Asymptotics of Cholesky GARCH Models and Time-Varying Conditional Betas

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with the dynamic conditional beta (DCB) $\boldsymbol{\beta}_{y\boldsymbol{x},t} = \boldsymbol{\Sigma}^{-1}_{\boldsymbol{x}\boldsymbol{x},t}\boldsymbol{\Sigma}_{\boldsymbol{x}y,t}$.

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Practical implementation: An ARCH-type model for the conditional variance $\begin{pmatrix} \boldsymbol{\Sigma}_{\boldsymbol{xx},t} & \boldsymbol{\Sigma}_{\boldsymbol{x}y,t} \\ \boldsymbol{\Sigma}_{y\boldsymbol{x},t} & \boldsymbol{\Sigma}_{yy,t} \end{pmatrix}$ of

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A Cholesky GARCH model directly specifies the DCB.

Notation

Let $\boldsymbol{\epsilon}_t = (\epsilon_{1t}, \ldots, \epsilon_{mt})'$ be a vector of $m \geq 2$ log-returns satisfying

$$\boldsymbol{\epsilon}_t = \boldsymbol{\Sigma}_t^{1/2}(\boldsymbol{\vartheta}_0)\boldsymbol{\eta}_t,$$

where $(\boldsymbol{\eta}_t)$ is *iid* $(0, I_n)$,

$$\boldsymbol{\Sigma}_t = \boldsymbol{\Sigma}_t(\boldsymbol{\vartheta}_0) = \boldsymbol{\Sigma}(\boldsymbol{\epsilon}_{t-1}, \boldsymbol{\epsilon}_{t-2}, \ldots; \boldsymbol{\vartheta}_0) > 0,$$

and $\boldsymbol{\vartheta}_0$ is a $d \times 1$ vector.

DCC of Engle

$$\boldsymbol{\Sigma}_t = \boldsymbol{D}_t \boldsymbol{R}_t \boldsymbol{D}_t = \left(\rho_{ijt}\sqrt{\sigma_{iit}\sigma_{jjt}}\right),$$

where $\boldsymbol{D}_t = diag(\sigma_{11t}^{1/2}, \ldots, \sigma_{mmt}^{1/2})$ contains the volatilities of the individual returns, and $\boldsymbol{R}_t = (\rho_{ijt})$ the conditional correlations.

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$$\boldsymbol{Q}_t = (1-\theta_1-\theta_2)\boldsymbol{S} + \theta_1 \boldsymbol{u}_{t-1}\boldsymbol{u}'_{t-1} + \theta_2 \boldsymbol{Q}_{t-1},$$

with $\boldsymbol{u}_t = (u_{1t} \ldots u_{mt})'$, $u_{it} = \epsilon_{it}/\sqrt{\sigma_{iit}}$, $\theta_1 + \theta_2 < 1$.

Engle (2016) DCB

Assuming

$$\begin{pmatrix} \boldsymbol{x}_t \\ y_t \end{pmatrix} \mid \mathcal{F}_{t-1} \sim \mathcal{N}\left\{ \begin{pmatrix} \mu_{\boldsymbol{x}_t} \\ \mu_{y_t} \end{pmatrix}, \begin{pmatrix} \Sigma_{\boldsymbol{xx},t} & \Sigma_{\boldsymbol{x}y,t} \\ \Sigma_{y\boldsymbol{x},t} & \Sigma_{yy,t} \end{pmatrix} \right\}$$

we have

$$y_t \mid \boldsymbol{x}_t \sim \mathcal{N}\left(\mu_{y_t} + \boldsymbol{\Sigma}_{y\boldsymbol{x},t}\boldsymbol{\Sigma}_{\boldsymbol{xx},t}^{-1}(\boldsymbol{x}_t - \mu_{\boldsymbol{x}_t}), \Sigma_{yy,t} - \Sigma_{y\boldsymbol{x},t}\Sigma_{\boldsymbol{xx},t}^{-1}\Sigma_{\boldsymbol{x}y,t}\right)$$

$\Rightarrow \beta_{y\boldsymbol{x},t} = \boldsymbol{\Sigma}_{\boldsymbol{xx},t}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{x}y,t}$ can be obtained by first estimating a DCC GARCH model.

Drawbacks of DCC-based DCB

1) The stationarity and ergodicity conditions of the DCC are not weel known[1].
2) The correlation constraints are complicated.
3) The asymptotic properties of the QMLE are unknown.
4) The effects of the DCC parameters on β_t are hardly interpretable.

We now introduce a class of Cholesky GARCH (CHAR) models that avoids all these drawbacks.

Cholesky Decomposition of $\boldsymbol{\Sigma} = \mathrm{Var}(\epsilon)$

Letting $v_1 := \epsilon_1$, we have

$$\epsilon_2 = \ell_{21} v_1 + v_2 = \beta_{21}\epsilon_1 + v_2,$$

where $\beta_{21} = \ell_{21}$ is the beta in the regression of ϵ_2 on ϵ_1, and v_2 is orthogonal to ϵ_1. Recursively, we have

$$\epsilon_i = \sum_{j=1}^{i-1} \ell_{ij} v_j + v_i = \sum_{j=1}^{i-1} \beta_{ij}\epsilon_j + v_i, \quad \text{for } i = 2, \ldots, m,$$

where v_i is uncorrelated to $v_1, \ldots, v_{i-1}$, and thus uncorrelated to $\epsilon_1, \ldots, \epsilon_{i-1}$.

Cholesky Decomposition of $\boldsymbol{\Sigma} = \text{Var}(\epsilon)$

In matrix form,

$$\epsilon = \boldsymbol{L}\boldsymbol{v} \qquad \text{and} \qquad \boldsymbol{B}\epsilon = \boldsymbol{v},$$

where $\boldsymbol{L}$ and $\boldsymbol{B} = \boldsymbol{L}^{-1}$ are lower unitriangular and $\boldsymbol{G} := \text{var}(\boldsymbol{v})$ is triangular. We obtain the Cholesky decomposition

$$\boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{G}\boldsymbol{L}'$$

(see Pourahmadi, 1999).

Example: $\boldsymbol{\Sigma} = \boldsymbol{LGL}'$, $m = 3$

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 \\ -\beta_{21} & 1 & 0 \\ -\beta_{31} & -\beta_{32} & 1 \end{bmatrix}, \quad \boldsymbol{G} = \begin{bmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{bmatrix},$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} g_1 & l_{21} g_1 & l_{31} g_1 \\ l_{21} g_1 & l_{21}^2 g_1 + g_2 & l_{21} l_{31} g_1 + l_{32} g_2 \\ l_{31} g_1 & l_{21} l_{31} g_1 + l_{32} g_2 & l_{31}^2 g_1 + l_{32}^2 g_2 + g_3 \end{bmatrix}.$$

Remark: In $\boldsymbol{\Sigma} = \boldsymbol{DRD}$ the constraints on the elements of $\boldsymbol{R}$ are

$$\rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2 - 2\rho_{12}\rho_{13}\rho_{23} \leq 1.$$

In $\boldsymbol{\Sigma} = \boldsymbol{LGL}'$ there is no constraint on the ℓ_{ij}'s.

Sequential construction of $\boldsymbol{\Sigma}_t = \boldsymbol{L}_t \boldsymbol{G}_t \boldsymbol{L}_t'$

Let the factors $\boldsymbol{v}_t = \boldsymbol{G}_t^{1/2} \boldsymbol{\eta}_t$, $\boldsymbol{\eta}_t$ iid $(0, \boldsymbol{I}_m)$ $(v_{it} = \sqrt{g_{it}}\eta_{it})$.

Taking $\boldsymbol{\Sigma}_t^{1/2} = \boldsymbol{L}_t \boldsymbol{G}_t^{1/2}$, we have $\boldsymbol{\epsilon}_t = \boldsymbol{\Sigma}_t^{1/2} \boldsymbol{\eta}_t = \boldsymbol{L}_t \boldsymbol{v}_t$.

Step 1:
$$v_{1t} = \epsilon_{1t}$$
$$g_{1,t} = \text{Var}\left(v_{1t} \mid \mathcal{F}_{t-1}\right)$$

Step 2:
$$\epsilon_{2t} = l_{21,t} v_{1t} + v_{2t}$$
$$v_{2t} = \epsilon_{2t} - l_{21,t} v_{1t}$$
$$g_{2,t} = \text{Var}\left(v_{2t} \mid \mathcal{F}_{t-1}\right)$$

Step 3:
$$\epsilon_{3t} = l_{31,t} v_{1t} + l_{32,t} v_{2t} + v_{3t}$$
$$v_{3t} = \epsilon_{3t} - l_{31,t} v_{1t} - l_{32,t} v_{2t}$$
$$g_{3,t} = \text{Var}\left(v_{3t} \mid \mathcal{F}_{t-1}\right)$$

Alternative construction of $\boldsymbol{\Sigma}_t^{-1} = \boldsymbol{B}_t'\boldsymbol{G}_t^{-1}\boldsymbol{B}_t$

$\epsilon_t = \boldsymbol{L}_t \boldsymbol{v}_t$ and therefore $\boldsymbol{B}_t \epsilon_t = \boldsymbol{v}_t$, where the subdiagonal elements of the lower unitriangular matrix $\boldsymbol{B}_t = \boldsymbol{L}_t^{-1}$ are $-\beta_{ij,t}$

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$$g_{3,t} = \mathrm{Var}\left(v_{3t} \mid \mathcal{F}_{t-1}\right)$$

The order of the series

Replace $\boldsymbol{\epsilon}_t = (\epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t})'$ by $\boldsymbol{\epsilon}_t^* = (\epsilon_{2t}, \epsilon_{1t}, \epsilon_{3t})'$ (the position of the last component is imposed by the problem)

$$
\begin{aligned}
\text{Step } 1^*: \quad v_{1t}^* &= \epsilon_{2t}, \quad g_{1,t}^* = \mathrm{Var}\left(v_{1t}^* \mid \mathcal{F}_{t-1}\right) \\
\text{Step } 2^*: \quad \epsilon_{1t} &= \beta_{12,t}\epsilon_{2t} + v_{2t}^* \quad (v_{2t}^* \neq v_{2t}) \\
g_{2,t}^* &= \mathrm{Var}\left(v_{2t}^* \mid \mathcal{F}_{t-1}\right) \\
\text{Step } 3^*: \quad \epsilon_{3t} &= \beta_{32,t}\epsilon_{2t} + \beta_{31,t}\epsilon_{1t} + v_{3t}^* \quad (v_{3t}^* = v_{3t}) \\
g_{3,t} &= \mathrm{Var}\left(v_{3t} \mid \mathcal{F}_{t-1}\right)
\end{aligned}
$$

In particular, $\beta_{12,t} = \beta_{21,t}\frac{g_{1t}}{\beta_{21,t}^2 g_{1t} + g_{2t}}$: for most parametric specifications, the order matters.

The order of the series (matrix form)

$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} g_1^{-1} + \beta_{21}^2 g_2^{-1} + \beta_{31,t}^2 g_{3,t}^{-1} & -\beta_{21} g_2^{-1} + \beta_{31,t}\beta_{32,t} g_{3,t}^{-1} & -\beta_{31,t} g_{3,t}^{-1} \\ -\beta_{21} g_2^{-1} + \beta_{31,t}\beta_{32,t} g_{3,t}^{-1} & \beta_{32,t}^2 g_{3,t}^{-1} + g_2^{-1} & -\beta_{32,t} g_{3,t}^{-1} \\ -\beta_{31,t} g_{3,t}^{-1} & -\beta_{32,t} g_{3,t}^{-1} & g_{3,t}^{-1} \end{bmatrix}.$$

$$\boldsymbol{\epsilon}_t^* = \Delta \boldsymbol{\epsilon}_t, \qquad \Delta \boldsymbol{\Sigma}^{*-1} \Delta =$$

$$\begin{bmatrix} \beta_{31,t}^2 g_{3,t}^{-1} + g_2^{*-1} & -\beta_{12} g_2^{*-1} + \beta_{31,t}\beta_{32,t} g_{3,t}^{-1} & -\beta_{31,t} g_{3,t}^{-1} \\ -\beta_{12} g_2^{*-1} + \beta_{31,t}\beta_{32,t} g_{3,t}^{-1} & g_1^{*-1} + \beta_{12}^2 g_2^{*-1} + \beta_{32,t}^2 g_{3,t}^{-1} & -\beta_{32,t} g_{3,t}^{-1} \\ -\beta_{31} g_{3,t}^{-1} & -\beta_{32,t} g_{3,t}^{-1} & g_{3,t}^{-1} \end{bmatrix}.$$

We have $\Delta \boldsymbol{\Sigma}^{*-1} \Delta = \boldsymbol{\Sigma}^{-1}$ when

$\beta_{12} = \beta_{21} g_1 / (\beta_{21}^2 g_1 + g_2)$, $g_1^* = \beta_{21}^2 g_1 + g_2$ and $g_2^* = g_1 - \beta_{21}^2 g_1^2 / (\beta_{21}^2 g_1 + g_2)$

When the first parameters are time-invariant, the order does not matter.

A general model for the factors

Assume

$$\boldsymbol{v}_t = \boldsymbol{G}_t^{1/2}\eta_t, \qquad (\eta_t) \text{ iid } (0, I_n),$$

where $\boldsymbol{G}_t = \text{diag}(\boldsymbol{g}_t)$ follows a GJR-like equation

$$\boldsymbol{g}_t = \omega_0 + \sum_{i=1}^{q} \left\{ \boldsymbol{A}_{0i,+}\boldsymbol{v}_{t-i}^{2+} + \boldsymbol{A}_{0i,-}\boldsymbol{v}_{t-i}^{2-} \right\} + \sum_{j=1}^{p} \boldsymbol{B}_{0j}\boldsymbol{g}_{t-j},$$

with positive coefficients and

$$\boldsymbol{v}_t^{2+} = \left(\left\{ v_{1t}^{+} \right\}^2, \cdots, \left\{ v_{mt}^{+} \right\}^2 \right)', \quad \boldsymbol{v}_t^{2-} = \left(\left\{ v_{1t}^{-} \right\}^2, \cdots, \left\{ v_{mt}^{-} \right\}^2 \right)'.$$

Markovian representation of the factors

Letting $\boldsymbol{z}_t = \left(\boldsymbol{v}^{2+'}_{t:(t-q+1)}, \boldsymbol{v}^{2-'}_{t:(t-q+1)}, \boldsymbol{g}'_{t:(t-p+1)}\right)'$,
$\boldsymbol{h}_t = \left(\boldsymbol{\omega}'_0 \boldsymbol{\Upsilon}^{+'}_t, 0'_{m(q-1)}, \boldsymbol{\omega}'_0 \boldsymbol{\Upsilon}^{-'}_t, 0'_{(q-1)m}, \boldsymbol{\omega}'_0, 0'_{(p-1)m}\right)'$, with $\boldsymbol{\Upsilon}^+_t = \mathrm{diag}\left(\eta^{2+}_t\right)$ $\boldsymbol{\Upsilon}^-_t = \mathrm{diag}\left(\eta^{2-}_t\right)$ and obvious notations, we rewrite the model as

$$\boldsymbol{z}_t = \boldsymbol{h}_t + \boldsymbol{H}_t \boldsymbol{z}_{t-1},$$

where, in the case $p = q = 1$,

$$\boldsymbol{H}_t = \begin{pmatrix} \boldsymbol{\Upsilon}^+_t \boldsymbol{A}_{01,+} & \boldsymbol{\Upsilon}^+_t \boldsymbol{A}_{01,-} & \boldsymbol{\Upsilon}^+_t \boldsymbol{B}_{01} \\ \boldsymbol{\Upsilon}^-_t \boldsymbol{A}_{01,+} & \boldsymbol{\Upsilon}^-_t \boldsymbol{A}_{01,-} & \boldsymbol{\Upsilon}^-_t \boldsymbol{B}_{01} \\ \boldsymbol{A}_{01,+} & \boldsymbol{A}_{01,-} & \boldsymbol{B}_{01} \end{pmatrix}.$$

Stationarity of the factors

In view of

$$\boldsymbol{z}_t = \boldsymbol{h}_t + \boldsymbol{H}_t \boldsymbol{z}_{t-1},$$

there exists a stationary and ergodic sequence $(\boldsymbol{v}_t)$ satisfying $\boldsymbol{v}_t = \boldsymbol{G}^{1/2}\boldsymbol{\eta}_t$ if and only if

$$\gamma_0 = \inf_{t \geq 1} \frac{1}{t} E(\log \|\boldsymbol{H}_t \boldsymbol{H}_{t-1} \dots \boldsymbol{H}_1\|) < 0.$$

Stationarity of $\beta_t := -\text{vech}^0 \boldsymbol{B}_t$

If $(\boldsymbol{v}_t)$ is stationary and ergodic ($\gamma_0 < 0$), and

$$\det\left\{\boldsymbol{I}_{m_0} - \sum_{i=1}^{s} \boldsymbol{C}_{0i} z^i\right\} \neq 0 \text{ for all } |z| \leq 1,$$

then

$$\beta_t = \boldsymbol{c}_0\left(\boldsymbol{v}_{t-1}, \ldots, \boldsymbol{v}_{t-r}, \boldsymbol{g}_{t-1}^{1/2}, \ldots, \boldsymbol{g}_{t-r}^{1/2}\right) + \sum_{j=1}^{s} \boldsymbol{C}_{0j} \beta_{t-j}.$$

defines a stationary and ergodic sequence (and thus the existence of a stationary CHAR model).

Existence of moments

If in addition

$$E\|\eta_1\|^{2k_1} < \infty \quad \text{and} \quad \varrho(E\boldsymbol{H}_1^{\otimes k_1}) < 1,$$

for some integer $k_1 > 0$, and

$$\|\boldsymbol{c}_0(\boldsymbol{x}) - \boldsymbol{c}_0(\boldsymbol{y})\| \leq K\,\|\boldsymbol{x} - \boldsymbol{y}\|^a$$

for some constants $K > 0$ and $a \in (0, 1]$, then the CHAR model satisfies $E\,\|\epsilon_1\|^{2k_1} < \infty$.

A simpler parameterization

A tractable submodel is

$$g_{it} = \omega_{0i} + \gamma_{0i+} \left(\epsilon^{+}_{1,t-1}\right)^2 + \gamma_{0i-} \left(\epsilon^{-}_{1,t-1}\right)^2 + \sum_{k=2}^{i} \alpha^{(k)}_{0i} v^2_{k,t-1} + b_{0i} g_{i,t-1}$$

with positivity coefficients, and

$$\beta_{ij,t} = \varpi_{0ij} + \varsigma_{0ij+} \epsilon^{+}_{1,t-1} + \varsigma_{0ij-} \epsilon^{-}_{1,t-1} + \sum_{k=2}^{i} \tau^{(k)}_{0ij} v_{k,t-1} + c_{0ij} \beta_{ij,t-1}$$

without positivity constraints. Notice the triangular structure and note that the asymmetry is introduced via the first (observed) factor only.

Stationarity for the previous specification

There exists a strictly stationary, non anticipative and ergodic solution to the CHAR model when

1) $E \log \left\{ \omega_{01} + \gamma_{01+} \left(\eta^{+}_{1,t-1}\right)^2 + \gamma_{01-} \left(\eta^{-}_{1,t-1}\right)^2 + b_{01} \right\} < 0$,
2) $E \log \left\{ \alpha^{(i)}_{0i} \eta^2_{it} + b_{0i} \right\} < 0$ for $i = 2, \ldots, m$,
3) $|c_{0ij}| < 1$ for all (i, j).

Moreover, the stationary solution satisfies $E\|\epsilon_1\|^{2s_0} < \infty$, $E\|\boldsymbol{g}_1\|^{s_0} < \infty$, $E\|\boldsymbol{v}_1\|^{s_0} < \infty$, $E\|\boldsymbol{\beta}_1\|^{s_0} < \infty$ and $E\|\boldsymbol{\Sigma}_1\|^{s_0} < \infty$ for some $s_0 > 0$.

Invertibility of the CHAR

Under the stationarity conditions

$$\boldsymbol{g}_t = \boldsymbol{g}(\boldsymbol{\eta}_u, u < t), \qquad \boldsymbol{\beta}_t = \boldsymbol{\beta}(\boldsymbol{\eta}_u, u < t).$$

For practical use, we need (uniform) invertibility:

$$\boldsymbol{g}_t(\boldsymbol{\vartheta}) = \boldsymbol{g}(\boldsymbol{\vartheta}; \epsilon_u, u < t), \qquad \boldsymbol{\beta}_t(\boldsymbol{\vartheta}) = \boldsymbol{\beta}(\boldsymbol{\vartheta}; \boldsymbol{\epsilon}_u, u < t)$$

with some abuse of notation.

Invertibility of the CHAR

For practical use, we need uniform invertibility:

$$\beta_t(\vartheta) = \beta(\vartheta; \epsilon_u, u < t).$$

More precisely, we need

$$\sup_{\vartheta \in \Theta} \left\| \beta_t(\vartheta) - \widetilde{\beta}_t(\vartheta) \right\| \leq K\rho^t,$$

where $\widetilde{\beta}_t(\vartheta) = \beta(\vartheta; \epsilon_{t-1}, \ldots, \epsilon_1, \widetilde{\epsilon}_0, \widetilde{\epsilon}_{-1}, \ldots)$ for fixed initial values $\widetilde{\epsilon}_0, \widetilde{\epsilon}_{-1}, \ldots$, and $\beta_t(\vartheta_0) = \beta_t$.

Invertibility conditions for the "triangular" model

In vector form, the model

$$\widetilde{v}_{kt}(\boldsymbol{\vartheta}) = \epsilon_{kt} - \sum_{j=1}^{k-1} \widetilde{\beta}_{kj,t}(\boldsymbol{\vartheta})\epsilon_{jt},$$

$$\widetilde{\beta}_{ij,t}(\boldsymbol{\vartheta}) = \omega_{ij,t-1} + \sum_{k=2}^{i} \tau_{ij}^{(k)} \widetilde{v}_{k,t-1}(\boldsymbol{\vartheta}) + c_{ij}\widetilde{\beta}_{ij,t-1}(\boldsymbol{\vartheta}),$$

with $\omega_{ijt} = \varpi_{ij} + \varsigma_{ij+}\epsilon_{1t}^{+} + \varsigma_{ij-}\epsilon_{1t}^{-}$, writes

$$\begin{aligned}
\widetilde{\boldsymbol{\beta}}_t(\boldsymbol{\vartheta}) &= \boldsymbol{w}_{t-1} + \boldsymbol{T}\widetilde{\boldsymbol{B}}_{t-1}(\boldsymbol{\vartheta})\boldsymbol{\epsilon}_{t-1} + \boldsymbol{C}\widetilde{\boldsymbol{\beta}}_{t-1}(\boldsymbol{\vartheta}) \\
&= \boldsymbol{w}_{t-1} + \boldsymbol{T}\boldsymbol{\epsilon}_{t-1} + \left\{\boldsymbol{C} - (\boldsymbol{\epsilon}_{t-1}' \otimes \boldsymbol{T})\mathbf{D}_m^0\right\}\widetilde{\boldsymbol{\beta}}_{t-1}(\boldsymbol{\vartheta}) \\
&:= \boldsymbol{w}_{t-1}^{*} + \boldsymbol{S}_{t-1}\widetilde{\boldsymbol{\beta}}_{t-1}(\boldsymbol{\vartheta}).
\end{aligned}$$

Invertibility conditions for the "triangular" model

By the Cauchy rule, the triangular model

$$\widetilde{\beta}_t(\boldsymbol{\vartheta}) = \boldsymbol{w}^*_{t-1} + \boldsymbol{S}_{t-1}\widetilde{\beta}_{t-1}(\boldsymbol{\vartheta}).$$

is uniformly invertible under the conditions

$$E \log^+ \sup_{\boldsymbol{\vartheta}\in\Theta} \|\boldsymbol{w}_1 + \boldsymbol{T}\boldsymbol{\epsilon}_1\| < \infty,$$

$$\gamma_S := \limsup_{n\to\infty} \frac{1}{n} \log \sup_{\boldsymbol{\vartheta}\in\Theta} \left\| \prod_{i=1}^{n} \boldsymbol{S}_{t-i} \right\| < 0.$$

Full QMLE of the general CHAR

A QMLE of the CHAR parameter ϑ_0 is

$$\widehat{\vartheta}_n = \arg\min_{\vartheta \in \Theta} \widetilde{O}_n(\vartheta), \qquad \widetilde{O}_n(\vartheta) = n^{-1} \sum_{t=1}^{n} \widetilde{q}_t(\vartheta),$$

where $\widetilde{\boldsymbol{\Sigma}}_t(\vartheta) = \boldsymbol{\Sigma}\left(\epsilon_{t-1}, \ldots, \epsilon_1, \widetilde{\epsilon}_0, \widetilde{\epsilon}_{-1}, \ldots; \vartheta\right)$ and

$$\widetilde{q}_t(\vartheta) = \epsilon_t' \widetilde{\boldsymbol{B}}_t'(\vartheta) \widetilde{\boldsymbol{G}}_t^{-1}(\vartheta) \widetilde{\boldsymbol{B}}_t(\vartheta) \epsilon_t + \sum_{i=1}^{m} \log \widetilde{g}_{it}(\vartheta).$$

- Does not require matrix inversion.
- CAN under general regularity conditions.

Regularity conditions for the triangular model

Let $\omega_{it} = \omega_i + \gamma_{i+} \left(\epsilon_{1t}^{+}\right)^2 + \gamma_{i-} \left(\epsilon_{1t}^{-}\right)^2$. The regularity conditions for CAN of the QMLE of the triangular model

$$g_{it}(\vartheta) = \omega_{i,t-1} + \sum_{k=2}^{i} \alpha_i^{(k)} v_{k,t-1}^2(\vartheta) + b_i g_{i,t-1}(\vartheta)$$

$$\beta_{ij,t}(\vartheta) = \omega_{ij,t-1} + \sum_{k=2}^{i} \tau_{ij}^{(k)} v_{k,t-1}(\vartheta) + c_{ij} \beta_{ij,t-1}(\vartheta)$$

are

B1 $|c_{0ij}| < 1$ and other stationarity conditions.

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are

B2 For $i = 2, \ldots, m$, the distribution of η_{it}^2 conditionally on $\{\eta_{jt}, j \neq i\}$ is non-degenerate. The support of η_{1t} contains at least two positive points and two negative points.

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Let $\omega_{it} = \omega_i + \gamma_{i+}\left(\epsilon_{1t}^{+}\right)^2 + \gamma_{i-}\left(\epsilon_{1t}^{-}\right)^2$. The regularity conditions for CAN of the QMLE of the triangular model

$$g_{it}(\vartheta) = \omega_{i,t-1} + \sum_{k=2}^{i} \alpha_i^{(k)} v_{k,t-1}^2(\vartheta) + b_i g_{i,t-1}(\vartheta)$$

$$\beta_{ij,t}(\vartheta) = \omega_{ij,t-1} + \sum_{k=2}^{i} \tau_{ij}^{(k)} v_{k,t-1}(\vartheta) + c_{ij}\beta_{ij,t-1}(\vartheta)$$

are

B3 Positivity conditions: $\gamma_{i+}, \gamma_{i-}, \alpha_i^{(k)} \geq 0$ for all ϑ, $\omega_{0i} > 0$, $|b_{0i}| < 1$ and $|c_{0ij}| < 1$.

Regularity conditions for the triangular model

Let $\omega_{it} = \omega_i + \gamma_{i+}\left(\epsilon_{1t}^{+}\right)^2 + \gamma_{i-}\left(\epsilon_{1t}^{-}\right)^2$. The regularity conditions for CAN of the QMLE of the triangular model

$$g_{it}(\vartheta) = \omega_{i,t-1} + \sum_{k=2}^{i} \alpha_i^{(k)} v_{k,t-1}^2(\vartheta) + b_i g_{i,t-1}(\vartheta)$$

$$\beta_{ij,t}(\vartheta) = \omega_{ij,t-1} + \sum_{k=2}^{i} \tau_{ij}^{(k)} v_{k,t-1}(\vartheta) + c_{ij}\beta_{ij,t-1}(\vartheta)$$

are

B4 Identifiability conditions: $(\gamma_{0i+}, \gamma_{0i-}, \alpha_{0i}^{(2)}, \ldots, \alpha_{0i}^{(i)}) \neq 0$ and $(\varsigma_{0ij+}, \varsigma_{0ij-}, \tau_{0ij}^{(2)}, \ldots, \tau_{0ij}^{(i)}) \neq 0$.

Regularity conditions for the triangular model

Let $\omega_{it} = \omega_i + \gamma_{i+}\left(\epsilon_{1t}^{+}\right)^2 + \gamma_{i-}\left(\epsilon_{1t}^{-}\right)^2$. The regularity conditions for CAN of the QMLE of the triangular model

$$g_{it}(\vartheta) = \omega_{i,t-1} + \sum_{k=2}^{i} \alpha_i^{(k)} v_{k,t-1}^2(\vartheta) + b_i g_{i,t-1}(\vartheta)$$

$$\beta_{ij,t}(\vartheta) = \omega_{ij,t-1} + \sum_{k=2}^{i} \tau_{ij}^{(k)} v_{k,t-1}(\vartheta) + c_{ij}\beta_{ij,t-1}(\vartheta)$$

are

B5 Uniform invertibility condition.

Regularity conditions for the triangular model

Let $\omega_{it} = \omega_i + \gamma_{i+}\left(\epsilon_{1t}^{+}\right)^2 + \gamma_{i-}\left(\epsilon_{1t}^{-}\right)^2$. The regularity conditions for CAN of the QMLE of the triangular model

$$g_{it}(\boldsymbol{\vartheta}) = \omega_{i,t-1} + \sum_{k=2}^{i} \alpha_i^{(k)} v_{k,t-1}^2(\boldsymbol{\vartheta}) + b_i g_{i,t-1}(\boldsymbol{\vartheta})$$

$$\beta_{ij,t}(\boldsymbol{\vartheta}) = \omega_{ij,t-1} + \sum_{k=2}^{i} \tau_{ij}^{(k)} v_{k,t-1}(\boldsymbol{\vartheta}) + c_{ij}\beta_{ij,t-1}(\boldsymbol{\vartheta})$$

are

B6 Moments conditions (larger than 6) on

$$\left\|\boldsymbol{\epsilon}_t\right\|, \quad \sup_{\boldsymbol{\vartheta}\in V(\boldsymbol{\vartheta}_0)} \left\|\boldsymbol{\beta}_t(\boldsymbol{\vartheta})\right\|, \quad \sup_{\boldsymbol{\vartheta}\in V(\boldsymbol{\vartheta}_0)} \left\|\frac{\partial \boldsymbol{\beta}_t(\boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta}'}\right\|.$$

Equation-by-Equation (EbE) estimator

Consider the triangular model. In a first step, the parameter $\boldsymbol{\vartheta}_0^{(1)} = (\omega_{01}, \gamma_{01+}, \gamma_{01-}, b_{01})$ is estimated by

$$\widehat{\boldsymbol{\vartheta}}_n^{(1)} = \underset{\boldsymbol{\vartheta}^{(1)} \in \Theta^{(1)}}{\arg\min} \sum_{t=1}^{n} \widetilde{q}_{1t}(\boldsymbol{\vartheta}^{(1)}),$$

where

$$\widetilde{q}_{1t}(\boldsymbol{\vartheta}^{(1)}) \quad = \quad \frac{\epsilon_{1t}^2}{\widetilde{g}_{1t}(\boldsymbol{\vartheta}^{(1)})} + \log \widetilde{g}_{1t}(\boldsymbol{\vartheta}^{(1)}),$$

and $\widetilde{g}_{1t}(\boldsymbol{\vartheta}^{(1)}) = \omega_1 + \gamma_{1+}\left(\epsilon_{1t}^{+}\right)^2 + \gamma_{1-}\left(\epsilon_{1t}^{-}\right)^2 + b_1 \widetilde{g}_{1,t-1}(\boldsymbol{\vartheta}^{(1)})$.

EbE second step

Let $\vartheta_0^{(2)} = (\varphi_0^{(2)}, \theta_0^{(2)})$, where $\widetilde{\beta}_{21,t} = \widetilde{\beta}_{21,t}(\varphi_0^{(2)})$ and $\widetilde{g}_{2t} = \widetilde{g}_{2t}(\theta_0^{(2)})$.
Independently or in parallel to $\vartheta_0^{(1)}$, one can estimate $\vartheta_0^{(2)}$ by

$$\widehat{\vartheta}_n^{(2)} = \underset{\vartheta^{(2)} \in \Theta^{(2)}}{\arg\min} \sum_{t=1}^{n} \widetilde{q}_{2t}(\vartheta^{(2)}),$$

where, for $t = 1, \ldots, n$,

$$\begin{aligned}
\widetilde{q}_{2t}(\vartheta^{(2)}) &= \frac{\widetilde{v}_{2t}^2(\varphi^{(2)})}{\widetilde{g}_{2t}(\vartheta^{(2)})} + \log \widetilde{g}_{2t}(\vartheta^{(2)}), \\
\widetilde{g}_{2t}(\vartheta^{(2)}) &= \omega_{2,t-1} + \alpha_2^{(2)} \widetilde{v}_{2,t-1}^2(\varphi^{(2)}) + b_2 \widetilde{g}_{2,t-1}(\varphi^{(2)}), \\
\widetilde{v}_{2t}(\varphi^{(2)}) &= \epsilon_{2t} - \widetilde{\beta}_{21,t}(\varphi^{(2)}) \epsilon_{1t}, \\
\widetilde{\beta}_{21,t}(\varphi^{(2)}) &= \omega_{21,t-1} + \tau_{21}^{(2)} \widetilde{v}_{2,t-1}(\varphi^{(2)}) + c_{21} \widetilde{\beta}_{21,t-1}(\varphi^{(2)}).
\end{aligned}$$

EbE remaining steps

For $i \geq 3$, $\widetilde{\beta}_{ij,t}$ depends on $\varphi_0^{(+i)} = \left(\varphi_0^{(i)}, \varphi_0^{(-i)}\right)$, where $\varphi_0^{(-i)}$ has been estimated in the previous steps. The volatility $\widetilde{g}_{2t}$ depends on $\vartheta_0^{(+i)} = (\theta_0^{(i)}, \varphi_0^{(+i)})$, and $\vartheta_0^{(i)} = (\theta_0^{(i)}, \varphi_0^{(i)})$ can be estimated by

$$\widehat{\vartheta}_n^{(i)} = \arg\min_{\vartheta^{(i)} \in \Theta^{(i)}} \sum_{t=1}^{n} \widetilde{q}_{it}(\vartheta^{(i)}, \widehat{\varphi}_n^{(-i)}), \quad \widetilde{q}_{it}(\vartheta^{(+i)}) = \frac{\widetilde{v}_{it}^2(\varphi^{(+i)})}{\widetilde{g}_{it}(\vartheta^{(+i)})} + \log \widetilde{g}_{it}(\vartheta^{(+i)}),$$

$$\widetilde{g}_{it}(\vartheta^{(+i)}) = \omega_{i,t-1} + \sum_{k=2}^{i} \alpha_i^{(k)} \widetilde{v}_{k,t-1}^2(\varphi^{(+k)}) + b_i \widetilde{g}_{i,t-1}(\vartheta^{(+i)}),$$

$$\widetilde{v}_{kt}(\varphi^{(+k)}) = \epsilon_{kt} - \sum_{j=1}^{k-1} \widetilde{\beta}_{kj,t}(\varphi^{(+k)}) \epsilon_{jt},$$

$$\widetilde{\beta}_{ij,t}(\varphi^{(+i)}) = \omega_{ij,t-1} + \sum_{k=2}^{i} \tau_{ij}^{(k)} \widetilde{v}_{k,t-1}(\varphi^{(+k)}) + c_{ij} \widetilde{\beta}_{ij,t-1}(\varphi^{(+i)}),$$

QML vs. EbE

1) If $m = 2$, the one-step full QMLE and the two-step EbEE are exactly the same.
2) For $m \geq 3$, the two estimators are generally different.
3) The QML and EbE estimators are CAN under similar assumptions ▸ CAN of the EbEE.
4) The EbEE is simpler, but is not always less efficient than the full QMLE.

We illustrate the last point on a simplistic example.

Example

Consider a static model with $g_{it} = 1$ for $i \in \{1, 2, 3\}$.

$$\begin{aligned} \epsilon_{1t} &= v_{1t} \\ \epsilon_{2t} &= l_{21} v_{1t} + v_{2t} \\ \epsilon_{3t} &= \underbrace{l_{31}}_{0} v_{1t} + l_{32} v_{2t} + v_{3t} = l_{32} v_{2t} + v_{3t} \end{aligned}$$

In terms of β's:

$$\begin{aligned} \epsilon_{2t} &= \beta_{21} \epsilon_{1t} + v_{2t} \\ \epsilon_{3t} &= \underbrace{-\beta_{21}\beta_{32}}_{\beta_{31}} \epsilon_{1t} + \beta_{32} \epsilon_{2t} + v_{3t} \end{aligned}$$

$\rightarrow \boldsymbol{\vartheta} = (\beta_{21}, \beta_{32})$

Example

Assume for instance that the variable v_{2t} is independent of the vector $(v_{1t}, v_{3t})'$ and that this vector is distributed as the product $\eta \boldsymbol{u}$, where the random variable η and the vector $\boldsymbol{u}$ are independent, e.g., $\boldsymbol{u} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}_2)$ and $\eta \sim \sqrt{(\nu-2)/\nu} St_\nu$, $\nu > 4$.

$$\text{QMLE: } \sqrt{n}\left(\widehat{\boldsymbol{\vartheta}}_n - \boldsymbol{\vartheta}_0\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left\{0, \begin{pmatrix} \frac{1+\beta_{32}^2 E\eta^4}{(1+\beta_{32}^2)^2} & 0 \\ 0 & E\eta^4 \end{pmatrix}\right\}.$$

$$\text{EbEE: } \sqrt{n}\left(\widehat{\beta}_{21} - \beta_{0,21}\right) = \frac{n^{-1/2}\sum_{t=1}^n \eta_{2t}\epsilon_{1t}}{n^{-1}\sum_{t=1}^n \epsilon_{1t}^2} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1).$$

Illustration

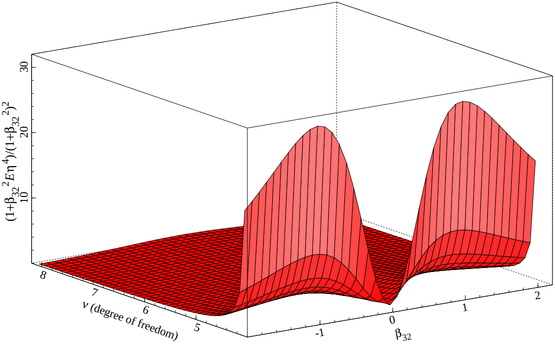
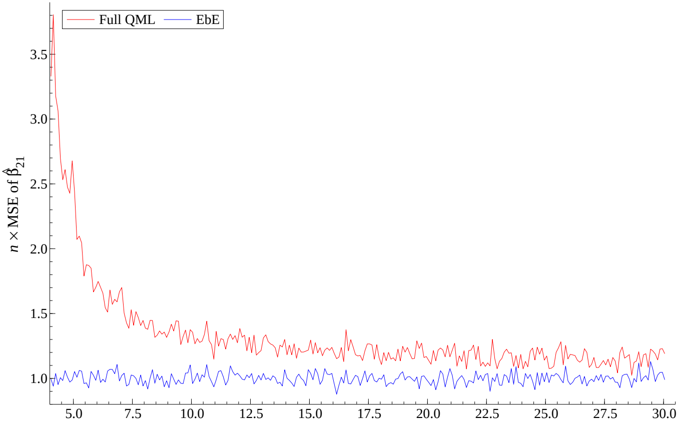


Figure: Ratio between the asymptotic variance of the QML estimator of β_{21} and its EbE counterpart as a function of the degree of freedom ν and β_{32}.

Illustration



Data Generating process

$$
\begin{aligned}
\boldsymbol{\epsilon}_t &= \boldsymbol{\Sigma}_t^{1/2}(\vartheta_0)\boldsymbol{\eta}_t, \\
\boldsymbol{\Sigma}_t &= \boldsymbol{L}_t\boldsymbol{G}_t\boldsymbol{L}_t', \\
g_{i,t} &= 0.1 + 0.1 v_{i,t-1}^2 + 0.8 g_{i,t-1}, \\
\beta_{ij,t} &= 0.1 + 0.2 v_{i,t-1} + 0.8\beta_{ij,t-1},
\end{aligned}
$$

where $(\boldsymbol{\eta}_t)$ is iid $\mathcal{N}(0, I_m)$, $t = 1, \ldots, n$.

Results for $m = 5, \eta \sim \mathcal{N}(0, I_m)$, 1000 replications

	FULL QML				EbE			
	BIAS	RMSE-STD	5% CP	95% CP	BIAS	RMSE-STD	5% CP	95% CP
n=1000								
ω	0.0202	0.0158	4.444	91.695	0.0190	0.0100	4.070	92.251
α	0.0037	0.0024	4.018	91.111	0.0036	-0.0004	3.659	91.696
β	-0.0245	0.0178	7.834	94.007	-0.0230	0.0092	7.379	94.347
ϖ	0.0008	0.0011	5.769	91.538	0.0007	0.0001	4.604	92.559
τ	0.0012	0.0011	7.306	93.939	0.0011	-0.0001	6.208	94.779
c	-0.0017	0.0022	8.115	93.805	-0.0016	0.0003	7.009	94.913
ALL	0.0000	0.0050	6.520	92.820	0.0000	0.0021	5.639	93.644
n=2000								
ω	0.0098	0.0014	3.824	92.745	0.0087	0.0000	3.305	93.138
α	0.0019	0.0014	4.412	92.966	0.0017	-0.0004	3.766	93.410
β	-0.0119	0.0034	6.642	94.804	-0.0104	-0.0006	6.130	95.460
ϖ	0.0002	-0.0009	6.483	91.324	0.0002	-0.0003	4.550	93.410
τ	0.0003	0.0001	7.145	93.493	0.0004	-0.0002	5.690	94.812
c	-0.0005	-0.0014	8.039	94.069	-0.0004	-0.0006	6.266	95.690
ALL	0.0000	0.0002	6.468	93.143	0.0000	-0.0004	5.135	94.426

$g_{i,t} = \omega_i + \alpha_i v_{i,t-1}^2 + \beta_i g_{i,t-1}$ and $\beta_{ij,t} = \varpi_{ij} + \tau_{ij} v_{i,t-1} + c_{ij}\beta_{ij,t-1}$

Results for $m = 5, \eta_i \sim t(0, 1, 7)$, 1000 replications

	FULL QML				EbE			
	BIAS	RMSE-STD	5% CP	95% CP	BIAS	RMSE-STD	5% CP	95% CP
n=1000								
ω	0.0236	0.0153	4.468	89.811	0.0225	0.0050	4.227	90.284
α	0.0056	0.0015	2.931	89.362	0.0057	-0.0066	2.860	90.074
β	-0.0309	0.0181	9.409	92.719	-0.0293	-0.0013	8.980	93.270
ϖ	0.0010	-0.0002	5.922	91.052	0.0010	-0.0014	4.532	92.650
τ	0.0014	-0.0013	7.080	94.173	0.0012	-0.0016	5.910	95.152
c	-0.0019	0.0005	8.818	93.995	-0.0018	-0.0024	7.308	95.226
ALL	-0.0001	0.0037	6.717	92.259	-0.0001	-0.0015	5.730	93.298
n=2000								
ω	0.0062	0.0018	3.624	92.584	0.0093	0.0005	3.141	91.734
α	0.0009	0.0007	3.490	91.678	0.0020	-0.0017	2.748	91.189
β	-0.0079	0.0025	7.282	94.899	-0.0120	-0.0006	7.546	94.984
ϖ	0.0002	0.0001	8.188	90.336	0.0003	-0.0004	4.308	94.079
τ	0.0002	0.0002	7.953	92.953	0.0005	-0.0006	4.984	95.507
c	-0.0004	0.0001	9.463	91.695	-0.0007	-0.0008	5.965	95.725
ALL	-0.0001	0.0006	7.289	92.125	0.0000	-0.0006	4.883	94.281

$g_{i,t} = \omega_i + \alpha_i v_{i,t-1}^2 + \beta_i g_{i,t-1}$ and $\beta_{ij,t} = \varpi_{ij} + \tau_{ij} v_{i,t-1} + c_{ij}\beta_{ij,t-1}$

Results for $m = 10$, $n = 4000$ and 1000 replications

Summary statistics on the 165 parameters	
Mean biais	-0.000873171
Mean (rmse-Mean STD)	0.00155948
Mean Coverage Prob 5%	5.042389091
Mean Coverage Prob 95%	93.6329697

Asset Pricing for Industry Portfolios

We consider the 12 industry portfolios used by Engle (2016), examined in the context of the Fama French 3 factor model.

The three factors are: *MKT* (Market factor = excess log-returns of the SP500), *SMB* (small minus big size factor) and *HML* (high minus low value factor)

Data are from Ken French's web site and cover the period 1994-2016.

We follow Patton and Verardo (2012) in building hedged portfolios to offset some unwanted exposures to predetermined factors.

Competing models

Let $\epsilon_t = (\mathbf{x}_t', y_t)'$ with $\mathbf{x}_t = (MKT_t, SMB_t, HML_t)'$ and $y_t = r_{kt}$.

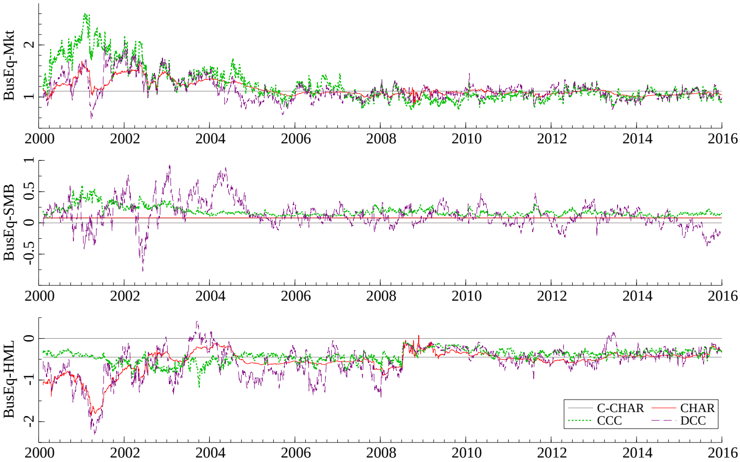
Hedging strategy:

$$E_{t-1}(r_{kt} \mid \boldsymbol{x}_t) = \beta_{k,MKT,t} MKT_t + \beta_{k,SMB,t} SMB_t + \beta_{k,HML,t} HML_t.$$

Competing models:

1) CCC-GARCH(1,1)
2) DCC-GARCH(1,1)
3) CHAR with constant betas
4) CHAR with time varying betas $\beta_{ij,t} = \varpi_{ij} + \tau_{ij} v_{i,t-1} v_{j,t-1} + c_{ij} \beta_{ij,t-1}$

Buseq: Business Equipment – Computers, Software, and Electronic Equipment



Buseq: One-step ahead forecasts

$$Z_{k,t+1|t} = \beta_{k,MKT,t+1|t} MKT_{t+1} + \beta_{k,SMB,t+1|t} SMB_{t+1} + \beta_{k,HML,t+1|t} HML_{t+1},$$

$$TE_{k,t+1} = r_{k,t+1} - Z_{k,t+1|t}$$

Robust (HAC) t-statistic of the regression of the tracking errors on a constant

	C-CHAR	CHAR	CCC	DCC
BusEq	-1.025	-0.442	0.133	-0.889
Chems	0.683	0.150	1.187	0.252
Durbl	-0.059	-0.402	-0.123	-0.720
Enrgy	-1.191	-1.865	-0.528	-1.665
Hlth	1.664	1.123	1.777	1.325
Manuf	-0.101	-0.269	0.064	-0.197
Money	-0.699	-0.648	-0.837	-0.541
NoDur	**2.731**	**2.180**	**2.961**	**2.343**
Other	-0.193	-0.436	-0.394	-0.415
Shops	1.746	1.947	**2.447**	1.768
Telcm	1.392	1.604	**2.247**	1.858
Utils	1.053	0.858	1.549	1.010

Note: Robust (HAC) t-statistics for the null hypothesis that the coefficient in the regression of $TE_{k,t+1}$ on a constant is zero. Values in bold are greater (in absolute value) than the critical value at the 5% significance level.

Results of the MCS test – MSE loss function

	C-CHAR	CHAR	CCC	DCC
BusEq		✓		
Chems		✓		
Durbl		✓		
Enrgy		✓		✓
Hlth		✓		✓
Manuf		✓		
Money		✓		✓
NoDur		✓		
Other		✓		
Shops		✓		
Telcm		✓		
Utils		✓		

Models included in the MCS in the beta hedging exercise. Models highlighted with the symbol ✓ are contained in the model confidence set using a MSE loss function. The significance level for the MCS is set to 20%, and 10,000 bootstrap samples (with a block length of 5 observations).

Transaction costs : $\frac{\Delta\beta_{CHAR}}{\Delta\beta_{DCC-DCB}}$

	MKT	SMB	HML
BusEq	0.356	0.380	0.341
Chems	0.310	0.263	0.376
Durbl	0.419	0.464	0.693
Enrgy	0.373	0.337	0.456
Hlth	0.461	0.667	0.397
Manuf	0.442	0.402	0.430
Money	0.390	0.397	0.366
NoDur	0.414	0.383	0.296
Other	0.273	0.343	0.335
Shops	0.344	0.297	0.395
Telcm	0.334	0.414	0.640
Utils	0.465	0.408	0.431

$\Delta_{\beta_{k,j}} = \sum_{t=2}^{1,678} |\beta_{k,j,t+1|t} - \beta_{k,j,t|t-1}|$.
For each column, the figures correspond to the ratio between the value of $\Delta_{\beta_{k,j}}$ obtained for the CHAR and the DCC-DCB models.

Conclusion

Compare to other multivariate GARCH (in particular BEKK and DCC), the Cholesky-GARCH models introduced here have several advantages.

1) Precise stationarity and moment conditions exist.
2) The parameters are directly interpretable in terms of DCB.
3) There is no complicated correlation constraint.
4) The estimation can be done without matrix invertion.
5) The asymptotic theory of the QMLE is available.
6) EbE estimation is possible for triangular models.
7) The model works nicely in practice, in particuilar for beta hedging.

Conclusion

Compare to other multivariate GARCH (in particular BEKK and DCC), the Cholesky-GARCH models introduced here have several advantages.

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7) The model works nicely in practice, in particuilar for beta hedging.

Thanks for your attention!

CAN of the EbE estimator

Theorem (CAN of the EbEE)

Under **B1-B5** *the EbEE* $\widehat{\boldsymbol{\vartheta}}_n = \left(\widehat{\boldsymbol{\vartheta}}_n^{(1)'}, \dots, \widehat{\boldsymbol{\vartheta}}_n^{(m)'}\right)'$ *satisfies*

$$\widehat{\boldsymbol{\vartheta}}_n \to \boldsymbol{\vartheta}_0, \quad \text{almost surely as } n \to \infty.$$

Under the additional assumption **B6***, as* $n \to \infty$*,*

$$\sqrt{n}\left(\widehat{\boldsymbol{\vartheta}}_n^{(i)} - \boldsymbol{\vartheta}_0^{(i)}\right) \stackrel{\mathcal{L}}{\to} \mathcal{N} \left\{0, \boldsymbol{\Sigma}^{(i)} := \left(\boldsymbol{J}^{(i)}\right)^{-1} \boldsymbol{I}^{(i)} \left(\boldsymbol{J}^{(i)}\right)^{-1}\right\}$$

for $i = 1$ *and* $i = 2$*.*

$$\boldsymbol{J}_n^{(i)} = \frac{\partial^2 \widetilde{O}_n^{(i)}(\widehat{\boldsymbol{\vartheta}}_n^{(+i)})}{\partial \boldsymbol{\vartheta}^{(i)} \partial \boldsymbol{\vartheta}^{(i)'}}, \qquad \boldsymbol{I}_n^{(i)} = \frac{1}{n} \sum_{t=1}^{n} \frac{\partial \widetilde{q}_{it}(\widehat{\boldsymbol{\vartheta}}_n^{(+i)})}{\partial \boldsymbol{\vartheta}^{(i)}} \frac{\partial \widetilde{q}_{it}(\widehat{\boldsymbol{\vartheta}}_n^{(+i)})}{\partial \boldsymbol{\vartheta}^{(i)'}}$$

CAN of the EbE estimator

$\beta_{32,t} = \varpi_{032} + \ldots + \tau_{032}^{(2)} v_{2,t-1} + c_{032}\beta_{32,t-1}$, where $v_{2,t-1} = \varepsilon_{2,t-1} - \beta_{21,t-1}\varepsilon_{1,t-1}$.

Theorem (CAN of the EbEE)

Denoting by $\mathbf{\Sigma}_{\boldsymbol{\varphi}^-}^{(+i)}$ (or by $\mathbf{\Sigma}_{\boldsymbol{\varphi}^+}^{(+(i-1))}$) the bottom-right sub-matrix of $\mathbf{\Sigma}^{(+i)}$ (or of $\mathbf{\Sigma}^{(+(i-1))}$) corresponding to the asymptotic variance of $\widehat{\varphi}_n^{(-i)}$ (which is equal to $\widehat{\varphi}_n^{(+(i-1))}$), and using the convention $\mathbf{\Sigma}^{(+2)} = \mathbf{\Sigma}^{(2)}$, for $i = 3, \ldots, m$ we also have

$$\sqrt{n}\left(\widehat{\boldsymbol{\vartheta}}_n^{(+i)} - \boldsymbol{\vartheta}_0^{(+i)}\right) \quad \xrightarrow{\mathcal{L}} \mathcal{N} \quad \left(0, \mathbf{\Sigma}^{(+i)}\right)$$

with

$$\mathbf{\Sigma}^{(+i)} = \begin{pmatrix} \mathbf{\Sigma}_{\boldsymbol{\vartheta}}^{(i)} & -\left(\boldsymbol{J}^{(i)}\right)^{-1} \boldsymbol{K}^{(i)} \mathbf{\Sigma}_{\boldsymbol{\varphi}^+}^{(+(i-1))} \\ -\left(\boldsymbol{J}^{(i)\prime}\right)^{-1} \boldsymbol{K}^{(i)\prime} \mathbf{\Sigma}_{\boldsymbol{\varphi}^+}^{(+(i-1))} & \mathbf{\Sigma}_{\boldsymbol{\varphi}^+}^{(+(i-1))} \end{pmatrix}$$

where $\mathbf{\Sigma}_{\boldsymbol{\vartheta}}^{(i)} = \left(\boldsymbol{J}^{(i)}\right)^{-1} \left\{\boldsymbol{I}^{(i)} + \boldsymbol{K}^{(i)} \mathbf{\Sigma}_{\boldsymbol{\varphi}^+}^{(+(i-1))} \boldsymbol{K}^{(i)\prime}\right\} \left(\boldsymbol{J}^{(i)}\right)^{-1}$ *and* $\boldsymbol{K}_n^{(i)} = \dfrac{\partial^2 \tilde{O}_n^{(i)}(\widehat{\boldsymbol{\vartheta}}_n^{(+i)})}{\partial \boldsymbol{\vartheta}^{(i)} \partial \boldsymbol{\varphi}^{(-i)\prime}}$

Return