsymbol{\eta}_t$ where

$$\boldsymbol{\Sigma}_t(\boldsymbol{\vartheta}_0) = \boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0) = \begin{pmatrix} \sigma_{01} & & 0 \\ & \ddots & \\ 0 & & \sigma_{0m} \end{pmatrix}.$$

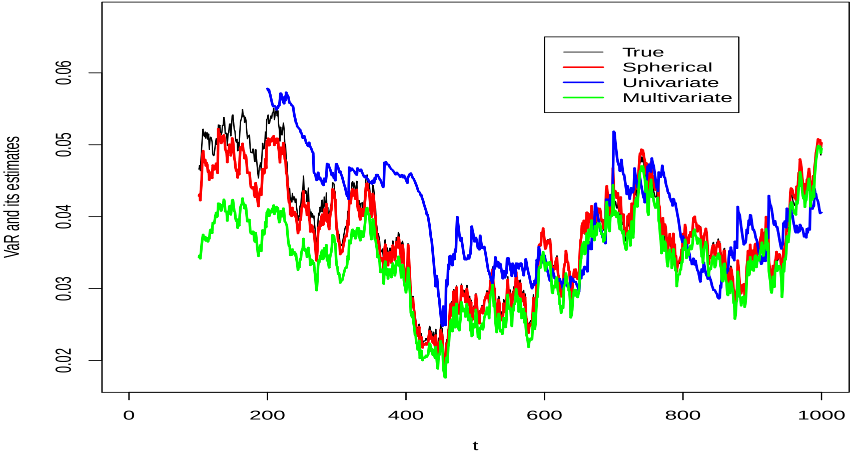
We have $\boldsymbol{\vartheta}_0 = (\sigma_{01}^2, \dots, \sigma_{0m}^2)'$ and the conditional VaR is constant:

$$\text{VaR}_{t-1}^{(\alpha)}(\epsilon^{(P)}) = \text{VaR}^{(\alpha)}(\epsilon^{(P)}).$$

- Univariate method: $(1-2\alpha)$-quantile of $|r_t|$;
- Spherical method: $\sqrt{\boldsymbol{a}'\boldsymbol{\Sigma}^2(\widehat{\boldsymbol{\vartheta}}_n)\boldsymbol{a}}\,\xi_{n,\alpha}$, where $\xi_{n,\alpha}$ is the $(1-2\alpha)$-quantile of $\widehat{\eta}_{it}$;
- "Multivariate FHS" method = univariate HS method: opposite of the α-quantile of r_t.

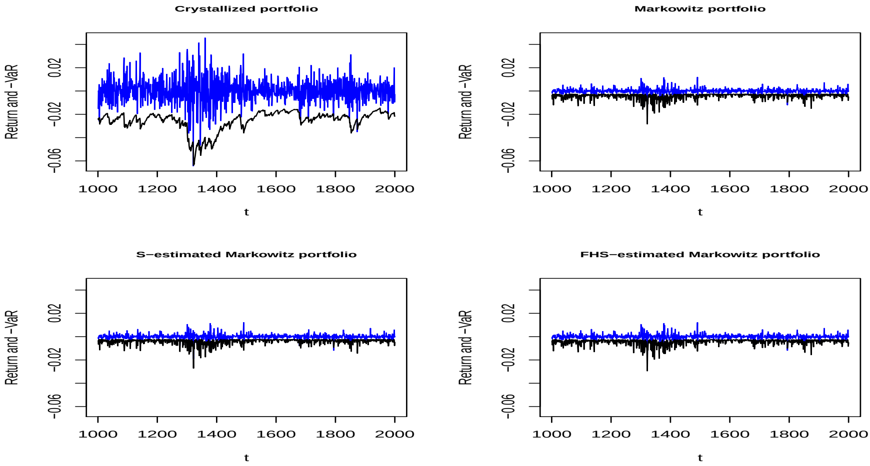
Return

The VaR and its 3 estimates



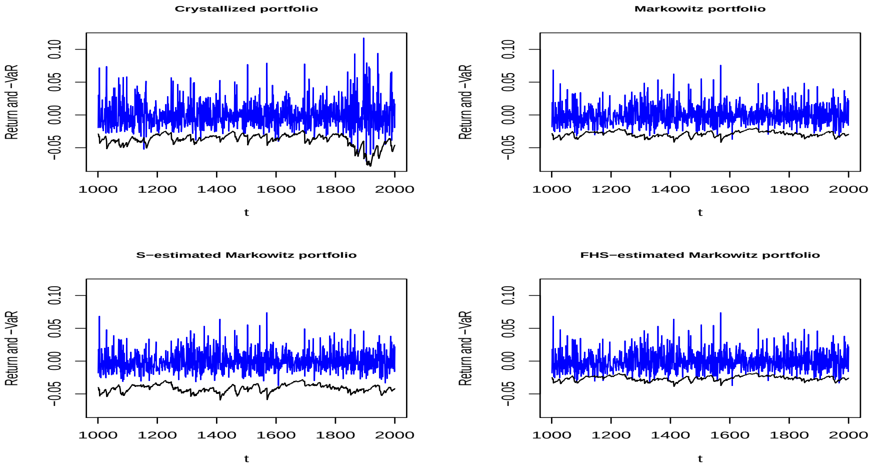
Other illustrations and backtests
Return

VaR of crystallized and minimal variance portfolios



Spherical innovations

VaR of crystallized and minimal variance portfolios



Non spherical innovations

Numerical experiments

Three competing VaR estimators (assuming $\mu_t = 0$)

- $\widehat{\text{VaR}}^{(\alpha)}_{S,t-1}(\epsilon^{(P)}) = \|\mathbf{a}'_{t-1}\widetilde{\boldsymbol{\Sigma}}_t(\widehat{\boldsymbol{\vartheta}}_n)\|\xi_{n,1-2\alpha}$

  based on an elliptic distribution for the conditional distribution of the risk factor returns.

- $\widehat{\text{VaR}}^{(\alpha)}_{FHS,t-1}(\epsilon^{(P)}) = -\xi_{n,\alpha}(t,\widehat{\boldsymbol{\vartheta}}_n)$

  the filtered historical simulation VaR based on a multivariate GARCH-type model.

- $\widehat{\text{VaR}}^{(\alpha)}_{U,t-1}(\epsilon^{(P)}) = -\widetilde{\sigma}_t(\widehat{\boldsymbol{\zeta}}_n)\widehat{F}_\nu(\alpha)$

  based on a univariate volatility model for the return r_t of the portfolio: $r_t = \sigma_t(\zeta)\nu_t$ where $\sigma_t(\zeta) = \sigma(\epsilon^{(P)}_{t-1},\ldots;\zeta)$.

Return

Static model

Consider the static model $r_t = \boldsymbol{a}'\boldsymbol{\epsilon}_t = \boldsymbol{a}'\boldsymbol{\Sigma}_t(\boldsymbol{\vartheta}_0)\boldsymbol{\eta}_t$ where

$$\boldsymbol{\Sigma}_t(\boldsymbol{\vartheta}_0) = \boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0) = \begin{pmatrix} \sigma_{01} & & 0 \\ & \ddots & \\ 0 & & \sigma_{0m} \end{pmatrix}.$$

We have $\boldsymbol{\vartheta}_0 = (\sigma_{01}^2, \dots, \sigma_{0m}^2)'$ and the conditional VaR is constant:

$$\mathrm{VaR}_{t-1}^{(\alpha)}(\epsilon^{(P)}) = \mathrm{VaR}^{(\alpha)}(\epsilon^{(P)}).$$

- Univariate (naive or VHS) method: $(1-2\alpha)$-quantile of $|r_t|$;
- Spherical method: $\sqrt{\boldsymbol{a}'\boldsymbol{\Sigma}^2(\widehat{\boldsymbol{\vartheta}}_n)\boldsymbol{a}}\xi_{n,\alpha}$, where $\xi_{n,\alpha}$ is the $(1-2\alpha)$-quantile of the $|\widehat{\eta}_{it}|$'s;
- "Multivariate FHS" method = univariate (V)HS method: opposite of the α-quantile of r_t.

Conclusions drawn from the example

For the simple (but unrealistic) static model:

1. All the methods are consistent (under sphericity);
2. When $\boldsymbol{\eta}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}_m)$, the theoretical ARE can be explicitly computed and compared; Details
3. The empirical and theoretical ARE's are in perfect agreement;
4. The method based on the sphericity assumption is often much more efficient. Details

The framework of a crystallized portfolio

An equally weighted portfolio of 3 assets:

$$V_t = \sum_{i=1}^{3} p_{it}.$$

The vector of the log-returns

$$\boldsymbol{\epsilon}_t \sim \text{iid } \mathcal{N}(\mathbf{0}, \boldsymbol{DRD}),$$

with

$$\boldsymbol{D} = \begin{pmatrix} 0.01 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0.04 \end{pmatrix}, \quad \boldsymbol{R} = \begin{pmatrix} 1 & -0.855 & 0.855 \\ -0.855 & 1 & -0.810 \\ 0.855 & -0.810 & 1 \end{pmatrix}.$$

Non-stationarity of the portfolio returns

The composition of the log-return portfolio is not constant: $a_{i,t-1} = \frac{p_{i,t-1}}{\sum_{j=1}^{3} p_{j,t-1}}$ and $r_t = \boldsymbol{a}_{t-1}' \boldsymbol{\epsilon}_t$ is non-stationary.

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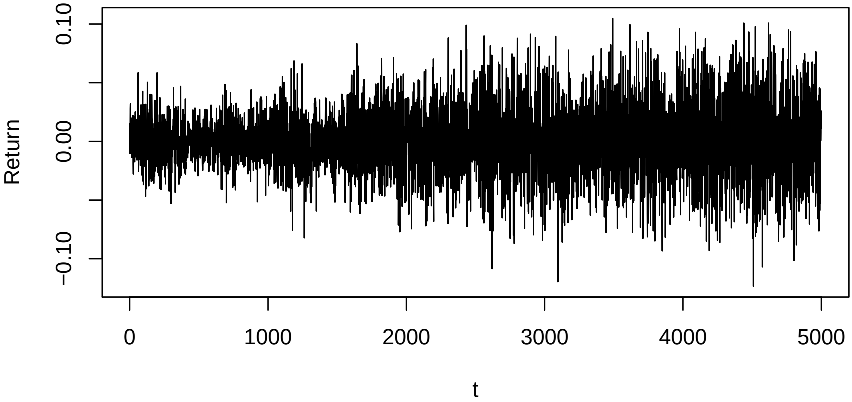
Indeed, the ratio

$$\frac{a_{1,t}}{a_{2,t}} = \frac{p_{1,t}}{p_{2,t}} = \frac{p_{1,0}}{p_{2,0}} \exp\left\{\sum_{k=1}^{t}\left(\epsilon_{1,k} - \epsilon_{2,k}\right)\right\}$$

is non stationary by Chung-Fuchs's theorem: the non-singularity of $\boldsymbol{\Sigma}$ entails that the variance of $\epsilon_{1,k} - \epsilon_{2,k}$ is non degenerated. This property holds under more general assumptions, for instance if the sequence $(\epsilon_{1,k} - \epsilon_{2,k})$ is mixing and nondegenerated.

A trajectory of (r_t)



The return process (r_t) (non stationary)

Time-varying composition of the portfolio

Time-varying composition of the portfolio

The VaR and its 3 estimates

Other illustrations and backtests

Conclusions drawn from the example

The naive univariate approach is not suitable because

1. the return of the portfolio is not stationary in general;
2. the dynamics is multivariate;
3. the information is also multivariate

$$\mathrm{VaR}_{t-1}^{(\alpha)}(\epsilon^{(P)}) = \mathrm{VaR}^{(\alpha)}\left(r_t \mid \boldsymbol{p}_u, u < t\right) \neq \mathrm{VaR}^{(\alpha)}\left(r_t \mid \epsilon_u^{(P)}, u < t\right).$$

Asymptotic comparison of two VaR estimators

Asymptotic variances of the two estimators of $\mathrm{VaR}^{(\alpha)}$:

$\sigma_U^2(\alpha,\mathbf{a})$: univariate; $\sigma_S^2(\alpha,\mathbf{a})$: spherical distribution method.

When $\boldsymbol{\eta}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}_m)$, we have

$$\frac{\sigma_S^2(\alpha,\mathbf{a})}{\sigma_U^2(\alpha,\mathbf{a})} = \frac{1}{m} - \frac{\xi_{1-2\alpha}^2\phi^2(\xi_{1-2\alpha})}{m\alpha(1-2\alpha)} + \frac{\xi_{1-2\alpha}^2\phi^2(\xi_{1-2\alpha})}{m\alpha(1-2\alpha)} \frac{\frac{1}{m}\sum_{i=1}^m a_i^4\sigma_{0i}^4}{\left(\frac{1}{m}\sum_{i=1}^m a_i^2\sigma_{0i}^2\right)^2}.$$

- $1/m$ because sphericity allows to use m times more residuals,
- negative second term because it is easier to estimate the quantile from residuals than from innovations (in the Gaussian case),
- the third term is the price paid for the estimation of $\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)$.

Asymptotic comparison of two VaR estimators

When $\boldsymbol{\eta}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}_m)$, we have

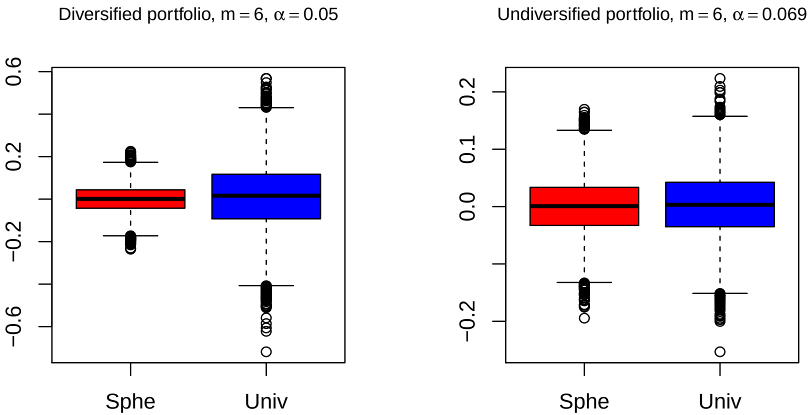
$$\frac{1}{m} \leq \frac{\sigma_S^2(\alpha, \mathbf{a})}{\sigma_U^2(\alpha, \mathbf{a})} \leq \frac{1}{m}\left[1 + (m-1)\frac{\xi_{1-2\alpha}^2 \phi^2(\xi_{1-2\alpha})}{\alpha(1-2\alpha)}\right] < 1$$

for $m \geq 2$.

- the bound $1/m$ is obtained for $a_i\sigma_{0i} = a_j\sigma_{0j}$ for all i and j (and any α),
- the upper bound is obtained with a totally undiversified portfolio of one asset.

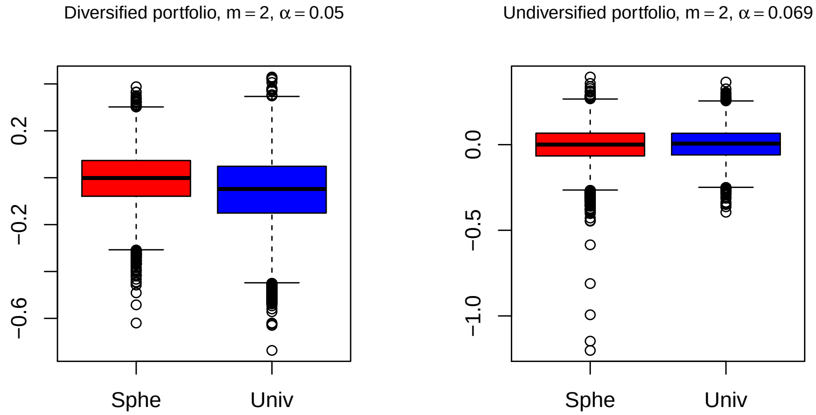
Static model

On $10,000$ replications of simulations of length $n = 500$



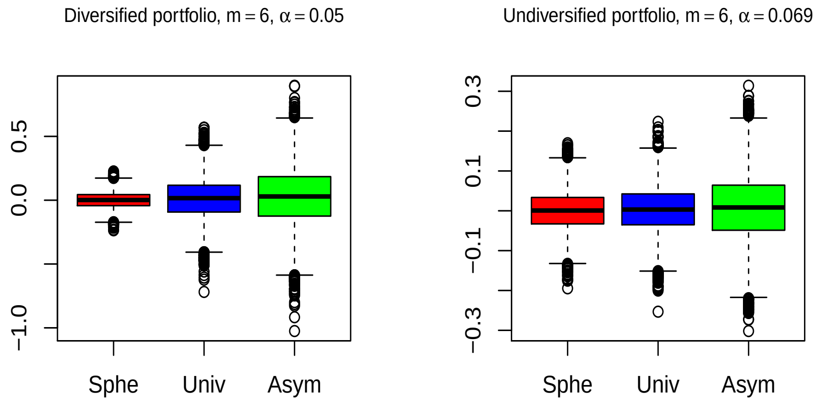
Estimation errors of the spherical distribution method (red) and univariate method (blue) when η_t is Gaussian.

An extreme case in favor of the univariate method



As previously, but $m = 2$ and $\boldsymbol{\eta}_t \sim t_2(5)$.

The 3 methods



The "multivariate" method (in green) is called asymmetric.

Static model