ISOMETRIC EMBEDDINGS OF COMPACT SPACES INTO BANACH SPACES

Y. DUTRIEUX AND G. LANCIEN

Abstract. We show the existence of a compact metric space K such that whenever K embeds isometrically into a Banach space Y, then any separable Banach space is linearly isometric to a subspace of Y. We also address the following related question: if a Banach space Y contains an isometric copy of the unit ball or of some special compact subset of a separable Banach space X, does it necessarily contain a subspace isometric to X? We answer positively this question when X is a polyhedral finite-dimensional space, c_0 or ℓ_1.

1. Introduction

This paper is motivated by questions about universal Banach spaces. In 1925, P.S. Urysohn [9] was the first to give an example of a separable metric space $\mathbb{U}$ such that every separable metric space is isometric to a subset of $\mathbb{U}$ (we say that $\mathbb{U}$ is isometrically universal). However the foundation of the questions about universal Banach spaces is the theorem of S. Banach and S. Mazur [2] asserting that every separable Banach space is linearly isometric to a subspace of $C([0,1])$ and therefore every separable metric space is isometric to a subset of $C([0,1])$. It is then natural to wonder what are the Banach spaces that are isometrically universal for smaller classes of Banach spaces or metric spaces. For instance, G. Godefroy and N.J. Kalton proved very recently in [5] that if a separable Banach space contains an isometric copy of every separable strictly convex Banach space, then it contains an isometric copy of every separable Banach space. On the other hand, in another recent work [6], N.J. Kalton and the second named author showed that every metric space with relatively compact balls embeds almost isometrically into the Banach space c_0. The main result of this paper is that a Banach space containing isometrically every compact metric space must contain a subspace linearly isometric to $C([0,1])$.

The techniques that we use come from classical results on isometries between Banach spaces. The first of them is of course the well known result of S. Mazur and S. Ulam [8] who proved that a surjective isometry between two Banach spaces is necessarily affine. In other words, the linear structure of a Banach space is completely determined by its isometric structure. Then, one naturally

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wonders about what can be said when a Banach space X is isometric to a subset of a Banach space Y. The first fundamental result in this direction is due to T. Figiel who showed in [3] that if $j : X \to Y$ is an isometric embedding such that $j(0) = 0$ and Y is the closed linear span of $j(X)$, then there is a (necessarily unique) linear quotient $Q : Y \to X$ of norm one and so that $Q \circ j = Id_X$. Let us notice now that the existence of the above map Q is clearly equivalent to the following:

$$\forall x_1, .., x_n \in X \ \ \forall \lambda_1, .., \lambda_n \in \mathbb{R} \quad \| \sum_{k=1}^{n} \lambda_k j(x_k) \| \geq \| \sum_{k=1}^{n} \lambda_k x_k \|$$

More recently, as an application of their work on Lipschitz-free Banach spaces, G. Godefroy and N.J. Kalton [4] used Figiel's result to prove that if a separable Banach space is isometric to a subset of another Banach space Y, then it is actually linearly isometric to a subspace of Y. Let us mention that this is not true in the non separable case and that counterexamples are given in [4].

In section 2 we recall the necessary background on Lipschitz-free Banach spaces. We also state the version of Theorem 3.1 of [4] that we shall use in the sequel. In section 3 we prove the main result of the paper. More precisely, we produce a compact subset K_0 of $C([0,1])$ such that any Banach space containing an isometric copy of K_0 must contain a subspace which is linearly isometric to $C([0,1])$. We also show how our technique can be combined with the results of G.M. Lövblom in [7] on almost isometries between $C(K)$-spaces.

Finally, let us say that M is an *isometrically representing subset* of the Banach space X if any Banach space Y containing an isometric copy of M contains a subset which is isometric to X. Notice that if M is an isometrically representing subset of a separable Banach space X, then it follows from the result of Godefroy and Kalton that any Banach space containing an isometric copy of M has a subspace which is linearly isometric to X. In the last section we produce compact isometrically representing subsets for the finite dimensional polyhedral spaces and for ℓ_1. We also show that the unit ball of c_0 isometrically represents the whole space.

2. Preliminary results

We begin this section with a localized version of Theorem 3.1 and Corollary 3.3 in [4]. We use the notation of [4] but recall it for the sake of completeness.

Let (E, d) be a metric space with a specified point that we denote as 0. For Y a Banach space and $f : E \to Y$ we write

$$\|f\|_L = \sup \left\{ \frac{\|f(y) - f(x)\|}{d(x,y)} \ ; \ x \neq y \text{ in } E \right\}$$

The space $Lip_0(E)$ is the space of all $f : E \to \mathbb{R}$ such that $f(0) = 0$ and $\|f\|_L < \infty$ equipped with the norm $\|\cdot\|_L$. It turns to have a canonical predual $\mathcal{F}(X)$ which is the closed linear span in the dual of $Lip_0(E)$ of the evaluation functionals $\delta(x)$ defined by $\delta(x)(f) = f(x)$, for all f in $Lip_0(E)$ and x in E. If Y is a Banach space and $g : E \to Y$ is a Lipschitz map, then there exists a unique linear operator $\overline{g} : \mathcal{F}(E) \to Y$ such that $\overline{g} \circ \delta = g$. Moreover $\|\overline{g}\| = \|g\|_L$. In particular, when E is a Banach space, applying this to the identity map on E, we see that δ admits a norm-one linear left inverse β. For more information on the subject, the reader is invited to refer to [4] and its bibliography and in particular to the book of N. Weaver [10].

When F is a subset of E which contains 0, we denote by $\mathcal{F}_E(F)$ the closed linear span in $\mathcal{F}(E)$ of the evaluation functionals $\delta(x)$, $x \in F$. Since, by inf-convolution, any real valued Lipschitz function on F can be extended to the whole space E with the same Lipschitz constant, it is clear that the spaces $\mathcal{F}_E(F)$ and $\mathcal{F}(F)$ are canonically isometric.

In our first lemma, we rephrase Theorem 3.1 of [4] for our particular purpose.

Lemma 2.1. *Let X be a separable Banach space. Let F be a closed convex subset of X such that $0 \in F$. We assume that the closed linear span of F is X. Then there exists an isometric linear embedding $T : X \to \mathcal{F}_X(F)$ such that $\beta \circ T$ is the identity map on X.*

Proof. Since X is separable, there exists a sequence $(x_n)_{n\geq 1}$ in F which is total in X and such that the set $\{\sum_{k=1}^{\infty} t_k x_k \ ; \ 0 \leq t_k \leq 1 \text{ for all } k\}$ is a compact subset of F. We introduce the Hilbert cube

$$H_n = \left\{(t_k)_{k=1}^{\infty} \ ; \ 0 \leq t_k \leq 1 \text{ for all } k \text{ and } t_n = 0\right\}$$

endowed with the product Lebesgue measure λ_n. Following the proof of Theorem 3.1 in [4], we define

$$\phi_n = \int_{H_n} \left[\delta\left(x_n + \sum t_k x_k\right) - \delta\left(\sum t_k x_k\right)\right] d\lambda_n(t)$$

Our choice of (x_n) ensures that $\phi_n \in \mathcal{F}_X(F)$. As proved in Theorem 3.1 in [4], the map $x_n \mapsto \phi_n$ extends to a norm-one linear operator T from X to $\mathcal{F}_X(F)$ such that $\beta \circ T$ is the identity map on X. □

From this, we derive the main statement of this section.

Theorem 2.2. *Let X and Y be Banach spaces. Assume that X is separable. Let F be a closed convex subset of X, containing 0 and such that the closed linear span of F is X. Let $j : F \to Y$ be an isometric embedding such that $j(0) = 0$ and denote Z the closed linear span of $j(F)$. Finally, assume that*

$$(2.1) \qquad \forall x_1,..,x_n \in F \quad \forall \lambda_1,..,\lambda_n \in \mathbb{R} \qquad \|\sum_{k=1}^{n} \lambda_k j(x_k)\| \geq \|\sum_{k=1}^{n} \lambda_k x_k\|$$

Then X is linearly isometric to a subspace of Z.

Proof. Let $T : X \to \mathcal{F}_X(F)$ be the operator given by Lemma 2.1 and let $\overline{\jmath} : \mathcal{F}_X(F) \to Z$ be the linear operator defined by $\overline{\jmath} \circ \delta = j$. It follows from the remark made in the introduction that there is a linear operator $Q : Z \to X$ satisfying $\|Q\| \leq 1$ and $(Q \circ j)(x) = x$ for all x in F. So for any $x \in F$, we have $Q \circ \overline{\jmath} \circ \delta(x) = Q \circ j(x) = x = \beta \circ \delta(x)$. Hence, by linearity and continuity, $Q \circ \overline{\jmath}(\mu) = \beta(\mu)$ for all $\mu \in \mathcal{F}_X(F)$. Since $T(X) \subset \mathcal{F}_X(F)$, we have $Q \circ \overline{\jmath} \circ T(x) = \beta \circ T(x) = x$ for any $x \in F$ and thus, again by linearity and continuity, for any $x \in X$. Finally, the fact that Q is a contraction implies that $\overline{\jmath} \circ T : X \to Z$ is a linear isometric embedding. □

3. Isometric Embeddings of spaces of continuous functions

We begin this section with the main result of this paper.

Theorem 3.1. *Let (R, d) be a compact metric space. Then there is a compact subset K of $C(R)$ such that whenever K embeds isometrically into a Banach space Y, then $C(R)$ is linearly isometric to a subspace of Y.*

Proof. We may assume that the diameter of (R, d) is less than or equal to 1. Then we consider $K = \{f \in C(R),\ \|f\|_\infty \leq 1 \text{ and } \|f\|_L \leq 1\}$. Let Y be a Banach space and assume that $j : K \to Y$ is an isometry such that $j(0) = 0$. We denote $F = K/5$ and Z the closed linear span in Y of $j(F)$. In view of Theorem 2.2, it is enough to prove inequality (2.1). Our argument is adapted from the original work of T. Figiel [3].

For $s, t \in R$, we define $\varphi_t(s) = 1 - d(s, t)$. For t in R, the functions φ_t and $-\varphi_t$ clearly belong to K and $\|j(\varphi_t) - j(-\varphi_t)\| = 2$. Thus we can pick $y_t^* \in Y^*$ such that $\|y_t^*\| = 1$ and $y_t^*(j(\varphi_t) - j(-\varphi_t)) = 2$. Since j is an isometry and $j(0) = 0$, we clearly have:

$$(3.2) \qquad \forall t \in R \quad \forall \lambda \in [-1, 1] \quad (y_t^* \circ j)(\lambda\varphi_t) = \lambda$$

The key point of our proof is to show that

$$(3.3) \qquad \forall t \in R \quad \forall \varphi \in F \quad (y_t^* \circ j)(\varphi) = \varphi(t)$$

So let us assume that there exist $t \in R$ and $\varphi \in F$ such that $(y_t^* \circ j)(\varphi) \neq \varphi(t)$. We set $\psi = \varphi - (y_t^* \circ j)(\varphi)\varphi_t$. Since $\|\psi\|_\infty < 1/2$ and $\psi(t) \neq 0$, there exists $u \in \{-2, 2\}$ so that

$$0 < (\varphi_t - u\psi)(t) < 1$$

Besides, $\|\psi\|_L < 1/2$ and the diameter of R is less than or equal to 1, so for any $s \in R$:

$$-1 < 1 - d(s,t) - u\psi(s) = (\varphi_t - u\psi)(s) \leq (\varphi_t - u\psi)(t) < 1$$

Hence

$$\|\psi - \frac{\varphi_t}{u}\|_\infty = \|\varphi - \big((y_t^* \circ j)(\varphi) + \frac{1}{u}\big)\varphi_t\|_\infty < \frac{1}{2} \tag{3.4}$$

Note that $\lambda = (y_t^* \circ j)(\varphi) + \frac{1}{u} \in [-1,1]$. So (3.2) yields:

$$\frac{1}{2} = |(y_t^* \circ j)(\varphi) - \lambda| = |(y_t^* \circ j)(\varphi) - (y_t^* \circ j)(\lambda\varphi_t)| \leq \|\varphi - \lambda\varphi_t\|_\infty$$

This is in contradiction with inequality (3.4) and finishes the proof of (3.3).

We are now able to prove inequality (2.1) and therefore to conclude our proof. Indeed, we have that for all $\psi_1, .., \psi_n \in F$ and all $\lambda_1, .., \lambda_n \in \mathbb{R}$:

$$\begin{aligned}\|\sum_{k=1}^n \lambda_k j(\psi_k)\| &\geq \sup_{t\in R} |y_t^*(\sum_{k=1}^n \lambda_k j(\psi_k))| \\ &= \sup_{t\in R} |\sum_{k=1}^n \lambda_k \psi_k(t)| = \|\sum_{k=1}^n \lambda_k \psi_k\|_\infty\end{aligned}$$

□

From the universality of $C([0,1])$, we immediately deduce the following.

Corollary 3.2. *Consider the following compact subset of* $C([0,1])$*:*

$$K_0 = \{f \in C([0,1]);\ \|f\|_\infty \leq 1 \text{ and } \|f\|_L \leq 1\}$$

If a Banach space Y contains an isometric copy of K_0, then it contains an isometric copy of any separable metric space and any separable Banach space is linearly isometric to a subspace of Y.

It is now natural to ask if a metric space that is isometrically universal for all metric compact spaces is isometrically universal for all separable metric spaces. The next proposition shows that, for elementary reasons, this is not the case.

Proposition 3.3. *There exists a separable metric space V such that every separable and bounded metric space is isometric to a subset of V but so that $\mathbb{R}$ cannot be isometrically embedded into V.*

Proof. Let B denote the unit ball of $C([0,1])$. Notice first that rB contains an isometric copy of all separable metric spaces with diameter less than r. Let V be the disjoint union of the sets $V_n = nB$, for $n \in \mathbb{N}$. We now define a metric d on V as follows. On V_n, d is the natural distance in $C([0,1])$. For $n \neq m$,

$f \in V_n$ and $g \in V_m$, we set $d(f,g) = \|f\|_\infty + 1 + \|g\|_\infty$. It is clear that (V,d) is a separable metric space which is isometrically universal for all separable bounded metric spaces. On the other hand, any connected component of V is bounded. Therefore $\mathbb{R}$ does not embed isometrically into V. □

We shall now combine Theorem 3.1 with a result of G.M. Lövblom [7] on almost isometries between $C(K)$ spaces to obtain

Corollary 3.4. *Let R and S be compact metric spaces. Assume there exists a Lipschitz embedding F of the unit ball of $C(R)$ into $C(S)$ such that*

$$\forall f,g \in B_{C(R)} \quad \frac{15}{16}\|f-g\| \le \|F(f)-F(g)\| \le \|f-g\|$$

Then $C(R)$ is linearly isometric to a subspace of $C(S)$.

Proof. We may assume that $F(0) = 0$. Using Theorem 2.1 in [7] for the particular value of $\varepsilon = \frac{1}{16}$, we obtain an isometry $j : \frac{1}{2}B_{C(R)} \to B_{C(S)}$. In particular, the set of functions on R such that both the supremum and the Lipschitz norms are less than or equal to $\frac{1}{2}$ isometrically embeds into $C(S)$. Then it follows from our previous proof that $C(R)$ embeds linearly isometrically into $C(S)$. □

4. Isometrically representing subsets

In this section, we address the following problem: given a separable Banach space X, we look for a small subset K of X such that whenever K isometrically embeds into a Banach space Y, then X embeds linearly isometrically into Y. We remind the reader that we call such a set K an *isometrically representing subset* of X. We shall restrict ourselves to considering K to be a compact subset of X or the unit ball of X.

We start with a finite dimensional result.

Theorem 4.1. *Let X be a finite dimensional polyhedral Banach space. Then the unit ball of X is an isometrically representing subset of X.*

Proof. Let $j : B_X \to Y$ be an isometric embedding such that $j(0) = 0$.
Let $x_1^*, .., x_l^* \in S_{X^*}$ so that $B_X = \cap_{i=1}^{l}\{x \in X;\ |x_i^*(x)| \le 1\}$. After removing some of the x_i^*'s if necessary, we may and do assume that

$$\forall i \in \{1,..,l\} \quad \exists x_i \in S_X \quad x_i^*(x_i) = 1 \text{ and } \forall j \ne i\ |x_j^*(x_i)| < 1$$

Then

$$\exists r \in (0, \frac{1}{2}] \quad \forall x \in X \quad \forall i \in \{1,..,l\} \quad \|x - x_i\| \le 4r \Rightarrow \|x\| = x_i^*(x) \tag{4.5}$$

We now imitate the proof of Theorem 3.1 with the x_i's replacing the functions φ_t. So for $1 \le i \le l$, we can pick $y_i^* \in S_{Y^*}$ so that for any $\lambda \in [-1,1]$,

$(y_i^* \circ j)(\lambda x_i) = \lambda$. Assume now that $x \in rB_X$ and $(y_i^* \circ j)(x) \neq x_i^*(x)$. Then consider $w = x - (y_i^* \circ j)(x)x_i$. Since $\|2w\| \leq 4r$ and $x_i^*(w) \neq 0$, it follows from (4.5) that there is $u \in \{-2, 2\}$ such that $x_i^*(x_i - uw) = \|x_i - uw\| < 1$. Following the lines of our previous proof, we then get a contradiction. Thus we have

$$\forall x \in rB_X \quad \forall i \in \{1, .., l\} \quad (y_i^* \circ j)(x) = x_i^*(x) \tag{4.6}$$

Therefore, we obtain that for all $x_1, .., x_n \in rB_X$ and all $\lambda_1, .., \lambda_n \in \mathbb{R}$:

$$\begin{aligned} \|\sum_{k=1}^n \lambda_k j(x_k)\| &\geq \sup_{1 \leq i \leq l} |y_i^*(\sum_{k=1}^n \lambda_k j(x_k))| \\ &= \sup_{1 \leq i \leq l} |x_i^*(\sum_{k=1}^n \lambda_k x_k)| = \|\sum_{k=1}^n \lambda_k x_k\| \end{aligned}$$

Hence, we are in situation to apply Theorem 2.2 and conclude that X is linearly isometric to a subspace of Z.

□

As a consequence, we obtain a similar result for c_0.

Corollary 4.2. *The Banach space c_0 is isometrically represented by its unit ball.*

Proof. We shall prove that

$$\forall x_1, .., x_n \in \frac{1}{8} B_{c_0} \quad \forall \lambda_1, .., \lambda_n \in \mathbb{R} \qquad \|\sum_{k=1}^n \lambda_k j(x_k)\| \geq \|\sum_{k=1}^n \lambda_k x_k\|_\infty \tag{4.7}$$

We may as well assume that $x_1, .., x_n$ are finitely supported sequences, so that there exists $N \in \mathbb{N}$ such that $x_1, .., x_n$ belong to the linear span of $\{e_1, .., e_N\}$, where $(e_k)_{k \geq 1}$ is the canonical basis of c_0. If $(e_k^*)_{k \geq 1}$ denotes the dual basis of $(e_k)_{k \geq 1}$, notice that for any $x \in c_0$ satisfying $\|x - e_k\| \leq \frac{1}{2}$, we have that $\|x\| = e_k^*(x)$. Then the inequality (4.7) follows directly from our previous proof. Again, Theorem 2.2 finishes the argument. □

In the case of ℓ_1, we need to use a completely different method to obtain the following.

Proposition 4.3. *The Banach space ℓ_1 admits a compact isometrically representing subset.*

Proof. For $A \subset \mathbb{N}$, we define $\mu(A) = \sum_{k \in A} 2^{-k} = \|\sum_{k \in A} 2^{-k} e_k\|_{\ell_1}$ where (e_k) stands for the canonical basis of ℓ_1. We denote by K the space of all subsets of $\mathbb{N}$ endowed with the metric $d(A, B) = \mu(A \backslash B) + \mu(B \backslash A)$. It is clear that K is isometric to a compact subset of ℓ_1.

Assume now that $j : K \to Y$ is an isometric embedding of K into some Banach space Y. We may assume that $j(\emptyset) = 0$. For any $n \in \mathbb{N}$, we denote $y_n = 2^n j(\{n\})$. Notice that $\|y_n\| = 1$. For any $\alpha = (\alpha_k) \in \ell_1$, we define $T\alpha = \sum \alpha_k y_k \in Y$. T is clearly a norm-one operator. Moreover, given $\alpha \in \ell_1$, we set $P = \{k \; ; \; \alpha_k > 0\}$, $Q = \{k \; ; \; \alpha_k < 0\}$ and pick $y^* \in S_{Y^*}$ such that $y^*(j(P) - j(Q)) = d(P, Q)$. Since all the triangle inequalities are equalities, we infer that $y^*(j(\{p\})) = 2^{-p}$ and $y^*(j(\{q\})) = -2^{-q}$ for any $p \in P$, $q \in Q$. Hence $y^*(T\alpha) = \sum |\alpha_k|$ and T is a linear isometric embedding. □

Questions. We leave open the following questions:

(1) Is a Banach space always isometrically represented by its unit ball?
(2) Does every separable Banach space admit a compact isometrically representing subset?
(3) If two separable Banach spaces have the same compact subsets up to isometry, do they isometrically embed into each other?

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Université de Franche-Comté, Laboratoire de Mathématiques UMR 6623, 16 route de Gray, 25030 Besançon Cedex, FRANCE.
E-mail address: yves.dutrieux@univ-fcomte.fr

Université de Franche-Comté, Laboratoire de Mathématiques UMR 6623, 16 route de Gray, 25030 Besançon Cedex, FRANCE.
E-mail address: gilles.lancien@univ-fcomte.fr