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# Basis in the big disk algebra and in the corresponding Hardy-space

F. Lancien

*Université de Besancon, Équipe de Mathématiques, U.A. n. 040741,*
*25030 Besancon Cedex, France*

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Abstract

It is well known that the disk algebra $A(\mathbb{T})$ has a basis [1] and that $H^1(\mathbb{T})$ has an unconditional basis [9]. Recently W. Lusky gave new proofs of these results using the commuting bounded approximation property ([7] and [8]).

With similar methods we prove the existence of a basis in the so called "big disk algebra" $\mathcal{A}(\mathbb{T}^N)$, the space of continuous functions on the multidimensional torus $\mathbb{T}^N$ which are "analytic" with respect to the lexicographic order on the dual group $\mathbb{Z}^N$ and in the space $\mathcal{H}^1(\mathbb{T}^N)$, the analog for $L^1$ functions on $\mathbb{T}^N$.

## I. Introduction

We fix here some notations and recall Lusky's result. Then in Part II we prove the existence of a basis in the "big disk algebra" $\mathcal{A}(\mathbb{T}^N)$ and in the space $\mathcal{H}^1(\mathbb{T}^N)$.

For $G$ a compact group with Haar measure $\mu$ and dual group $\Gamma$, we will denote by $\Gamma_+$ the positive part of $\Gamma$ with respect to a total order on it *i.e.* $\Gamma = \Gamma_+ \cup (-\Gamma_+)$, $\Gamma_+ \cap (-\Gamma_+) = \{0\}$. The Fourier transform of a function $f$ on $G$ is $\hat{f}(\chi) = \int_G \chi(-x) f(x) d\mu(x)$ for $\chi$ in $\Gamma$. Let

$$\mathcal{A}(G) = \big\{ f \in \mathcal{C}(G) \;\text{ s.t. }\; \hat{f}(\chi) = 0 \;\text{ for all }\; \chi \;\text{ in }\; \Gamma \setminus \Gamma_+ \big\},$$
$$\mathcal{H}^1(G) = \big\{ f \in L^1(G,\mu) \;\text{ s.t. }\; \hat{f}(\chi) = 0 \;\text{ for all }\; \chi \;\text{ in }\; \Gamma \setminus \Gamma_+ \big\}.$$

The groups that we will consider here are the torus $\mathbb{T} = \{z \in \mathbb{C}, |z| = 1\}$, the finite dimensional torus $\mathbb{T}^N = \big\{(z_1 \dots z_N), z_i \in \mathbb{T}, i = 1 \dots N\big\}$ and the infinite dimensional torus $\mathbb{T}^{\mathbb{N}} = \{(z_1 \dots z_N \dots), z_i \in \mathbb{T},\ i = 1,2\dots\}$. Their duals are respectively $\mathbb{Z}, \mathbb{Z}^N$ and $\mathbb{Z}^{(\mathbb{N})}$ the set of sequences of integer with finitely many non zero terms.

Throughout this paper we will use operators on $\mathcal{C}(G), L^1(G)$, defined by multipliers on $\Gamma$. Namely, operators verifying: $(Tf)^\wedge(\chi) = t(\chi)\hat{f}(\chi), \forall\chi \in \Gamma$. We will denote by capital letters the operators $T_n, S_n \dots$ and by the corresponding small letters their associated multipliers $t_n, s_n \dots$ on $\Gamma$.

On $\mathbb{T}$ we use the standard notations:
the $k^{th}$ partial sum of the Fourier series is $S_k f(t) = \sum_{|j|\le k} \hat{f}(j)e^{ijt}$,
the $n^{th}$ Cesaro mean is $\sigma_n f = \frac{1}{n+1}\sum_{|k|\le n} S_k f$.

For multidimensional tori, we will consider operators which are compositions of operators acting only on some of the variables. We will use the following notation. For $K$ an operator on $L^1(\mathbb{T}^k)$, we will denote by $K^{l_1,\dots,l_k}$ the operator on $L^1(\mathbb{T}^N)$ or $L^1(\mathbb{T}^{\mathbb{N}})$ which acts as $K$ on the $l_1^{th}, \dots$ and $l_k^{th}$ variables and fixes the other variables.

We now give Lusky's result about the existence of a basis in Banach spaces having the commuting bounded approximation property. Let first recall that a Banach space $X$ is said to have the *commuting bounded approximation property* (in short CBAP) if there exist operators $R_n : X \to X$ such that:

(0) $R_n$ has finite rank,
(1) $(R_n)$ is bounded,
(2) $R_n x \to x$ for all $x$ in $X$,
(3) $R_n R_m = R_{\min(n,m)}, n \ne m$.

If each $R_n$ is a projection, $(R_n)$ is called a *finite dimensional Schauder decomposition* (in short FDD).

It is known that the CBAP does not imply the existence of a basis or even of a FDD ([11], one can also see [3] and [13] for related results). But W. Lusky has proven ([7]) that if $X$ has the CBAP with the sequence $(R_n)$ satisfying moreover

(4) $R_n - R_{n-1}$ factors through $\ell_p^{m_n}$ uniformly for some $p$ in $[1,\infty]$,

then $X \oplus \ell_p$, if $1 \le p < \infty$, or $X \oplus c_0$, if $p = \infty$, has a basis.

More precisely (4) means that there exist operators $A_n : X \to \ell_p^{m_n}$, $B_n : \ell_p^{m_n} \to X$ with $R_n - R_{n-1} = B_n A_n$ and $\sup_n \|A_n\| < \infty$, $\sup_n \|B_n\| < \infty$.

## II. Bases in $\mathcal{A}(\mathbb{T}^N)$ and $\mathcal{H}^1(\mathbb{T}^N)$

We consider now $\mathbb{T}^N$ the $N^{th}$ dimensional torus, whose dual group is $\mathbb{Z}^N$. We take the lexicographic order on $\mathbb{Z}^N$ $\big(i.e.\ (k_1, \dots, k_N)$ is positive if $k_1 > 0$ or, $k_1 = 0$ and

$k_2 > 0$, or $k_1 = k_2 = 0$ and $k_3 > 0, \dots\big)$. Then a function is said to be "analytic" for this order, if the support of its Fourier transform is included in the non negative part of $\mathbb{Z}^N$. For more details about Fourier analysis on groups with ordered dual we refer to chapter 8 of [12]. We consider $\mathcal{A}(\mathbb{T}^N)$, the algebra of "analytic" continuous functions on $\mathbb{T}^N$ and $\mathcal{H}^1(\mathbb{T}^N)$ the space of "analytic" $L^1$ functions on $\mathbb{T}^N$.

**Theorem 1**

$$\mathcal{A}(\mathbb{T}^N) \ \textit{and} \ \mathcal{H}^1(\mathbb{T}^N) \ \textit{have a basis}\,.$$

*Remarks.* 1) Of course, since $\mathcal{H}^1(\mathbb{T}^N)$ contains $L^1(\mathbb{T})$ as soon as $N \geq 2$, there is no hope to extend to the multidimensional case the existence of an unconditional basis in $H^1(\mathbb{T})$ (see [5]).

2) Let us remark also that $\mathcal{H}^1(\mathbb{T}^N)$ is not isomorphic to $L^1(\mathbb{T}^N)$. Indeed a complemented subspace of an $L^1$ space having R.N.P. is isomorphic to $l^1$ (see [2]). Here $H^1(\mathbb{T})$ which has R.N.P. is complemented in $\mathcal{H}^1(\mathbb{T}^N)$ (using the projection $S_0^2 \dots S_0^N$).

3) A proof similar to the proof of the fact that $A(B^N)$ is not isomorphic to $A(D)$, given in [10], shows that $\mathcal{A}(\mathbb{T}^N)$ is not isomorphic to $A(D)$.

*Proof.* We wish to explain first the strategy we will follow in the proof.

First we note that $\mathcal{A}(\mathbb{T}^N) \oplus c_0 \simeq \mathcal{A}(\mathbb{T}^N)$ and $\mathcal{H}^1(\mathbb{T}^N) \oplus \ell_1 \simeq H^1(\mathbb{T}^N)$. To see the first isomorphism one can construct a copy of $c_0$ in $\mathcal{A}(\mathbb{T}^N)$ by using Weierstra$\beta$' theorem. For the second, the non equiintegrability of the ball of $\mathcal{H}^1(\mathbb{T}^N)$ gives peak functions which provide a projection onto $\ell^1$.

So we want to find operators $T_n : \mathcal{A}(\mathbb{T}^N) \mapsto \mathcal{A}(\mathbb{T}^N)$ $\big($resp. $\mathcal{H}^1(\mathbb{T}^N) \mapsto \mathcal{H}^1(\mathbb{T}^N)\big)$, which satisfy properties $(0)-(4)$, in order to use Lusky's theorem. In fact the same operators are used in both cases, so we will write the argument only for $\mathcal{A}(\mathbb{T}^N)$.

Since most of these properties are easier to check on multipliers, we will work with restrictions of operators on $\mathcal{C}(\mathbb{T}^N)$ defined by multipliers. We will look for

(1) a bounded sequence of operators $T_n$ on $\mathcal{C}(\mathbb{T}^N)$,

such that the corresponding multipliers $t_n$ on $\mathbb{Z}^N$ satisfy:

(0') $\operatorname{supp} t_n$ is finite, for all $n$,

(2') $\bigcup \operatorname{supp} t_n = \mathbb{Z}^N$ ,

(3') $t_n = 1$ on $\operatorname{supp}\, t_m$, for $m < n$ .

For property (4), we will use as in [7], the following idea. Once (3) holds, we have $T_n - T_{n-1} = (T_n - T_{n-1})\,T_{n+1}$. Since each $T_n$ is of finite rank, it is possible to find finite dimensional subspaces $E_n$ of $\mathcal{C}(\mathbb{T}^N)$ with $T_{n+1}A(\mathbb{T}^N) \subseteq E_n$ and $\sup_n d\big(E_n, \ell_\infty^{\dim E_n}\big) < \infty$ . Then the following factorization holds

$$\begin{array}{ccccc} \mathcal{A}(\mathbb{T}^N) & & \xrightarrow{T_n - T_{n+1}} & & \mathcal{A}(\mathbb{T}^N) \\ T_{n+1} & \searrow & & \nearrow & \big(T_n - T_{n-1}\big)R \\ & & E_n \subseteq \mathcal{C}(\mathbb{T}^N) & & \end{array}$$

provided that $(T_n - T_{n-1})R$ is uniformly bounded, where $R$ is the Riesz projection (which itself is not bounded on $\mathcal{C}(\mathbb{T}^N)$). For this, we will choose $t_n$ in such a way that:

(4') there exists a sequence of uniformly bounded multipliers $(b_n)_n$ on $\mathcal{C}(\mathbb{T}^N)$ with supp $b_n \subset \mathbb{Z}_+^N$ and $b_n = 1$ on supp $(t_n - t_{n-1})$ $\big($i.e. $b_n$ acts as the Riesz projection does on supp $(t_n - t_{n-1})\big)$.

When such a sequence of multipliers $t_n$ will be constructed, this by Lusky's theorem will imply that $\mathcal{A}(\mathbb{T}^N) \oplus c_0$ thus $\mathcal{A}(\mathbb{T}^N)$ has a basis. The same proof will work for $\mathcal{H}^1(\mathbb{T}^N)$ with $L^1(\mathbb{T}^N)$, $\ell_1^{\dim E_n}$, and $\ell_1$ instead of $\mathcal{C}(\mathbb{T}^N)$, $\ell_1^{\dim E_n}$, and $c_0$ and will prove the existence of a basis on $\mathcal{H}^1(\mathbb{T}^N)$.

We now proceed to the explicit construction of the required multipliers on $\mathbb{T}^N$. We define them by induction on the dimension.

We recall briefly the case of the dimension 1 which is treated in [8], because this is the starting point of our inductive construction. We consider $V_n$ the De La Vallée - Poussin operator of order $2^n, V_n = 2\sigma_{2^{n+1}} - \sigma_{2^n}$, thus $(V_n)$ is bounded on $\mathcal{C}(\mathbb{T})$. The corresponding multiplier is:

$$\begin{aligned} v_n(k) &= 1 \quad \text{for} \quad |k| \le 2^n \\ &= \frac{2^{n+1} - |k|}{2^n} \quad \text{for} \quad 2^n \le |k| \le 2^{n+1} \\ &= 0 \quad \text{for} \quad |k| \ge 2^{n+1}\,. \end{aligned}$$

It is easy to check that $v_n$ satisfies the conditions (0'), (2') and (3'). In order to have (4') we use $D_n$ defined by the following multiplier:

$$\begin{aligned} d_n(k) =& \frac{k - 2^{n-1}}{2^{n-1}} \quad \text{for} \quad 0 \le k \le 2^{n-1} \\ &= 1 \quad \text{for} \quad 2^{n-1} \le k \le 4.2^{n-1} \\ &= \frac{5.2^{n-1} - k}{2^{n-1}} \quad \text{for} \quad 4.2^{n-1} \le k \le 5.2^{n-1} \\ &= 0 \quad \text{otherwise}\,. \end{aligned}$$

Then $D_n$ has the following properties: $D_n\mathcal{C}(\mathbb{T}) \subseteq \mathcal{A}(\mathbb{T}), (V_n - V_{n-1})D_n|_{\mathcal{A}(\mathbb{T})} = (V_n - V_{n-1})|_{\mathcal{A}(\mathbb{T})}$ and $\sup_n \|D_n\| < \infty$. This gives (4).

On $\mathbb{Z}^2$, we form new multipliers by "crossbreeding" two sequences of multipliers on $\mathbb{Z}$. For this, we use operators that act on each variable, as described in § I. We define $T_n$ by

$$T_n = \sum_{i=0}^{n} V^1_{n-i}\big(S^2_{2^i} - S^2_{2^{i-1}}\big) + S^1_0\big(V^2_{2^n} - S^2_{2^n}\big)$$

with the convention $S_{2^{-1}} = 0$. (In fact we decompose $V_n$ with respect to dyadic blocks: $V_n = \sum_{i=0}^{n}\big(S_{2^i} - S_{2^{i-1}}\big) + V_n - S_{2^n}$ , then we apply this operator in the second variable, composed on each dyadic bloc with decreasing $V_k's$ acting on the first variable).

The multiplier associated to the operator $T_n$, is:

$$\begin{aligned} t_n(k_1,k_2) &= v_{n-i}(k_1) \quad \text{if} \quad 2^{i-1} \le k_2 \le 2^i, \quad i = 0,1,\dots n \\ &= v_n(k_2) \qquad \text{if} \quad 2^n \le k_2 \le 2^{n+1},\ k_1 = 0 \\ &= 0 \quad \text{otherwise}. \end{aligned}$$

This multiplier clearly satisfies supp $t_n$ is finite, $\bigcup \operatorname{supp} t_n = \mathbb{Z}^2$ and $t_n = 1$ on supp $t_{n-1}$. Hence we have (0'), (2') and (3').

Moreover since $V_n = 2\sigma_{2^{n+1}} - \sigma_{2^n}$, the operator $T_n$ is the difference of two operators of the form $\frac{1}{2^n+1}\sum_{i=0}^{n}\sum_{2^{i-1}<k\le 2^{-1}} S^1_k V^2_{n-i}$ .

Since $\sup_n \frac{1}{2^n+1}\sum_{i=0}\big\|\sum_{2^{i-1}<k\le 2^i} S_k\big\| < \infty$ and $\sup_n \|V_n\| < \infty$ , we obtain that $(T_n)$ is uniformly bounded, thus we have (1).

To achieve condition (4), we take $B_n$ constructed in a way similar to $T_n$, with $D^2_k$ is place of $V^2_k$ :

$$B_n = \sum_{i=0}^{n} D^1_{n-i}\big(S^2_{2^i} - S^2_{2^{i-1}}\big) + S^1_0\big(V^2_{n+1} - S^2_{2^n}\big)D^2_n\,.$$

Then for the same reason as above for $(T_n)$, since $\sup_n \|D_k\| < \infty$ , $(B_n)$ is uniformly bounded. Moreover from the fact that $d_n = 1$ on supp $(v_n - v_{n-1}) \cap \mathbb{Z}_+$ and supp $d_n \subset \mathbb{Z}_+$, it follows that $b_n = 1$ on supp $(t_n - t_{n-1}) \cap \mathbb{Z}^2_+$ and supp $b_n \subset \mathbb{Z}^2_+ = \big\{(k,\ell) \in \mathbb{Z}^2\ s.t.\ \ k \ge 0 \text{ or } (k = 0 \text{ and } \ell \ge 0)\big\}$ .

Thus $T_n - T_{n-1}|_{\mathcal{A}(\mathbb{T}^2)} = (T_n - t_{n-1})B_n|_{\mathcal{A}(\mathbb{T}^2)}$ where $B_n : C(\mathbb{T}^2) \to \mathcal{A}(\mathbb{T}^2)$. So we have the following factorization: $T_n - T_{n-1}|_{\mathcal{A}(\mathbb{T}^2)} = (T_n - T_{n-1})B_n|_{\mathcal{A}(\mathbb{T}^2)}T_{n+1}|_{\mathcal{A}(\mathbb{T}^2)}$, which proves (4).

By Lusky's theorem we obtain that $\mathcal{A}(\mathbb{T}^2) \oplus c_0$ has a basis, thus $\mathcal{A}(\mathbb{T}^2)$ has a basis. Now, we define inductively operators on $\mathcal{C}(\mathbb{Z}^l)$ by:

$$
\begin{aligned}
T_n^1 &= V_n \\
T_n^2 &= T_n \\
&\vdots \\
T_n^{1,2,\ldots l} &= \sum_{i=0}^{n} T_{n-i}^{1,\ldots l-1}\big(S_{2^i}^l - S_{2^{i-1}}^l\big) + S_0^{1,\ldots l-1}\big(V_{2^n}^l - S_{2^n}^l\big)
\end{aligned}
$$

and

$$
\begin{aligned}
B_n^1 &= U_n \\
B_n^2 &= B_n \\
&\vdots \\
B_n^{1,2,\ldots l} &= \sum_{i=0}^{n} B_{n-i}^{1,\ldots l-1}\big(S_{2^i}^l - S_{2^{i-1}}^l\big) + S_0^{1,\ldots l-1}\big(V_{2^n}^l - S_{2^n}^l\big)B_n^l\,.
\end{aligned}
$$

At this point it is easy to see that if $T_n^{1,2,\ldots l-1}$ and $B_n^{1,2,\ldots l-1}$ satisfy the required properties on $\mathbb{T}^{l-1}$, then $T_n^{1,2,\ldots l}$ and $B_n^{1,2,\ldots l}$ enjoy the same properties on $\mathbb{T}^l$. The arguments are exactly the same as in the proof of the two dimensional case once the one dimensional case is treated. Thus $(T_n^{1,2,\ldots l-1})_n$ satisfies properties $(0)-(4)$, which by Lusky's theorem proves the existence of a basis in $\mathcal{A}(\mathbb{T}^N)$. □

## III. Bases in $\mathcal{C}_\Lambda(\mathbb{T}^\mathbb{N})$ and $L^1_\Lambda(\mathbb{T}^\mathbb{N})$

We now consider for the group $G = \mathbb{T}^N$ or $\mathbb{T}^\mathbb{N}$, the space of continuous or $L^1$ functions whose Fourier transform has its support in some subset $\Lambda$ of the dual group $\Gamma$, we denote them respectively $\mathcal{C}_\Lambda(G)$ and $L^1_\Lambda(G)$ (the spaces $\mathcal{A}(G)$ and $\mathcal{H}^1(G)$ correspond to the case $\Lambda = \Gamma_+$). We recall that Cohen's theorem about idempotent measures asserts that the characteristic function of a subset $\Lambda$ of the dual group $\Gamma$ is the Fourier transform of some measure on the group $G$ if and only if $\Lambda$ is in the coset ring of $\Gamma$ (see by instance [12]). Using this result and sequences of generalized De La Vallée - Poussin multipliers one can prove the following.

**Theorem 2**

*(a) If $\Lambda$ is in the coset ring of $\mathbb{Z}^N$, then $\mathcal{C}_\Lambda(\mathbb{T}^N)$ and $L^1_\Lambda(\mathbb{T}^N)$ have a basis.*
*(b) If $\Lambda$ is in the coset ring of $\mathbb{Z}^\mathbb{N}$, then $\mathcal{C}_\Lambda(\mathbb{T}^\mathbb{N})$ and $L^1_\Lambda(\mathbb{T}^\mathbb{N})$ have a basis.*

*Proof.* The multipliers that we use for (a) are products of the characteristic function of the subset $\Lambda$ and of the De La Vallée - Poussin multipliers on $\mathbb{Z}^N$.

For (b) we use the following generalizations of the De La Vallée - Poussin multipliers. For $(\alpha_n)$ an increasing sequence of positive integers, let denote $\mathcal{V}_n$ the operator defined by the following multiplier on $\mathbb{Z}$:

$$\begin{aligned} \nu_n(k) &= 1 \quad \text{for} \quad |k| \leq \alpha_n \\ &= \frac{\alpha_{n+1} - k}{\alpha_{n+1} - \alpha_n} \quad \text{for} \quad \alpha_n \leq |k| \leq \alpha_{n+1} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

then $\mathcal{V}_n$ can be seen has the difference of two Cesaro means and thus $\|\mathcal{V}_n\| \leq \frac{\alpha_{n+1}+\alpha_n}{\alpha_{n+1}-\alpha_n}$. Let also $\delta_n$ be the analog of $d_n$ for $\nu_n$ in place of $v_n$ i.e. $\delta_n(k) = 1$ for $k > 0$ in supp $(\nu_n - \nu_{n-1})$ and supp $\delta_n$ included in $\mathbb{Z}_+$, then $\|\mathcal{D}_n\| \leq \frac{\alpha_{n+1}}{\alpha_n}$.

Note that the operators $V_n$ and $D_n$ we considered before correspond to $\alpha_n = 2^n$, in which case $\sup_n \|V_n\| \leq K$ and $\sup_n \|D_n\| \leq K$. Here we will take $\alpha_n^j = (1+j^2)^n$ so that $\|\mathcal{V}_n^j\| \leq \frac{\alpha_{n+1}^j+\alpha_n^j}{\alpha_{n+1}^j-\alpha_n^j} = 1 + 2/j^2$. We consider $R_n$ the operator defined by the following multiplier:

$$r_n\big(k_1, \dots k_n, \dots\big) = \nu_n^1(k_1) \dots \nu_n^n(k_n) s_0\big(k_{n+1}\big) s_0\big(k_{n+2}\big) \dots$$

then $\|R_n\| \leq \prod_{j=1}^{\infty} \|\mathcal{V}_n^j\| \prod_{j=1}^{\infty}(1 + 2/j^2) \leq K$ .

As for the proof of Theorem 1, one can check that $(R_n)_n$ composed with the projection $P_\Lambda$ (defined by the characteristic function of the subset $\Lambda$) is a sequence of operators which satisfies on $\mathcal{C}_\Lambda(\mathbb{T}^{\mathbb{N}})$ and $L^1_\Lambda(\mathbb{T}^{\mathbb{N}})$ the properties $(0) - (4)$ of Lusky's theorem. In particular we use the multipliers $\delta_n$ for the proof of (4). □

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