Multiplicity of positive solutions of nonlinear Schrödinger equations concentrating at a potential well

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Abstract

We consider singularly perturbed nonlinear Schrödinger equations

$$-\varepsilon^2\Delta u + V(x)u = f(u), \quad u > 0, \quad v \in H^1(\mathbb{R}^N) \tag{0.1}$$

where $V \in C(\mathbb{R}^N, \mathbb{R})$ and f is a nonlinear term which satisfies the so-called Berestycki-Lions conditions. We assume that there exists a bounded domain $\Omega \subset \mathbb{R}^N$ such that

$$m_0 \equiv \inf_{x\in\Omega} V(x) < \inf_{x\in\partial\Omega} V(x)$$

and we set $K = \{x \in \Omega \mid V(x) = m_0\}$. For $\varepsilon > 0$ small we prove the existence of at least $\mathrm{cupl}(K) + 1$ solutions to (0.1) concentrating, as $\varepsilon \to 0$ around K. We remark that, under our assumptions of f, the search of solutions to (0.1) cannot be reduced to the study of the critical points of a functional restricted to a Nehari manifold.

1 Introduction

In these last years a great deal of work has been devoted to the study of semiclassical standing waves for the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\Delta\psi - V(x)\psi + f(\psi) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N \tag{1.2}$$

where $\hbar$ denotes the Plank constant, i the imaginary unit, m is a positive number, $f(\exp(i\theta)\xi) = \exp(i\theta)f(\xi)$ for $\theta, \xi \in \mathbb{R}$. A solution of the form $\psi(x,t) = \exp(-iEt/\hbar)v(x)$ is called a standing wave. Then, assuming $m = \frac{1}{2}$, $\psi(x,t)$ is a solution of (1.2) if and only if the function v satisfies

$$\hbar^2\Delta v - (V(x) - E)v + f(v) = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{1.3}$$

For small $\hbar > 0$, these standing waves are referred to as semi-classical states. The limit $\hbar \to 0$ somehow described the transition from Quantum Mechanics to Classical Mechanics. For the physical background for this equation, we refer to the introduction in [1, 15].

In this paper we are interested on positive solutions of (1.3) in $H^1(\mathbb{R}^N)$ for small $\hbar > 0$. For simplicity and without loss of generality, we write $V - E$ as V, i.e., we shift E to 0 and we set $\hbar = \varepsilon$. Thus, we consider the following equation

$$-\varepsilon^2\Delta v + V(x)v = f(v), \quad v > 0, \quad v \in H^1(\mathbb{R}^N) \tag{1.4}$$

when $\varepsilon > 0$ is small. Throughout the paper, we assume that $N \geq 3$ and that the potential V satisfies

(V1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} V(x) = \underline{V} > 0$.

We observe that defining $u(x) = v(\varepsilon x)$ equation (1.4) is equivalent to

$$-\Delta u + V(\varepsilon x)u = f(u), \quad u > 0, \quad u \in H^1(\mathbb{R}^N). \tag{1.5}$$

We shall mainly work on equation (1.5). Note also that for each $x_0 \in \mathbb{R}$ and $R > 0$, $V(\varepsilon x)$ converges uniformly to $V(x_0)$ on $B(x_0/\varepsilon, R)$ as $\varepsilon \to 0$. Thus for each $x_0 \in \mathbb{R}^N$ we have a formal limiting problem

$$-\Delta U + V(x_0)U = f(U) \quad \text{in} \quad \mathbb{R}^N, \quad U \in H^1(\mathbb{R}^N). \tag{1.6}$$

The approaches to find solutions of (1.5) when $\varepsilon > 0$ is small and to study their behavior as $\varepsilon \to 0$ can be, roughly, classified into two categories.

A first approach relies on a reduction type method. The first result in this direction was given by Floer and Weinstein [36] where $N = 1$, $f(u) = u^3$ and x_0 is a non degenerate minimum or maximum of V. Later Oh [44, 45] generalized this result to higher dimensions N and $f(u) = |u|^{p-1}u$, $1 < p < (N+2)/(N-2)$, $N \geq 3$, $1 < p < \infty$ if $N = 1, 2$ for non degenerate minima or maxima of V. Successively in [1], a Liapunov-Schmidt type procedure was used to reduce (1.5), for $\varepsilon > 0$ small, to as finite-dimensional equation that inherits the variational structure of the original problem. Existence of semiclassical standing waves solutions were proved concentrating at local minima or local maxima of V, with nondegenerate m -th derivative, for some integer m. This result was generalized by Li [43], where a degeneracity of any order of the derivative is allowed. The reduction type methods permitted to obtain other precise and striking results in [2, 21, 22, 25, 35, 40]. However this approach relies on the uniqueness and non-degeneracy of the ground state solutions, namely of the positive least energy solutions for the autonomous problems (1.6). This uniqueness and nondegeneracy property is true for the model nonlinearity $f(\xi) = |\xi|^{p-1}\xi$, $1 < p < (N+2)/(N-2)$, $N \geq 3$ and for some classes of nonlinearities, see [23]. However, as it is shown in [29], it does not hold in general. For a weakening of the nondegeneracy requirement, within the frame of the reduction methods, see nevertheless [24].

An alternative type of approach was initiated by Rabinowitz [46]. It is purely variational and do not require the nondegeneracy condition for the limit problems (1.6). In [46] Rabinowitz proved, by a mountain pass argument, the existence of positive solutions of (1.5) for small $\varepsilon > 0$ whenever

$$\liminf_{|x| \to \infty} V(x) > \inf_{x \in \mathbb{R}^N} V(x). \tag{1.7}$$

These solutions concentrate around the global minimum points of V when $\varepsilon \to 0$, as it was shown by X. Wang [48]. Later, del Pino and Felmer [31] by introducing a penalization approach prove a localized version of the result of

Rabinowitz and Wang (see also [32, 33, 34, 39] for related results). In [31], assuming (V1) and,

(V2) There is a bounded domain $\Omega \subset \mathbb{R}^N$ such that

$$m_0 = \inf_{x\in\Omega} V(x) < \inf_{x\in\partial\Omega} V(x)$$

they show the existence of a single peak solution concentrating around the minimum points of V in Ω. They assume that f satisfies the assumptions

(f1) $f \in C(\mathbb{R},\mathbb{R})$.

(f2) $f(0) = \lim_{\xi\to 0} \frac{f(\xi)}{\xi} = 0$.

(f3) There exists $p \in (1, \frac{N+2}{N-2})$ such that

$$\lim_{\xi\to\infty} \frac{f(\xi)}{\xi^p} = 0.$$

In addition they require the *global Ambrosetti-Rabinowitz condition:*

$$\text{for some } \mu > 2, \quad 0 < \mu \int_0^t f(s)ds < f(t)t, \quad t > 0 \tag{1.8}$$

and the monotonicity condition

$$(0,\infty) \to \mathbb{R}; \quad \xi \mapsto \frac{f(\xi)}{\xi} \quad \text{is strictly increasing.} \tag{1.9}$$

After some weakening of the conditions (1.8) and (1.9) in [31, 42], it was finally shown in [11] (see also [12] for $N = 1, 2$) that (1.8)–(1.9) can be replaced by

(f4) There exists $\xi_0 > 0$ such that

$$F(\xi_0) > \frac{1}{2} m_0 \xi_0^2 \quad \text{where} \quad F(\xi) = \int_0^\xi f(\tau)\, d\tau.$$

Note that Berestycki and Lions [5] proved that there exists a least energy solution of (1.6) with $V(x_0) = m_0$ if (f1)-(f4) are satisfied . Also, using the Pohozaev's identity, they showed that conditions (f3) and (f4) are necessary for existence of a non-trivial solution of (1.6). Thus as far as the existence of solutions of (1.5), concentrating around local minimum points of V is concerned, the results of [11, 12] are somehow optimal.

Subsequently the approach developed in [11] was adapted in [18] to derive a related result for nonlinear Schrödinger equations with an external magnetic field. Also, very recently, it has been extended [13, 14, 30] to obtain the existence of a family of solutions of (1.5) concentrating, as $\varepsilon \to 0$, to a topologically non-trivial saddle point of V. In that direction previous results obtained either by the reduction or by the variational ones as developed by del Pino and Felmer, can be found in [25, 28, 34, 40].

In this work, letting K being a set of local minima of V, we are interested in the multiplicity of positive solutions concentrating around K. Starting from the paper of Bahri and Coron [6], many papers are devoted to study the effect of the domain topology on the existence and multiplicity of solutions for semilinear elliptic problems. We refer to [8, 9, 10, 16, 26, 27] for related studies for Dirichlet and Neumann boundary value problems. We also refer to [7] for the study of nodal solutions.

For problem (1.4), the topology of the domain is trivial, but the number of positive solutions to (1.4) is influenced by the topology of the level sets of the potential V. This fact was showed by the first author and Lazzo in [19]. Letting

$$K = \{x \in \mathbb{R}^N;\, V(x) = \inf_{x \in \mathbb{R}^N} V(x)\}$$

and assuming that $V \in C(\mathbb{R}^N, \mathbb{R})$ and (1.7) hold, they relate the number of positive solutions of (1.4) to the topology of the set K. It is assumed in [19] that $f(\xi) = |\xi|^{p-1}\xi$, $1 < p < (N+2)/(N-2)$ for $N > 2$ and $1 < p < +\infty$ if $N = 2$. In this case, one can reduce the search of solutions of (1.4) to the existence of critical points of a functional I_ε restricted to the Nehari manifold $\mathcal{N}_\varepsilon$. Here

$$\begin{aligned} I_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(\varepsilon x)u^2\,dx - \int_{\mathbb{R}^N} F(u)\,dx, \quad F(\xi) = \int_0^\xi f(\tau)\,d\tau, \\ \mathcal{N}_\varepsilon &= \{u \in H^1(\mathbb{R}^N) \setminus \{0\};\, I'_\varepsilon(u)u = 0\}. \end{aligned}$$

We remark that I_ε is bounded from below on $\mathcal{N}_\varepsilon$. The multiplicity of positive solutions is obtained through a study of the topology of the level set in $\mathcal{N}_\varepsilon$

$$\Sigma_{\varepsilon,h} = \{u \in \mathcal{N}_\varepsilon;\ I_\varepsilon(u) \in [c_\varepsilon, c_\varepsilon + h]\} \quad \text{for } h > 0,$$

where $c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u)$. In particular, in [19] two maps are introduced $\phi_\varepsilon : M \to \Sigma_{\varepsilon,h}$ and $\beta : \Sigma_{\varepsilon,h} \to M_\delta$ whose composition is homotopically equivalent to the embedding $j : M \to M_\delta$ for $h > 0$ and $\delta > 0$ small. Here M_δ denotes a δ-neighborhood of M.

Hence one has

$$\text{cat}(\Sigma_{\varepsilon,h}) \geq \text{cat}_{M_\delta}(M)$$

and this implies the existence of at least $\text{cat}_{M_\delta}(M)$ positive solutions to (1.4). Here $\text{cat}_X(A)$ denotes the Lusternick-Schnirelmann category of A in X for any topological pair (X, A).

This approach was then extended to nonlinear Schrödinger equations with competing potentials in [20] and in [17] for nonlinear Schrödinger equations with an external magnetic field.

In [25] Dancer, Lam and Yan proved the existence of at least $\text{cat}(K)$ positive solutions to (1.4) for $\varepsilon > 0$ small, when K is a connected compact local minimum or maximum set of V. They also assume that $f(\xi) = |\xi|^{p-1}\xi$, with $1 < p < (N+2)/(N-2)$ for $N > 2$ and $1 < p < \infty$ if $N = 2$. The proof of this result relying on a reduction method uses the special form of the nonlinear term f.

Successively in [3] Ambrosetti, Malchiodi and Secchi proved if V has a nondegenerate (in the sense of Bott) manifold M of critical points and $f(\xi) = |\xi|^{p-1}\xi$, with $1 < p < (N+2)/(N-2)$ for $N > 2$ and $1 < p < \infty$ if $N = 2$, that the equation (1.4) has at least $\text{cupl}(M) + 1$ critical points concentrating near points of M. Here $\text{cupl}(M)$ denotes the cup-length of M (see Definition (5.4)). If the critical points are local minima or local maxima, the above results can be sharped because M does not need to be a manifold and $\text{cupl}(M)$ can be replaced by $\text{cat}_{M_\delta}(M)$, for $\delta > 0$ small. The approach in [3] relies on a perturbative variational method, which requires the uniqueness and the nondegeneracy of the limiting problem.

In this paper our aim is to study the multiplicity of positive solutions concentrating around a set K of local minimum of V under the conditions $(f1) - (f4)$. In particular since we do not assume the monotonicity condition (1.9), we can not use a Nehari manifold. We will introduce a method to analyze the topological difference between two level sets of the indefinite functional I_ε in a small neighborhood of a set of expected solutions.

Our main result is

Theorem 1.1 *Suppose $N \geq 3$ and that (V1)–(V2) and (f1)–(f4) hold. Assume in addition that $\sup_{x\in\mathbb{R}^N} V(x) < \infty$. Then letting $K = \{x \in \Omega; V(x) = m_0\}$, for sufficiently small $\varepsilon > 0$, (1.4) has at least $\mathrm{cupl}(K)+1$ positive solutions v_ε^i, $i = 1, \ldots, \mathrm{cupl}(K)+1$ concentrating as $\varepsilon \to 0$ in K, where $\mathrm{cupl}(K)$ denotes the cup-length defined with Alexander-Spanier cohomology with coefficients in the field $\mathbb{F}$.*

Remark 1.2 *If $M = S^{N-1}$, the $N-1$ dimensional sphere in $\mathbb{R}^N$, then $\mathrm{cupl}(M) + 1 = \mathrm{cat}(M) = 2$. If $M = T^N$ is the N-dimensional torus, then $\mathrm{cupl}(M)+1 = \mathrm{cat}(M) = N+1$. However in general $\mathrm{cupl}(M)+1 \leq \mathrm{cat}(M)$.*

Remark 1.3 *When we say that the solutions v_ε^i, $i = 1, \ldots, \mathrm{cupl}(K)+1$ of Theorem 1.1 concentrate when $\varepsilon \to 0$ in K, we mean that there exists a maximum point x_ε^i of v_ε^i such that $\lim_{\varepsilon\to 0} dist(x_\varepsilon^i, K) = 0$ and that, for any such x_ε^i, $w_\varepsilon^i(x) = v_\varepsilon^i(\varepsilon(x + x_\varepsilon^i))$ converges, up to a subsequence, uniformly to a least energy solution of*

$$-\Delta U + m_0 U = f(U), \quad U > 0, \quad U \in H^1(\mathbb{R}^N).$$

We also have

$$v_\varepsilon^i(x) \leq C exp(-\frac{c}{\varepsilon}|x - x_\varepsilon^i|) \quad \text{for some } c,\, C > 0.$$

Remark 1.4 *In addition to condition $(V1)$ the boundedness of V from above is assumed in Theorem 1.1. Arguing as in [11, 13] we could prove Theorem 1.1 without this additional assumption. However, for the sake of simplicity, we assume here the boundedness of V.*

In the proof of Theorem 1.1, two ingredients play important roles. The first one is a suitable choice of a neighborhood $\mathcal{X}_{\varepsilon,\delta}$ of a set of expected solutions. We remark that in a situation where the search of solutions to (1.5) can be reduced to a variational problem on a Nehari manifold, we just need to study the (global) level set $\Sigma_{\varepsilon,h} \subset \mathcal{N}_\varepsilon$ and we can apply a standard deformation argument which is developed for functionals defined on Hilbert manifolds without boundary. On the contrary, in our setting, we need to find critical points in a neighborhood which has boundary. Thus we need to find a neighborhood positively invariant under a pseudo-gradient flow to develop our deformation argument. With the aid of ε-independent gradient estimate (Proposition 3.1 below), we find such a neighborhood. Moreover we analyze the topological difference between two level sets of the indefinite functional I_ε in the neighborhood $\mathcal{X}_{\varepsilon,\delta}$. To this aim a second crucial ingredient is the definition of two maps

$$\begin{aligned}
&\Phi_\varepsilon : ([1-s_0, 1+s_0] \times K, \{1 \pm s_0\} \times K) \to (\mathcal{X}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}, \mathcal{X}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}),\\
&\Psi_\varepsilon : (\mathcal{X}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}, \mathcal{X}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}) \to \\
&\qquad ([1-s_0, 1+s_0] \times \Omega([0,\nu_1]), ([1-s_0, 1+s_0] \setminus \{1\}) \times \Omega([0,\nu_1]))
\end{aligned}$$

where $s_0 \in (0,1)$ is small, $\Omega([0,\nu_1]) = \{y \in \Omega;\, m_0 \le V(y) \le m_0 + \nu_1\}$ for a suitable $\nu_1 > 0$ small, $E(m_0)$ is the least energy level associated to the limit problem

$$-\Delta u + m_0 u = f(u), \quad u \in H^1(\mathbb{R}^N) \tag{1.10}$$

and $\mathcal{X}_{\varepsilon,\delta}^c = \{u \in \mathcal{X}_{\varepsilon,\delta};\, J_\varepsilon(u) \le c\}$ for any $c \in \mathbb{R}$.

We show that $\Psi_\varepsilon \circ \Phi_\varepsilon$ is homotopically equivalent to the embedding $j(s,p) = (s,p)$. We emphasize that to define such maps, center of mass and a function P_0 which is inspired by the Pohozaev identity are important.

Also, differently from [19], it will be necessary to use the notions of category and cup-length for maps to derive our topological result (see Remark 4.3).

The paper is organized as follows. Section 2 is devoted to some preliminaries. In Section 3, we introduce, for technical reasons, a penalized functional J_ε. In Section 4 we define our neighborhood of expected solutions $\mathcal{X}_{\varepsilon,\delta}$ and we build our deformation argument. We also introduce the two maps Φ_ε and Ψ_ε and establish some of their important properties. Finally in Section 5, after having briefly recall the definitions and properties that we use of category and cup-length of a maps, we give the proof of Theorem 1.1 and of Remark 1.3.

2 Preliminaries

In what follows we use the notation:

$$\begin{aligned}\|u\|_{H^1} &= \left(\int_{\mathbb{R}^N} |\nabla u|^2 + u^2\,dx\right)^{1/2},\\ \|u\|_r &= \left(\int_{\mathbb{R}^N} |u|^r\,dx\right)^{1/r} \quad \text{for } r \in [1,\infty).\end{aligned}$$

We study the multiplicity of solutions to (1.5) via a variational method. That is, we look for critical points of the functional $I_\varepsilon \in C^1(H^1(\mathbb{R}^N),\mathbb{R})$ defined by

$$I_\varepsilon(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)u^2\,dx - \int_{\mathbb{R}^N} F(u)\,dx.$$

The critical points of I_ε are clearly solutions of (1.5). Without loss of generality, we may assume that $f(\xi) = 0$ for all $\xi \le 0$. Indeed it is then easy to see from the maximum principle that any nontrivial solution of (1.5) is positive.

2.1 Limit problems

We introduce the notation

$$\Omega(I) = \{y \in \Omega;\ V(y) - m_0 \in I\}$$

for an interval $I \subset [0, \inf_{x\in\partial\Omega} V(x) - m_0)$. We choose a small $\nu_0 > 0$ such that

(i) $0 < \nu_0 < \inf_{x\in\partial\Omega} V(x) - m_0$,

(ii) $F(\xi_0) > \frac{1}{2}(m_0 + \nu_0)\xi_0^2$,

(iii) $\Omega([0,\nu_0]) \subset K_d$, where $d > 0$ is a constant for which Lemma 5.5 (Section 5) holds.

For any $a > 0$ we also define a functional $L_a \in C^1(H^1(\mathbb{R}^N),\mathbb{R})$ by

$$L_a(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{a}{2}\|u\|_2^2 - \int_{\mathbb{R}^N} F(u)\,dx.$$

associated to the limit problem

$$-\Delta u + au = f(u), \quad u \in H^1(\mathbb{R}^N). \tag{2.1}$$

We denote by $E(a)$ the least energy level for (2.1). That is,

$$E(a) = \inf\{L_a(u);\, u \neq 0,\ L_a'(u) = 0\}.$$

In [5] it is proved that there exists a least energy solution of (2.1), for any $a > 0$, if (f1)–(f4) are satisfied (here we consider (f4) with $m_0 = a$). Also it is showed that each solution of (2.1) satisfies the Pohozaev's identity

$$\frac{N}{N-2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + N\int_{\mathbb{R}^N} a\frac{u^2}{2} - F(u)dx = 0. \tag{2.2}$$

We note that, under our choice of $\nu_0 > 0$, $E(a)$ is attained for $a \in [m_0, m_0 + \nu_0]$. Clearly $a \mapsto E(a);\ [m_0, m_0 + \nu_0] \to \mathbb{R}$ is continuous and strictly increasing. Choosing $\nu_0 > 0$ smaller if necessary, we may assume

$$E(m_0 + \nu_0) < 2E(m_0).$$

We choose $\ell_0 \in (E(m_0 + \nu_0), 2E(m_0))$ and we set

$$S_a = \{U \in H^1(\mathbb{R}^N) \setminus \{0\};\ L_a'(U) = 0,\ L_a(U) \leq \ell_0,\ U(0) = \max_{x \in \mathbb{R}^N} U(x)\}.$$

We also define

$$\widehat{S} = \bigcup_{a \in [m_0, m_0 + \nu_0]} S_a.$$

From the proof of [11, Proposition1] we see that $\widehat{S}$ is compact in $H^1(\mathbb{R}^N)$ and that its elements have a uniform exponential decay. Namely that there exist C, $c > 0$ such that

$$U(x) + |\nabla U(x)| \leq C\exp(-c|x|) \quad \text{for all} \quad U \in \widehat{S}. \tag{2.3}$$

We also have that, for $b \in [m_0, m_0 + \nu_0]$,

$$\lim_{b \to m_0} \sup_{\widetilde{U} \in S_b} \inf_{U \in S_{m_0}} ||U - \widetilde{U}||_{H^1} = 0. \tag{2.4}$$

To prove (2.4) we assume by contradiction that it is false. Then there exists sequences $(b_j) \subset [m_0, m_0 + \nu_0]$ such that $b_j \to m_0$ and (U_j) with $U_j \in S_{b_j}$ such that (U_j) is bounded away from S_{m_0}. But $(U_j) \subset \widehat{S}$ and $\widehat{S}$ is compact. Thus, up to a subsequence, $U_j \to U_0$ strongly. To reach a contradiction it suffices to show that $L'_{m_0}(U_0) = 0$ and $L_{m_0}(U_0) \leq \ell_0$. But since (U_j) is bounded

$$0 = L'_{b_j}(U_j) = L'_{m_0}(U_j) + o(1)$$

as $b_j \to m_0$. Also, since $U_j \to U_0$, we have by continuity of L'_{m_0} that $0 = L'_{m_0}(U_j) \to L'_{m_0}(U_0)$. Thus $L'_{m_0}(U_0) = 0$. Similarly we can show that $L_{m_0}(U_0) \leq \ell_0$ and this proves that (2.4) hold.

In what follows, we try to find our critical points in the following set:

$$\mathcal{S}(r) = \{U(x-y) + \varphi(x);\, y \in \mathbb{R}^N,\ U \in \widehat{S},\ \|\varphi\|_{H^1} < r\} \quad \text{for } r > 0.$$

2.2 A function P_0

For a later use, we define the $C^1(H^1(\mathbb{R}^N), \mathbb{R})$ functional

$$P_0(u) = \left(\frac{N(\int_{\mathbb{R}^N} F(u)\,dx - \frac{m_0}{2}\|u\|_2^2)}{\frac{N-2}{2}\|\nabla u\|_2^2} \right)^{1/2}. \tag{2.5}$$

By the Pohozaev identity (2.2), for a non trivial solution U of $L'_{m_0}(U) = 0$, we have $P_0(U) = 1$. Moreover a direct calculation shows that

$$P_0(U(x/s)) = s \quad \text{for all } s > 0.$$

A motivation to the introduction of P_0 is that it permits to estimate L_{m_0} from below. The proof of the following lemma is given in [13, Lemma 2.4] but we recall it here for completeness.

Lemma 2.1 *Suppose that $P_0(u) \in (0, \sqrt{\frac{N}{N-2}})$. Then we have*

$$L_{m_0}(u) \geq g(P_0(u))E(m_0),$$

where

$$g(t) = \frac{1}{2}(Nt^{N-2} - (N-2)t^N). \tag{2.6}$$

Proof. First we recall, see [41] for a proof, that $E(m_0)$ can be characterized as

$$E(m_0) = \inf\{L_{m_0}(u);\, u \neq 0,\, P_0(u) = 1\}. \tag{2.7}$$

Now suppose that $s = P_0(u) \in (0, \sqrt{\frac{N}{N-2}})$ and set $v(x) = u(sx)$. We have $P_0(v) = s^{-1}P_0(u) = 1$ and then, by (2.7), $L_{m_0}(v) \geq E(m_0)$. This implies that

$$\frac{1}{N}\int_{\mathbb{R}^N} |\nabla v|^2\, dx = \frac{2}{N-2}\int_{\mathbb{R}^N} (F(v) - \frac{m_0}{2}v^2)\, dx \geq E(m_0)$$

and thus

$$\begin{aligned} L_{m_0}(u) &= \frac{s^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla v|^2\, dx + s^N \int_{\mathbb{R}^N} \Big(\frac{m_0}{2}v^2 - F(v)\Big)\, dx \\ &= g(s)\frac{1}{N}\int_{\mathbb{R}^N} |\nabla v|^2\, dx \\ &\geq g(s)E(m_0). \end{aligned}$$

Since $s \in (0, \sqrt{\frac{N}{N-2}})$ implies $g(s) > 0$ this proves the lemma. ∎

Remark 2.2 *Clearly P_0 takes bounded sets to bounded sets. Thus from (2.4), choosing $\nu_0 > 0$ smaller if necessary, we may assume*

$$P_0(U) \in (0, \sqrt{\frac{N}{N-2}}) \quad \text{for all} \quad U \in \widehat{S} = \bigcup_{a \in [m_0, m_0+\nu_0]} S_a.$$

Also there exists $r_0 > 0$ such that

$$P_0(u) \in (0, \sqrt{\frac{N}{N-2}}) \quad \text{for all} \quad u \in \mathcal{S}(r_0).$$

Remark 2.3 *We remark that $g(t)$ defined in (2.6) satisfies $g(t) \leq 1$ for all $t > 0$ and that $g(t) = 1$ holds if and only if $t = 1$.*

2.3 Center of mass in $\mathcal{S}(r_0)$

Following [13, 14] we introduce a center of mass in $\mathcal{S}(r)$.

Lemma 2.4 *There exists r_0, $R_0 > 0$ and a map $\Upsilon : \mathcal{S}(r_0) \to \mathbb{R}^N$ such that*

$$|\Upsilon(u) - p| \le 2R_0$$

for all $u(x) = U(x-p) + \varphi(x) \in \mathcal{S}(r_0)$ with $p \in \mathbb{R}^N$, $U \in \widehat{S}$, $\|\varphi\|_{H^1} \le r_0$. Moreover, $\Upsilon(u)$ has the following properties

(i) $\Upsilon(u)$ is shift equivariant, that is,

$$\Upsilon(u(x-y)) = \Upsilon(u(x)) + y \quad \textit{for all } u \in \mathcal{S}(r_0) \textit{ and } y \in \mathbb{R}^N.$$

(ii) $\Upsilon(u)$ is locally Lipschitz continuous, that is, there exists constants C_1, $C_2 > 0$ such that

$$|\Upsilon(u) - \Upsilon(v)| \le C_1 \|u-v\|_{H^1} \quad \textit{for all } u, v \in \mathcal{S}(r_0) \textit{ with } \|u-v\|_{H^1} \le C_2.$$

The proof is given in [13, 14] in a more complicated situation. We give here a simple proof.

Proof. We set $r_* = \min_{U \in \widehat{S}} \|u\|_{H^1} > 0$ and choose $R_0 > 1$ such that for $U \in \widehat{S}$

$$\|U\|_{H^1(|x| \le R_0)} > \frac{3}{4} r_* \quad \text{and} \quad \|U\|_{H^1(|x| \ge R_0)} < \frac{1}{8} r_*.$$

This is possible by the uniform exponential decay (2.3). For $u \in H^1(\mathbb{R}^N)$ and $p \in \mathbb{R}^N$, we define

$$d(p,u) = \psi\left(\inf_{U \in \widehat{S}} \|u - U(x-p)\|_{H^1(|x-p| \le R_0)} \right),$$

where $\psi \in C_0^\infty(\mathbb{R}, \mathbb{R})$ is such that

$$\psi(r) = \begin{cases} 1 & r \in [0, \frac{1}{4} r_*], \\ 0 & r \in [\frac{1}{2} r_*, \infty), \end{cases}$$
$$\psi(r) \in [0,1] \quad \text{for all } r \in [0, \infty).$$

Now let

$$\Upsilon(u) = \frac{\int_{\mathbb{R}^N} q\, d(q,u)\, dq}{\int_{\mathbb{R}^N} d(q,u)\, dq} \quad \text{for} \quad u \in \mathcal{S}(\frac{1}{8}r_*).$$

We shall show that Υ has the desired property.

Let $u \in \mathcal{S}(\frac{1}{8}r_*)$ and write $u(x) = U(x-p) + \varphi(x)$ ($p \in \mathbb{R}^N$, $U \in \widehat{S}$, $\|\varphi\|_{H^1} \leq \frac{1}{8}r_*$). Then for $|q-p| \geq 2R_0$ and $\tilde{U} \in \widehat{S}$, we have

$$\begin{aligned} \|u - \tilde{U}(x-q)\|_{H^1(|x-q|\leq R_0)} &\geq \|\tilde{U}(x-q)\|_{H^1(|x-q|\leq R_0)} \\ &\quad -\|U(x-p)\|_{H^1(|x-p|\geq R_0)} - \frac{1}{8}r_* \\ &> \frac{3}{4}r_* - \frac{1}{8}r_* - \frac{1}{8}r_* = \frac{1}{2}r_*. \end{aligned}$$

Thus $d(q,u) = 0$ for $|q-p| \geq 2R_0$. We can also see that, for small $r > 0$

$$d(q,u) = 1 \quad \text{for } |q-p| < r.$$

Thus $B(p,r) \subset \operatorname{supp} d(\cdot,u) \subset B(p,2R_0)$. Therefore $\Upsilon(u)$ is well-defined and we have

$$\Upsilon(u) \in B(p,2R_0) \quad \text{for} \quad u \in \mathcal{S}(\frac{1}{8}r_*).$$

Shift equivariance and locally Lipschitz continuity of Υ can be checked easily. Setting $r_0 = \frac{1}{8}r_*$, we have the desired result. ∎

Using this lemma we have

Lemma 2.5 *There exists $\delta_1 > 0$, $r_1 \in (0,r_0)$ and $\nu_1 \in (0,\nu_0)$ such that for $\varepsilon > 0$ small*

$$I_\varepsilon(u) \geq E(m_0) + \delta_1$$

for all $u \in \mathcal{S}(r_1)$ with $\varepsilon\Upsilon(u) \in \Omega([\nu_1,\nu_0])$.

Proof. We set $\underline{M} = \inf_{U\in\widehat{S}} \|U\|_2^2 \in (0,\infty)$, $\overline{M} = \sup_{U\in\widehat{S}} \|U\|_2^2 \in (0,\infty)$. The fact that $\underline{M} > 0$ and $\overline{M} < \infty$ can be shown in a standard way using Pohozaev identify and assumptions (f1)–(f3). For latter use in (2.11) below, we choose $\nu_1 \in (0,\nu_0)$ such that

$$E(m_0+\nu_1) - \frac{1}{2}(\nu_0-\nu_1)\overline{M} > E(m_0). \tag{2.8}$$

First we claim that for some $\delta_1 > 0$

$$\inf_{U\in\widehat{S}} L_{m_0+\nu_1}(U) \geq E(m_0) + 3\delta_1. \tag{2.9}$$

Indeed, on one hand, if $U \in S_a$ with $a \in [m_0, m_0 + \nu_1]$, we have

$$\begin{aligned} L_{m_0+\nu_1}(U) &= L_a(U) + \frac{1}{2}(m_0 + \nu_1 - a)\|U\|_2^2 \\ &\geq E(a) + \frac{1}{2}(m_0 + \nu_1 - a)\underline{M} \end{aligned}$$

and thus

$$\inf_{U\in\bigcup_{a\in[m_0,m_0+\nu_1]} S_a} L_{m_0+\nu_1}(U) > E(m_0). \tag{2.10}$$

On the other hand, if $U \in S_a$ with $a \in [m_0 + \nu_1, m_0 + \nu_0]$,

$$\begin{aligned} L_{m_0+\nu_1}(U) &= L_a(U) + \frac{1}{2}(m_0 + \nu_1 - a)\|U\|_2^2 \\ &\geq E(a) - \frac{1}{2}(\nu_0 - \nu_1)\overline{M} \\ &\geq E(m_0 + \nu_1) - \frac{1}{2}(\nu_0 - \nu_1)\overline{M} \end{aligned}$$

and using (2.8), it follows that

$$\inf_{U\in\bigcup_{a\in[m_0+\nu_1,m_0+\nu_0]} S_a} L_{m_0+\nu_1}(U) > E(m_0). \tag{2.11}$$

Choosing $\delta_1 > 0$ small enough, (2.9) follows from (2.10) and (2.11).

Now observe that, since elements in $\widehat{S}$ have uniform exponential decays,

$$|I_\varepsilon(U(x-p)) - L_{V(\varepsilon p)}(U)| \to 0 \quad \text{as } \varepsilon \to 0$$

uniformly in $U \in \widehat{S}$, $\varepsilon p \in \Omega$. Thus, by (2.9), for $U \in \widehat{S}$, $\varepsilon p \in \Omega([\nu_1, \nu_0])$

$$\begin{aligned} I_\varepsilon(U(x-p)) &= L_{V(\varepsilon p)}(U) + o(1) \geq L_{m_0+\nu_1}(U) + o(1) \\ &\geq E(m_0) + 2\delta_1 \qquad \text{for } \varepsilon > 0 \text{ small.} \end{aligned} \tag{2.12}$$

If we suppose that $u(x) = U(x-p) + \varphi(x) \in \mathcal{S}(r_0)$ satisfies $\varepsilon\Upsilon(u) \in \Omega([\nu_1, \nu_0])$, then by Lemma 2.4, εp belongs to a $2\varepsilon R_0$-neighborhood of $\Omega([\nu_1, \nu_0])$. Thus by (2.12) it follows that

$$I_\varepsilon(U(x-p)) \geq E(m_0) + \frac{3}{2}\delta_1 \quad \text{for } \varepsilon > 0 \text{ small.}$$

Finally we observe that I'_ε is bounded on bounded sets uniformly in $\varepsilon \in (0,1]$ and that by the compactness of $\hat{S}$, $\{U(x-p);\, U \in \widehat{S},\ p \in \mathbb{R}^N\}$ is bounded in $H^1(\mathbb{R}^N)$. Thus choosing $r_1 \in (0, r_0)$ small, if $u(x) = U(x-p) + \varphi(x) \in \mathcal{S}(r_1)$, we have

$$I_\varepsilon(U(x-p) + \varphi(x)) \geq I_\varepsilon(U(x-p)) - \frac{1}{2}\delta_1 \geq E(m_0) + \delta_1.$$

Thus, the conclusion of lemma holds. ∎

3 A penalized functional J_ε

For technical reasons, we introduce a penalized functional J_ε following [11]. We assume that $\partial\Omega$ is smooth and for $h > 0$ we set

$$\Omega_h = \{x \in \mathbb{R}^N \setminus \Omega;\ \mathrm{dist}\,(x, \partial\Omega) < h\} \cup \Omega.$$

We choose a small $h_0 > 0$ such that

$$V(x) > m_0 \quad \text{for all } x \in \overline{\Omega_{2h_0} \setminus \Omega}.$$

Let

$$Q_\varepsilon(u) = \left(\varepsilon^{-2}\|u\|^2_{L^2(\mathbb{R}^N \setminus (\Omega_{2h_0}/\varepsilon))} - 1\right)_+^{\frac{p+1}{2}}$$

and

$$J_\varepsilon(u) = I_\varepsilon(u) + Q_\varepsilon(u).$$

We also define

$$\widehat{\rho}(u) = \inf\{\|u - U(x-y)\|_{H^1};\ y \in \mathbb{R}^N,\ U \in \widehat{S}\} : \mathcal{S}(r_0) \to \mathbb{R}.$$

As shown in Proposition 3.2 below, a suitable critical point of J_ε is a critical point of I_ε. A motivation to introduce J_ε is a property given in Lemma 3.4, which enables us to get a useful estimate from below.

The following proposition gives a uniform estimate of $\|J'_\varepsilon\|_{H^{-1}}$ in an annular neighborhood of a set of expected solutions, which is one of the keys of our argument.

Proposition 3.1 *There exists $r_2 \in (0, r_1)$ with the following property: for any $0 < \rho_1 < \rho_0 < r_2$, there exists $\delta_2 = \delta_2(\rho_0, \rho_1) > 0$ such that for $\varepsilon > 0$ small*

$$\|J'_\varepsilon(u)\|_{H^{-1}} \geq \delta_2$$

for all $u \in \mathcal{S}(r_2)$ with $J_\varepsilon(u) \leq E(m_0 + \nu_1)$ and $(\widehat{\rho}(u), \varepsilon\Upsilon(u)) \in ([0, \rho_0] \times \Omega([0, \nu_0])) \setminus ([0, \rho_1] \times \Omega([0, \nu_1]))$.

Proof. By (f1)–(f3), for any $a > 0$ there exists $C_a > 0$ such that

$$|f(\xi)| \leq a|\xi| + C_a|\xi|^p \quad \text{for all } \xi \in \mathbb{R}.$$

We fix a $a_0 \in (0, \frac{1}{2}\underline{V})$ and compute

$$\begin{aligned} I'_\varepsilon(u)u &= \|\nabla u\|_2^2 + \int_{\mathbb{R}^N} V(\varepsilon x)u^2\, dx - \int_{\mathbb{R}^N} f(u)u\, dx \\ &\geq \|\nabla u\|_2^2 + \underline{V}\|u\|_2^2 - a_0\|u\|_2^2 - C_{a_0}\|u\|_{p+1}^{p+1} \\ &\geq \|\nabla u\|_2^2 + \frac{1}{2}\underline{V}\|u\|_2^2 - C_{a_0}\|u\|_{p+1}^{p+1}. \end{aligned}$$

Now choosing $r_2 > 0$ small enough there exists $c > 0$ such that

$$\|\nabla u\|_2^2 + \frac{1}{2}\underline{V}\|u\|_2^2 - 2^p C_{a_0}\|u\|_{p+1}^{p+1} \geq c\|u\|_{H^1}^2 \quad \text{for all} \quad \|u\|_{H^1} \leq 2r_2. \tag{3.1}$$

(For a technical reason, especially to get (3.21) later, we add "2^p" in front of C_{a_0}.) In particular, we have

$$I'_\varepsilon(u)u \geq c\|u\|_{H^1}^2 \quad \text{for all} \quad \|u\|_{H^1} \leq 2r_2. \tag{3.2}$$

Now we set

$$n_\varepsilon = \left[\frac{h_0}{\varepsilon}\right] - 1$$

and for each $i = 1, 2, \cdots, n_\varepsilon$ we fix a function $\varphi_{\varepsilon,i} \in C_0^\infty(\Omega)$ such that

$$\begin{aligned} &\varphi_{\varepsilon,i}(x) = \begin{cases} 1 & \text{if } x \in \Omega_{\varepsilon,i}, \\ 0 & \text{if } x \notin \Omega_{\varepsilon,i+1}, \end{cases} \\ &\varphi_{\varepsilon,i}(x) \in [0,1],\ |\varphi'_{\varepsilon,i}(x)| \leq 2 \quad \text{for all } x \in \Omega. \end{aligned}$$

Here we denote for $\varepsilon > 0$ and $h \in (0, 2h_0/\varepsilon]$

$$\begin{aligned}\Omega_{\varepsilon,h} &= (\Omega_{\varepsilon h})/\varepsilon \\ &= \{x \in \mathbb{R}^N \setminus (\Omega/\varepsilon);\ \mathrm{dist}\,(x, (\partial\Omega)/\varepsilon) < h\} \cup (\Omega/\varepsilon).\end{aligned}$$

Now suppose that a sequence $(u_\varepsilon) \subset \mathcal{S}(r_2)$ satisfies for $0 < \rho_0 < \rho_1 < r_2$

$$J_\varepsilon(u_\varepsilon) \le E(m_0 + \nu_1), \tag{3.3}$$
$$\widehat{\rho}(u_\varepsilon) \in [0, \rho_0], \tag{3.4}$$
$$\varepsilon\Upsilon(u_\varepsilon) \in \Omega([0, \nu_0]), \tag{3.5}$$
$$\|J'_\varepsilon(u_\varepsilon)\|_{H^{-1}} \to 0. \tag{3.6}$$

We shall prove, in several steps, that for $\varepsilon > 0$ small

$$\widehat{\rho}(u_\varepsilon) \in [0, \rho_1] \quad \text{and} \quad \varepsilon\Upsilon(u_\varepsilon) \in \Omega([0, \nu_1]), \tag{3.7}$$

from which the conclusion of Proposition 3.1 follows.

Step 1: There exists a $i_\varepsilon \in \{1, 2, \cdots, n_\varepsilon\}$ *such that*

$$\|u_\varepsilon\|^2_{H^1(\Omega_{\varepsilon,i_\varepsilon+1} \setminus \Omega_{\varepsilon,i_\varepsilon})} \le \frac{4r_2^2}{n_\varepsilon}. \tag{3.8}$$

Indeed by (3.5) and the uniform exponential decay of $\widehat{S}$, we have $\|u_\varepsilon\|_{H^1(\mathbb{R}^N \setminus (\Omega/\varepsilon))} \le 2r_2$ for $\varepsilon > 0$ small. Thus

$$\sum_{i=1}^{n_\varepsilon} \|u_\varepsilon\|^2_{H^1(\Omega_{\varepsilon,i+1} \setminus \Omega_{\varepsilon,i})} \le \|u_\varepsilon\|^2_{H^1(\Omega_{\varepsilon,h_0/\varepsilon} \setminus (\Omega/\varepsilon))} \le 4r_2^2$$

and there exists $i_\varepsilon \in \{1, 2, \cdots, n_\varepsilon\}$ satisfying (3.8).

Step 2: For the i_ε *obtained in Step 1, we set*

$$u_\varepsilon^{(1)}(x) = \varphi_{\varepsilon,i_\varepsilon}(x) u_\varepsilon(x) \quad \textit{and} \quad u_\varepsilon^{(2)}(x) = u_\varepsilon(x) - u_\varepsilon^{(1)}(x).$$

Then we have, as $\varepsilon \to 0$,

$$I_\varepsilon(u_\varepsilon^{(1)}) = J_\varepsilon(u_\varepsilon) + o(1), \tag{3.9}$$
$$I'_\varepsilon(u_\varepsilon^{(1)}) \to 0 \quad \textit{in } H^{-1}(\mathbb{R}^N), \tag{3.10}$$
$$u_\varepsilon^{(2)} \to 0 \quad \textit{in } H^1(\mathbb{R}^N), \tag{3.11}$$
$$Q_\varepsilon(u_\varepsilon^{(2)}) \to 0. \tag{3.12}$$

Observe that

$$I_\varepsilon(u_\varepsilon) = I_\varepsilon(u_\varepsilon^{(1)}) + I_\varepsilon(u_\varepsilon^{(2)}) + o(1). \tag{3.13}$$

Indeed, by (3.8)

$$\begin{aligned}
& I_\varepsilon(u_\varepsilon) - (I_\varepsilon(u_\varepsilon^{(1)}) + I_\varepsilon(u_\varepsilon^{(2)})) \\
= \; & \int_{\Omega_{\varepsilon,i_\varepsilon+1}\setminus\Omega_{\varepsilon,i_\varepsilon}} \nabla(\varphi_{\varepsilon,i_\varepsilon} u_\varepsilon)\nabla((1-\varphi_{\varepsilon,i_\varepsilon})u_\varepsilon) + V(\varepsilon x)\varphi_{\varepsilon,i_\varepsilon}(1-\varphi_{\varepsilon,i_\varepsilon})(u_\varepsilon)^2\,dx \\
& - \int_{\Omega_{\varepsilon,i_\varepsilon+1}\setminus\Omega_{\varepsilon,i_\varepsilon}} F(u_\varepsilon) - F(u_\varepsilon^{(1)}) - F(u_\varepsilon^{(2)})\,dx \\
\to \; & 0 \qquad \text{as } \varepsilon \to 0.
\end{aligned}$$

Thus

$$J_\varepsilon(u_\varepsilon) = I_\varepsilon(u_\varepsilon^{(1)}) + I_\varepsilon(u_\varepsilon^{(2)}) + Q_\varepsilon(u_\varepsilon^{(2)}) + o(1). \tag{3.14}$$

We can also see that

$$\|I'_\varepsilon(u_\varepsilon) - I'_\varepsilon(u_\varepsilon^{(1)}) - I'_\varepsilon(u_\varepsilon^{(2)})\|_{H^{-1}} \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.15}$$

In a similar way, it follows from (3.6) that, since $(u_\varepsilon^{(2)})$ is bounded, that

$$I'_\varepsilon(u_\varepsilon^{(2)})u_\varepsilon^{(2)} + Q'_\varepsilon(u_\varepsilon^{(2)})u_\varepsilon^{(2)} = J'_\varepsilon(u_\varepsilon)u_\varepsilon^{(2)} + o(1) = o(1). \tag{3.16}$$

We note that $\|u_\varepsilon^{(2)}\|_{H^1} \le 2r_2$ and $(p+1)Q_\varepsilon(u) \le Q'_\varepsilon(u)u$ for all $u \in H^1(\mathbb{R}^N)$. Thus by (3.2)

$$c\|u_\varepsilon^{(2)}\|_{H^1}^2 + (p+1)Q_\varepsilon(u_\varepsilon^{(2)}) \to 0 \quad \text{as } \varepsilon \to 0,$$

which implies (3.11) and (3.12). Now (3.11) implies that $I_\varepsilon(u_\varepsilon^{(2)}) \to 0$ and thus (3.9) follows from (3.14).

Finally we show (3.10). We choose a function $\widetilde{\varphi} \in C_0^\infty(\mathbb{R}^N)$ such that

$$\widetilde{\varphi}(x) = \begin{cases} 1 & \text{for } x \in \Omega_{h_0}, \\ 0 & \text{for } x \in \mathbb{R}^N \setminus \Omega_{2h_0}. \end{cases}$$

Then we have, for all $w \in H^1(\mathbb{R}^N)$,

$$\begin{aligned}
I'_\varepsilon(u_\varepsilon^{(1)})w &= I'_\varepsilon(u_\varepsilon^{(1)})(\widetilde{\varphi}(\varepsilon x)w) \\
&= I'_\varepsilon(u_\varepsilon)(\widetilde{\varphi}(\varepsilon x)w) - (I'_\varepsilon(u_\varepsilon) - I'_\varepsilon(u_\varepsilon^{(1)}))(\widetilde{\varphi}(\varepsilon x)w) \\
&= J'_\varepsilon(u_\varepsilon)(\widetilde{\varphi}(\varepsilon x)w) - (I'_\varepsilon(u_\varepsilon) - I'_\varepsilon(u_\varepsilon^{(1)}))(\widetilde{\varphi}(\varepsilon x)w)
\end{aligned}$$

and it follows that

$$|I'_\varepsilon(u_\varepsilon^{(1)})w| \le \|J'_\varepsilon(u_\varepsilon)\|_{H^{-1}}\|\widetilde{\varphi}(\varepsilon x)w\|_{H^1} + \|I'_\varepsilon(u_\varepsilon) - I'_\varepsilon(u_\varepsilon^{(1)})\|_{H^{-1}}\|\widetilde{\varphi}(\varepsilon x)w\|_{H^1}.$$

We note that by (3.11) and (3.15), $\|I'_\varepsilon(u_\varepsilon) - I'_\varepsilon(u_\varepsilon^{(1)})\|_{H^{-1}} \to 0$. Therefore, by (3.6), $\|I'_\varepsilon(u_\varepsilon^{(1)})\|_{H^{-1}} \to 0$, that is (3.10) holds true;

Step 3: After extracting a subsequence — still we denoted by ε —, there exist a sequence $(p_\varepsilon) \subset \mathbb{R}^N$ and $U \in \widehat{S}$ such that

$$\varepsilon p_\varepsilon \to p_0 \quad \textit{for some } p_0 \in \Omega([0,\nu_1]), \tag{3.17}$$
$$\|u_\varepsilon^{(1)} - U(x - p_\varepsilon)\|_{H^1} \to 0, \tag{3.18}$$
$$I_\varepsilon(u_\varepsilon^{(1)}) \to L_{V(p_0)}(U) \quad \textit{as } \varepsilon \to 0. \tag{3.19}$$

Let $q_\varepsilon = \Upsilon(u_\varepsilon)$. We may assume that

$$u_\varepsilon^{(1)}(x + q_\varepsilon) \rightharpoonup \widetilde{U}(x) \quad \text{weakly in } H^1(\mathbb{R}^N)$$

for some $\widetilde{U} \in H^1(\mathbb{R}^N) \setminus \{0\}$ and also that $\varepsilon q_\varepsilon \to p_0$. From the definition of Υ and (3.10), it follows that $L'_{V(p_0)}(\widetilde{U}) = 0$. Setting

$$\widetilde{w}_\varepsilon(x) = u_\varepsilon^{(1)}(x + q_\varepsilon) - \widetilde{U}(x)$$

we shall prove that $\|\widetilde{w}_\varepsilon\|_{H^1} \to 0$. We have

$$\begin{aligned}
& I'_\varepsilon(u_\varepsilon^{(1)})\widetilde{w}_\varepsilon(x - q_\varepsilon) \\
= \; & I'_\varepsilon(\widetilde{U}(x - q_\varepsilon) + \widetilde{w}_\varepsilon(x - q_\varepsilon))\widetilde{w}_\varepsilon(x - q_\varepsilon) \\
= \; & \int_{\mathbb{R}^N} \nabla(\widetilde{U} + \widetilde{w}_\varepsilon)\nabla\widetilde{w}_\varepsilon + V(\varepsilon x + \varepsilon q_\varepsilon)(\widetilde{U} + \widetilde{w}_\varepsilon)\widetilde{w}_\varepsilon \, dx \\
& - \int_{\mathbb{R}^N} f(\widetilde{U} + \widetilde{w}_\varepsilon)\widetilde{w}_\varepsilon \, dx \\
= \; & L'_{V(p_0)}(\widetilde{U})\widetilde{w}_\varepsilon + \int_{\mathbb{R}^N} |\nabla\widetilde{w}_\varepsilon|^2 + V(\varepsilon x + \varepsilon q_\varepsilon)\widetilde{w}_\varepsilon^2 \, dx \\
& + \int_{\mathbb{R}^N} (V(\varepsilon x + \varepsilon q_\varepsilon) - V(p_0))\widetilde{U}\widetilde{w}_\varepsilon \, dx + \int_{\mathbb{R}^N} (f(\widetilde{U}) - f(\widetilde{U} + \widetilde{w}_\varepsilon))\widetilde{w}_\varepsilon \, dx \\
= \; & \int_{\mathbb{R}^N} |\nabla\widetilde{w}_\varepsilon|^2 + V(\varepsilon x + \varepsilon q_\varepsilon)\widetilde{w}_\varepsilon^2 \, dx + (I) + (II).
\end{aligned} \tag{3.20}$$

It is easy to see $(I) \to 0$ as $\varepsilon \to 0$. Now since $|f(\xi)| \le a_0|\xi| + C_{a_0}|\xi|^p$ for all $\xi \in \mathbb{R}$,

$$\begin{aligned} |(II)| &\le \int_{\mathbb{R}^N} (a_0(|\widetilde{U}| + |\widetilde{U} + \widetilde{w}_\varepsilon|) + C_{a_0}(|\widetilde{U}|^p + |\widetilde{U} + \widetilde{w}_\varepsilon|^p))|\widetilde{w}_\varepsilon|\, dx \\ &\le \int_{\mathbb{R}^N} a_0|\widetilde{w}_\varepsilon|^2 + 2^p C_{a_0}|\widetilde{w}_\varepsilon|^{p+1} + (2a_0|\widetilde{U}| + (1+2^p)C_{a_0}|\widetilde{U}|^p)|\widetilde{w}_\varepsilon|\, dx \\ &\le \int_{\mathbb{R}^N} a_0|\widetilde{w}_\varepsilon|^2 + 2^p C_{a_0}|\widetilde{w}_\varepsilon|^{p+1}\, dx + o(1). \end{aligned}$$

Here we used the fact that $\widetilde{w}_\varepsilon \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$. Thus, by (3.20) and (3.10), we have

$$\|\nabla \widetilde{w}_\varepsilon\|_2^2 + \underline{V}\|\widetilde{w}_\varepsilon\|_2^2 \le a_0\|\widetilde{w}_\varepsilon\|_2^2 + 2^p C_{a_0}\|\widetilde{w}_\varepsilon\|_{p+1}^{p+1} + o(1)$$

from which we deduce, using (3.1), that

$$\|\widetilde{w}_\varepsilon\|_{H^1} \to 0. \tag{3.21}$$

At this point we have obtained (3.18), (3.19) where p_ε and U are replaced with q_ε and $\widetilde{U}$. Since

$$I_\varepsilon(u_\varepsilon) = I_\varepsilon(u_\varepsilon^{(1)}) + o(1) = J_\varepsilon(u_\varepsilon) + o(1) \le E(m_0 + \nu_1) + o(1) \tag{3.22}$$

implies

$$E(V(p_0)) \le L_{V(p_0)}(\widetilde{U}) \le E(m_0 + \nu_1),$$

we have $p_0 \in \Omega([0, \nu_1])$ and $\widetilde{U}$ belongs to $S_{V(p_0)} \subset \widehat{S}$ after a suitable shift, that is, $U(x) := \widetilde{U}(x + y_0) \in \widehat{S}$ for some $y_0 \in \mathbb{R}^N$. Setting $p_\varepsilon = q_\varepsilon + y_0$, we get (3.17)–(3.19).

Step 4: Conclusion

In Steps 1–3, we have shown that a sequence $(u_n) \subset \mathcal{S}(r_2)$ satisfying (3.3)–(3.6) satisfies, up to a subsequence, and for some $U \in \widehat{S}$ (3.18)–(3.19) with $p_\varepsilon = \Upsilon(u_\varepsilon) + y_0$. This implies that

$$\begin{gathered} \varepsilon\Upsilon(u_\varepsilon) \to p_0 \in \Omega([0, \nu_1]), \\ \|u_\varepsilon(x) - U(x - p_\varepsilon)\|_{H^1} \to 0. \end{gathered}$$

In particular since $\widehat{\rho}(u_\varepsilon) \to 0$ we have $\widehat{\rho}(u_\varepsilon) \in [0, \rho_1]$ and (3.7) holds. This ends the proof of the Proposition. ∎

Proposition 3.2 *There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$ if $u_\varepsilon \in \mathcal{S}(r_2)$ satisfies*

$$J'_\varepsilon(u_\varepsilon) = 0, \tag{3.23}$$
$$J_\varepsilon(u_\varepsilon) \le E(m_0 + \nu_1), \tag{3.24}$$
$$\varepsilon\Upsilon(u_\varepsilon) \in \Omega([0, \nu_0]), \tag{3.25}$$

then

$$Q_\varepsilon(u_\varepsilon) = 0 \quad \text{and} \quad I'_\varepsilon(u_\varepsilon) = 0. \tag{3.26}$$

That is, u_ε is a solution of (1.4).

Proof. Suppose that u_ε satisfies (3.23)–(3.25). Since u_ε satisfies (3.24) we have

$$-\Delta u_\varepsilon + \Big(V(\varepsilon x) + (p+1)(\varepsilon^{-2}\|u_\varepsilon\|^2_{L^2(\mathbb{R}^N \setminus (\Omega_{2h_0}/\varepsilon))} - 1)_+^{\frac{p-1}{2}} \times \varepsilon^{-2}\chi_{\mathbb{R}^N \setminus (\Omega_{2h_0}/\varepsilon)}(x)\Big)u_\varepsilon = f(u_\varepsilon), \tag{3.27}$$

where $\chi_{\mathbb{R}^N \setminus (\Omega_{2h_0}/\varepsilon))}(x)$ is the characteristic function of the set $\mathbb{R}^N \setminus (\Omega_{2h_0}/\varepsilon)$. Clearly u_ε satisfies (3.3)–(3.6) and thus, by the proof of Proposition 3.1, we have

$$\|u_\varepsilon\|_{H^1(\mathbb{R}^N \setminus (\Omega_{h_0}/\varepsilon))} \le \|u_\varepsilon^{(2)}\|_{H^1(\mathbb{R}^N)} \to 0 \quad \text{as } \varepsilon \to 0.$$

Standard elliptic estimates then shown that

$$\|u_\varepsilon\|_{L^\infty(\mathbb{R}^N \setminus (\Omega_{\frac{3}{2}h_0}/\varepsilon))} \to 0 \quad \text{as } \varepsilon \to 0$$

and using a comparison principle, we deduce that for some c, $c' > 0$

$$|u_\varepsilon(x)| \le c' \exp(-c \,\mathrm{dist}\,(x, \Omega_{\frac{3}{2}h_0}/\varepsilon)).$$

In particular then

$$\|u_\varepsilon\|_{L^2(\mathbb{R}^N \setminus (\Omega_{2h_0}/\varepsilon))} < \varepsilon \quad \text{for} \quad \varepsilon > 0 \text{ small}$$

and we have (3.26). ∎

To find critical points of $J_\varepsilon(u)$, we need the following

Proposition 3.3 *For any fixed $\varepsilon > 0$, the Palais-Smale condition holds for J_ε in $\{u \in \mathcal{S}(r_2);\ \varepsilon\Upsilon(u) \in \Omega([0,\nu_0])\}$. That is, if a sequence $(u_j) \subset H^1(\mathbb{R}^N)$ satisfies for some $c > 0$*

$$\begin{aligned}
&u_j \in \mathcal{S}(r_2),\\
&\varepsilon\Upsilon(u_j) \in \Omega([0,\nu_0]),\\
&\|J'_\varepsilon(u_j)\|_{H^{-1}} \to 0,\\
&J_\varepsilon(u_j) \to c \quad \textit{as } j \to \infty,
\end{aligned}$$

then (u_j) has a strongly convergent subsequence in $H^1(\mathbb{R}^N)$.

Proof. Since $\mathcal{S}(r_2)$ is bounded in $H^1(\mathbb{R}^N)$, after extracting a subsequence if necessary, we may assume $u_j \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^N)$ for some $u_0 \in H^1(\mathbb{R}^N)$. We will show that $u_j \to u_0$ strongly in $H^1(\mathbb{R}^N)$. Denoting $B_R = \{x \in \mathbb{R}^N;\ |x| < R\}$, it suffices to show that

$$\lim_{R\to\infty}\lim_{j\to\infty}\|u_j\|^2_{H^1(\mathbb{R}^N\setminus B_R)} = 0. \tag{3.28}$$

To show (3.28) we first we note that, since $\varepsilon > 0$ is fixed, $\|u_j\|_{H^1(\mathbb{R}^N\setminus B_L)} < 2r_2$ for a large $L > 1$. In particular, for any $n \in \mathbb{N}$

$$\sum_{i=1}^{n}\|u_j\|^2_{H^1(A_i)} < 4r_2^2,$$

where $A_i = B_{L+i} \setminus B_{L+i-1}$.

Thus, for any $j \in \mathbb{N}$, there exists $i_j \in \{1, 2, \cdots, n\}$ such that

$$\|u_j\|^2_{H^1(A_{i_j})} < \frac{4r_2^2}{n}.$$

Now we choose $\zeta_i \in C^1(\mathbb{R},\mathbb{R})$ such that $\zeta_i(r) = 1$ for $r \le L+i-1$, $\zeta_i(r) = 0$ for $r \ge L+i$ and $\zeta_i'(r) \in [-2,0]$ for all $r > 0$. We set

$$\widetilde{u}_j(x) = (1-\zeta_{i_j}(|x|))u_j(x).$$

We have, for a constant $C > 0$ independent of n, j

$$
\begin{aligned}
J'_\varepsilon(u_j)\widetilde{u}_j &= I'_\varepsilon(u_j)\widetilde{u}_j + Q'_\varepsilon(u_j)\widetilde{u}_j, && (3.29)\\
I'_\varepsilon(u_j)\widetilde{u}_j &= I'_\varepsilon(\widetilde{u}_j)\widetilde{u}_j + \int_{A_{i_j}} \nabla(\zeta_{i_j}u_j)\nabla((1-\zeta_{i_j})u_j) + V(\varepsilon x)\zeta_{i_j}(1-\zeta_{i_j})u_j^2 \\
&\quad + [f((1-\zeta_{i_j})u_j) - f(u_j)](1-\zeta_{i_j})u_j\,dx \\
&\geq I'_\varepsilon(\widetilde{u}_j)\widetilde{u}_j - \frac{C}{n}, && (3.30)\\
Q'_\varepsilon(u_j)\widetilde{u}_j &= (p+1)\left(\varepsilon^{-2}\|u_j\|^2_{L^2(\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon))} - 1\right)_+^{\frac{p-1}{2}} \\
&\quad \times \int_{\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon))} (1-\zeta_{i_j})u_j^2\,dx \\
&\geq 0. && (3.31)
\end{aligned}
$$

Since $J'_\varepsilon(u_j)\widetilde{u}_j \to 0$, it follows from (3.29)–(3.31) that

$$I'_\varepsilon(\widetilde{u}_j)\widetilde{u}_j \leq \frac{C}{n} + o(1) \quad \text{as } j \to \infty.$$

Now recording that $\|\widetilde{u}_j\|_{H^1} < 2r_2$ we have by (3.2) for some $C > 0$

$$\|\widetilde{u}_j\|^2_{H^1} \leq \frac{C}{n} + o(1).$$

Thus, from the definition of $\widetilde{u}_j$, we deduce that

$$c\|u_j\|^2_{H^1(\mathbb{R}^N\setminus B_{L+n})} \leq \frac{C}{n} + o(1).$$

That is, (3.28) holds and (u_j) strongly converges. ∎

The following lemma will be useful to compute the relative category.

Lemma 3.4 *There exists $C_0 > 0$ independent of $\varepsilon > 0$ such that*

$$J_\varepsilon(u) \geq L_{m_0}(u) - C_0\varepsilon^2 \quad \textit{for all } u \in \mathcal{S}(r_1). \tag{3.32}$$

Proof. We have

$$
\begin{aligned}
J_\varepsilon(u) &= L_{m_0}(u) + \frac{1}{2}\int_{\mathbb{R}^N} (V(\varepsilon x) - m_0)u^2\,dx + Q_\varepsilon(u) \\
&\geq L_{m_0}(u) - \frac{1}{2}(m_0 - \underline{V})\|u\|^2_{L^2(\mathbb{R}^N\setminus(\Omega/\varepsilon))} + Q_\varepsilon(u).
\end{aligned}
$$

We distinguish the two cases: (a) $\|u\|^2_{L^2(\mathbb{R}^N\setminus(\Omega/\varepsilon))} \le 2\varepsilon^2$, (b) $\|u\|^2_{L^2(\mathbb{R}^N\setminus(\Omega/\varepsilon))} \ge 2\varepsilon^2$.

If case (a) occurs, we have

$$J_\varepsilon(u) \ge L_{m_0}(u) - (m_0 - \underline{V})\varepsilon^2$$

and (3.32) holds. If case (b) takes place, we have

$$Q_\varepsilon(u) \ge \left(\frac{1}{2}\varepsilon^{-2}\|u\|^2_{L^2(\mathbb{R}^N\setminus(\Omega/\varepsilon))}\right)^{\frac{p+1}{2}} \ge \frac{1}{2}\varepsilon^{-2}\|u\|^2_{L^2(\mathbb{R}^N\setminus(\Omega/\varepsilon))}$$

and thus

$$\begin{aligned} J_\varepsilon(u) &\ge L_{m_0}(u) + \frac{1}{2}(\varepsilon^{-2} - (m_0 - \underline{V}))\|u\|^2_{L^2(\mathbb{R}^N\setminus(\Omega/\varepsilon))} \\ &\ge L_{m_0}(u) \quad \text{for } \varepsilon > 0 \text{ small.} \end{aligned}$$

Therefore (3.32) also holds. ∎

4 A neighborhood of expected solutions

In this section we try to find critical points of J_ε. First we choose a neighborhood $\mathcal{X}_{\varepsilon,\delta}$ of a set of expected solutions, which is positively invariant under a pseudo-gradient flow and in which we develop a deformation argument. In the sequel we will estimate a change of topology between $\mathcal{X}_{\varepsilon,\delta} \cap \{u;\ J_\varepsilon(u) \le E(m_0) + \hat{\delta}\}$ and $\mathcal{X}_{\varepsilon,\delta} \cap \{u;\ J_\varepsilon(u) \le E(m_0) - \hat{\delta}\}$ using the relative category.

4.1 A neighborhood $\mathcal{X}_{\varepsilon,\delta}$

We fix $0 < \rho_1 < \rho_0 < r_2$ and we then choose δ_1, $\delta_2 > 0$ according to Lemma 2.5 and Proposition 3.1. We set for $\delta \in (0, \min\{\frac{\delta_2}{4}(\rho_0 - \rho_1), \delta_1\})$,

$$\mathcal{X}_{\varepsilon,\delta} = \{u \in \mathcal{S}(\rho_0);\ \varepsilon\Upsilon(u) \in \Omega([0,\nu_0]),\ J_\varepsilon(u) \le E(m_0) + \delta - \frac{\delta_2}{2}(\widehat{\rho}(u) - \rho_1)_+\}.$$

We shall try to find critical points of J_ε in $\mathcal{X}_{\varepsilon,\delta}$. In this aim first note that

(a) $u \in \mathcal{S}(\rho_0)$ and $\varepsilon\Upsilon(u) \in \Omega([\nu_1, \nu_0])$ imply, by Lemma 2.5, that

$$J_\varepsilon(u) \geq I_\varepsilon(u) \geq E(m_0) + \delta_1 \geq E(m_0) + \delta. \tag{4.1}$$

In particular,

$$\varepsilon\Upsilon(u) \in \Omega([0, \nu_1)) \quad \text{for } u \in \mathcal{X}_{\varepsilon,\delta}.$$

(b) For $u \in \mathcal{X}_{\varepsilon,\delta}$, if $\widehat{\rho}(u) = \rho_0$, i.e., $u \in \partial\mathcal{S}(\rho_0)$, then by the choice of δ

$$J_\varepsilon(u) \leq E(m_0) + \delta - \frac{\delta_2}{2}(\rho_0 - \rho_1) \leq E(m_0) - \delta. \tag{4.2}$$

Now we consider a deformation flow defined by

$$\begin{cases} \frac{d\eta}{d\tau} = -\phi(\eta)\frac{\mathcal{V}(\eta)}{\|\mathcal{V}(\eta)\|_{H^1}}, \\ \eta(0, u) = u, \end{cases} \tag{4.3}$$

where $\mathcal{V}(u) : \{u \in H^1(\mathbb{R}^N);\ J'_\varepsilon(u) \neq 0\} \to H^1(\mathbb{R}^N)$ is a locally Lipschitz continuous pseudo-gradient vector field satisfying

$$\|\mathcal{V}(u)\|_{H^1} \leq \|J'_\varepsilon(u)\|_{H^{-1}}, \quad J'_\varepsilon(u)\mathcal{V}(u) \geq \frac{1}{2}\|J'_\varepsilon(u)\|^2_{H^{-1}}$$

and $\phi(u) : H^1(\mathbb{R}^N) \to [0, 1]$ is a locally Lipschitz continuous function. We require that $\phi(u)$ satisfies $\phi(u) = 0$ if $J_\varepsilon(u) \notin [E(m_0) - \delta, E(m_0) + \delta]$.

Proposition 4.1 *For any $c \in (E(m_0) - \delta, E(m_0) + \delta)$ and for any neighborhood U of $\mathcal{K}_c \equiv \{u \in \mathcal{X}_{\varepsilon,\delta};\ J'_\varepsilon(u) = 0,\ J_\varepsilon(u) = c\}$ ($U = \emptyset$ if $\mathcal{K}_c = \emptyset$), there exist $d > 0$ and a deformation $\eta(\tau, u) : [0, 1] \times (\mathcal{X}_{\varepsilon,\delta} \setminus U) \to \mathcal{X}_{\varepsilon,\delta}$ such that*

(i) $\eta(0, u) = u$ for all u.

(ii) $\eta(\tau, u) = u$ for all $\tau \in [0, 1]$ if $J_\varepsilon(u) \notin [E(m_0) - \delta, E(m_0) + \delta]$.

(iii) $J_\varepsilon(\eta(\tau, u))$ is a non-increasing function of τ for all u.

(iv) $J_\varepsilon(\eta(1, u)) \leq c - d$ for all $u \in \mathcal{X}_{\varepsilon,\delta} \setminus U$ satisfying $J_\varepsilon(u) \leq c + d$.

Proof. We consider the flow defined by (4.3). The properties (i)-(iii) follows by standard arguments from the definition (4.3) and since $\phi(u) = 0$ if $J_\varepsilon(u) \notin [E(m_0) - \delta, E(m_0) + \delta]$. Clearly also since, by Proposition 3.3, J_ε satisfies the Palais-Smale condition for fixed $\varepsilon > 0$, property (iv) is standard. Thus to end the proof we just need to show that

$$\eta(\tau, \mathcal{X}_{\varepsilon,\delta}) \subset \mathcal{X}_{\varepsilon,\delta} \quad \text{for all } \tau \geq 0, \tag{4.4}$$

namely that $\mathcal{X}_{\varepsilon,\delta}$ is positively invariant under our flow. First note that because of property (iii), (4.1) implies that for $u \in \mathcal{X}_{\varepsilon,\delta}$, $\eta(t) = \eta(t,u)$ does not enter the set $\{u;\, \varepsilon\Upsilon(u) \in \Omega([\nu_1, \nu_0])\}$. Also, because of property (ii), (4.2), show that for $u \in \mathcal{X}_{\varepsilon,\delta}$, $\eta(t)$ remains in $\mathcal{S}(\rho_0)$. Thus to show (4.4) we just need to prove that the property

$$J_\varepsilon(u) \leq E(m_0) + \delta - \frac{\delta_2}{2}(\widehat{\rho}(u) - \rho_1)$$

is stable under the deformation. For this it suffices to show that for a solution $\eta(\tau)$ of (4.3), if $0 < s < t < 1$ satisfies

$$\begin{gathered} \widehat{\rho}(\eta(\tau)) \in [\rho_1, \rho_0] \quad \text{for all } \tau \in [s,t], \\ J_\varepsilon(\eta(s)) \leq E(m_0) + \delta - \tfrac{\delta_2}{2}(\widehat{\rho}(\eta(s)) - \rho_1), \end{gathered}$$

then

$$J_\varepsilon(\eta(t)) \leq E(m_0) + \delta - \frac{\delta_2}{2}(\widehat{\rho}(\eta(t)) - \rho_1).$$

We note that $(\widehat{\rho}(\eta(\tau)), \varepsilon\Upsilon(\eta(\tau))) \in [\rho_1, \rho_0] \times \Omega([0,\nu_1]) \subset ([0,\rho_0] \times \Omega([0,\nu_0])) \setminus ([0,\rho_1] \times \Omega([0,\nu_1]))$ for all $\tau \in [s,t]$. Thus by Proposition 3.1, we have for $\tau \in [s,t]$

$$\frac{d}{d\tau} J_\varepsilon(\eta(\tau)) = J_\varepsilon'(\eta)\frac{d\eta}{d\tau} = -\phi(\eta) J_\varepsilon'(\eta)\frac{\mathcal{V}(\eta)}{\|\mathcal{V}(\eta)\|_{H^1}} \leq -\phi(\eta)\frac{\delta_2}{2}$$

and

$$J_\varepsilon(\eta(t)) \leq J_\varepsilon(\eta(s)) - \frac{\delta_2}{2}\int_s^t \phi(\eta(\tau))\, d\tau. \tag{4.5}$$

On the other hand,

$$\|\eta(t) - \eta(s)\|_{H^1} \leq \int_s^t \|\frac{d\eta}{d\tau}\|_{H^1}\, d\tau \leq \int_s^t \phi(\eta(\tau))\, d\tau. \tag{4.6}$$

By (4.5)–(4.6), and using the fact that $|\widehat{\rho}(\eta(t)) - \widehat{\rho}(\eta(s))| \leq \|\eta(t) - \eta(s)\|_{H^1}$, we have

$$\begin{aligned} J_\varepsilon(\eta(t)) &\leq J_\varepsilon(\eta(s)) - \frac{\delta_2}{2}\|\eta(t) - \eta(s)\|_{H^1} \\ &\leq J_\varepsilon(\eta(s)) - \frac{\delta_2}{2}|\widehat{\rho}(\eta(t)) - \widehat{\rho}(\eta(s))| \\ &\leq E(m_0) + \delta - \frac{\delta_2}{2}(\widehat{\rho}(\eta(s)) - \rho_1) - \frac{\delta_2}{2}|\widehat{\rho}(\eta(t)) - \widehat{\rho}(\eta(s))| \\ &\leq E(m_0) + \delta - \frac{\delta_2}{2}(\widehat{\rho}(\eta(t)) - \rho_1). \end{aligned}$$

Thus (4.4) holds and the proof of the proposition is completed. ∎

4.2 Two maps Φ_ε and Ψ_ε

For $c \in \mathbb{R}$, we set

$$\mathcal{X}^c_{\varepsilon,\delta} = \{u \in \mathcal{X}_{\varepsilon,\delta};\ J_\varepsilon(u) \leq c\}.$$

For $\hat{\delta} > 0$ small, using relative category, we shall estimate the change of topology between $\mathcal{X}^{E(m_0)+\hat{\delta}}_{\varepsilon,\delta}$ and $\mathcal{X}^{E(m_0)-\hat{\delta}}_{\varepsilon,\delta}$.

We recall that $K = \{x \in \Omega; V(x) = m_0\}$. For $s_0 \in (0,1)$ small we introduce two maps:

$$\begin{aligned} &\Phi_\varepsilon : ([1-s_0, 1+s_0] \times K, \{1 \pm s_0\} \times K) \to (\mathcal{X}^{E(m_0)+\hat{\delta}}_{\varepsilon,\delta}, \mathcal{X}^{E(m_0)-\hat{\delta}}_{\varepsilon,\delta}), \\ &\Psi_\varepsilon : (\mathcal{X}^{E(m_0)+\hat{\delta}}_{\varepsilon,\delta}, \mathcal{X}^{E(m_0)-\hat{\delta}}_{\varepsilon,\delta}) \to \\ &\qquad ([1-s_0, 1+s_0] \times \Omega([0,\nu_1]), ([1-s_0, 1+s_0] \setminus \{1\}) \times \Omega([0,\nu_1])). \end{aligned}$$

Here we use notation from algebraic topology: $f : (A,B) \to (A',B')$ means $B \subset A$, $B' \subset A'$, $f : A \to A'$ is continuous and $f(B) \subset B'$.

Definition of Φ_ε:
Fix a least energy solution $U_0(x) \in \widehat{S}$ of $-\Delta u + m_0 u = f(u)$ and set

$$\Phi_\varepsilon(s,p) = U_0(\frac{x - \frac{1}{\varepsilon}p}{s}).$$

Definition of Ψ_ε:
We introduce a function $\tilde{P}_0 : H^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$\tilde{P}_0(u) = \begin{cases} 1+s_0 & \text{if } P_0(u) \geq 1+s_0, \\ 1-s_0 & \text{if } P_0(u) \leq 1-s_0, \\ P_0(u) & \text{otherwise,} \end{cases}$$

where $P_0(u)$ is given in (2.5).

We define our operator Ψ_ε by

$$\Psi_\varepsilon(u) = (\tilde{P}_0(u), \varepsilon\Upsilon(u)) \quad \text{for } u \in \mathcal{X}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}.$$

In what follows, we observe that Φ_ε and Ψ_ε are well-defined for a suitable choices of s_0 and $\hat{\delta}$. First we deal with Φ_ε.

We fix $s_0 \in (0,1)$ small so that $||U_0(\frac{x}{s}) - U_0(x)||_{H^1} < \rho_1$ for $s \in [1-s_0, 1+s_0]$. That is for each $s \in [1-s_0, 1+s_0]$,

$$U_0(\frac{x}{s}) = U_0(x) + \varphi_s(x), \quad \text{with } ||\varphi_s||_{H^1} < \rho_1.$$

Thus

$$U_0(\frac{x-p/\varepsilon}{s}) = U_0(x-p/\varepsilon) + \tilde{\varphi}_s(x) \quad \text{with } ||\tilde{\varphi}_s||_{H^1} < \rho_1.$$

Therefore, using the first property of Lemma 2.4, that is $|\Upsilon(u) - p| \leq 2R_0$ for $u(x) = U(x-p) + \varphi(x) \in \mathcal{S}(\rho_0)$, we get

$$\left|\Upsilon(U_0(\frac{x-p/\varepsilon}{s}) - p/\varepsilon\right| \leq 2R_0.$$

It follows that, for $s \in [1-s_0, 1+s_0]$

$$\varepsilon\Upsilon(U_0(\frac{x-p/\varepsilon}{s})) = p + o(1). \tag{4.7}$$

Also, using Lemma 2.1, we have for $p \in K$ and $s \in [1-s_0, 1+s_0]$

$$J_\varepsilon(U_0(\frac{x-p/\varepsilon}{s})) = L_{m_0}(U_0(\frac{x-p/\varepsilon}{s})) + o(1) = g(s)E(m_0) + o(1).$$

Thus, choosing $\hat{\delta} > 0$ small so that $g(1 \pm s_0)E(m_0) < E(m_0) - \hat{\delta}$, which is possible by Remark 2.3, we see that Φ_ε is well-defined as a map

$$([1-s_0, 1+s_0] \times K, \{1 \pm s_0\} \times K) \to (\mathcal{X}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}, \mathcal{X}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}).$$

Next we deal with the well-definedness of Ψ_ε.
By the definition Ψ_ε, $\Psi_\varepsilon(\mathcal{X}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}) \subset [1-s_0, 1+s_0] \times \Omega([0,\nu_1])$. Now we assume that $u \in \mathcal{X}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}$. By Lemma 3.4, we have

$$L_{m_0}(u) \le J_\varepsilon(u) + C_0\varepsilon^2 \le E(m_0) - \hat{\delta} + C_0\varepsilon^2.$$

By Lemma 2.1,

$$g(P_0(u))E(m_0) \le L_{m_0}(u) \le E(m_0) - \hat{\delta} + C_0\varepsilon^2.$$

Thus for $\varepsilon > 0$ small we see from Remark 2.3 that

$$P_0(u) \ne 1$$

and Ψ_ε is well-defined as a map $(\mathcal{X}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}, \mathcal{X}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}) \to ([1-s_0, 1+s_0] \times \Omega([0,\nu_1]), ([1-s_0, 1+s_0] \setminus \{1\}) \times \Omega([0,\nu_1]))$.

The next proposition will be important to estimate $\mathrm{cat}(\mathcal{X}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}, \mathcal{X}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}})$.

Proposition 4.2

$$\begin{aligned}\Psi_\varepsilon \circ \Phi_\varepsilon :\ & ([1-s_0, 1+s_0] \times K, \{1 \pm s_0\} \times K) \\ & \to ([1-s_0, 1+s_0] \times \Omega([0,\nu_1]), ([1-s_0, 1+s_0] \setminus \{1\}) \times \Omega([0,\nu_1]))\end{aligned}$$

is homotopic to the embedding $j(s,p) = (s,p)$. That is, there exists a continuous map

$$\eta :\ [0,1] \times [1-s_0, 1+s_0] \times K \to [1-s_0, 1+s_0] \times \Omega([0,\nu_1])$$

such that

$$\begin{aligned}&\eta(0,s,p) = (\Psi_\varepsilon \circ \Phi_\varepsilon)(s,p),\\ &\eta(1,s,p) = (s,p) \quad \textit{for all } (s,p) \in [1-s_0, 1+s_0] \times K,\\ &\eta(t,s,p) \in ([1-s_0, 1+s_0] \setminus \{1\}) \times \Omega([0,\nu_1])\\ &\qquad \textit{for all } t \in [0,1] \textit{ and } (s,p) \in \{1 \pm s_0\} \times K.\end{aligned}$$

Proof. By the definitions of Φ_ε and Ψ_ε, we have

$$\begin{aligned}(\Psi_\varepsilon \circ \Phi_\varepsilon)(s,p) &= \Big(\tilde{P}_0(U_0(\frac{x-p/\varepsilon}{s})), \varepsilon\Upsilon(U_0(\frac{x-p/\varepsilon}{s}))\Big)\\ &= \Big(s, \varepsilon\Upsilon(U_0(\frac{x-p/\varepsilon}{s}))\Big).\end{aligned}$$

We set

$$\eta(t,s,p) = \Big(s, (1-t)\varepsilon\Upsilon(\frac{x-p/\varepsilon}{s}) + tp\Big).$$

Recalling (4.7), we see that for $\varepsilon > 0$ small $\eta(t,s,p)$ has the desired properties and $\Psi_\varepsilon \circ \Phi_\varepsilon$ is homotopic to the embedding j. ∎

Remark 4.3 *As an application of the definition of category and of homotopic equivalence between maps, one can establish that if X, Ω^-, Ω^+ are closed sets such that $\Omega^- \subset \Omega^+$ and $\beta : X \to \Omega^+$, $\psi : \Omega^- \to X$ are two continuous maps such that $\beta \circ \psi$ is homotopically equivalent to the embedding $j : \Omega^- \to \Omega^+$, then $\mathrm{cat}_{\Omega^+}(\Omega^-) \le \mathrm{cat}_X(X)$. See, for instance, [17]. Conversely if X, X_0, Ω^-, Ω_0^-, Ω^+, Ω_0^+ are closed sets such that $X_0 \subset X$, $\Omega_0^- \subset \Omega^-$, $\Omega_0^+ \subset \Omega^+$, $\Omega^- \subset \Omega^+$ and there exists $\Psi : (X, X_0) \to (\Omega^+, \Omega_0^+)$ and $\Phi : (\Omega^-, \Omega_0^-) \to (X, X_0)$ two continuous maps such that $\Psi \circ \Phi$ is homotopically equivalent to the embedding $j : (\Omega^-, \Omega_0^-) \to (\Omega^+, \Omega_0^+)$, one can not infer that $\mathrm{cat}_X(X, X_0) \ge \mathrm{cat}_{\Omega^+}(\Omega^-, \Omega_0^+)$. Here $\mathrm{cat}_X(A, B)$ denotes the category of A in X relative to B, where (X, A) is a topological pair and B is a closed subset of X.*

Indeed, consider the topological pairs $(\Omega^-, \Omega_0^-) = (B, \emptyset)$, $(\Omega^+, \Omega_0^+) = (B, S)$, $(X, X_0) = (B, \{p\})$ where B is a ball, $S = \partial B$ and p is a point on the sphere $S = \partial B$. It is possible to construct two continuous maps $\Psi : (B, \{p\}) \to (B, S)$ and $\Phi : (B, \emptyset) \to (B, \{p\})$ such that $\Psi \circ \Phi : (B, \emptyset) \to (B; S)$ is homotopically equivalent to the embedding $j : (B, \emptyset) \to (B, S)$. However $\mathrm{cat}_X(X, X_0) = \mathrm{cat}_B(B, \{p\}) = 0$ and $\mathrm{cat}_{\Omega^+}(\Omega^-, \Omega_0^+) = \mathrm{cat}_B(B, S) = 1$.

From Remark 4.3, differently from [19], we can not infer, in general, that

$$\mathrm{cat}(\mathcal{X}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}, \mathcal{X}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}) \ge \mathrm{cat}(K, \partial K).$$

Therefore in the work it will be necessary to use the notions of category and cup-length for an map.

5 Proof of Theorem 1.1

In order to prove our theorem, we shall need some topological tools that we now present for the reader convenience. Following [7], see also [37, 38], we define

Definition 5.1 *Let $B \subset A$ and $B' \subset A'$ be topological spaces and $f : (A, B) \to (A', B')$ be a continuous map, that is $f : A \to A'$ is continuous and $f(B) \subset B'$. The category* $\mathrm{cat}(f)$ *of f is the least integer $k \geq 0$ such that there exist open sets A_0, A_1, $\cdots$, A_k with the following properties:*

(a) $A = A_0 \cup A_1 \cup \cdots \cup A_k$.

(b) $B \subset A_0$ *and there exists a map* $h_0 : [0,1] \times A_0 \to A'$ *such that*

$$\begin{aligned} &h_0(0,x) = f(x) && \text{for all } x \in A_0, \\ &h_0(1,x) \in B' && \text{for all } x \in A_0, \\ &h_0(t,x) = f(x) && \text{for all } x \in B \text{ and } t \in [0,1]. \end{aligned}$$

(c) For $i = 1, 2, \cdots, k$, $f|_{A_i} : A_i \to A'$ *is homotopic to a constant map.*

We also introduce the cup-length of $f : (A, B) \to (A', B')$. Let H^* denote Alexander-Spanier cohomology with coefficients in the field $\mathbb{F}$. We recall that the cup product $\smile$ turns $H^*(A)$ into a ring with unit 1_A, and it turns $H^*(A, B)$ into a module over $H^*(A)$. A continuous map $f : (A, B) \to (A', B')$ induces a homomorphism $f^* : H^*(A') \to H^*(A)$ of rings as well as a homomorphism $f^* : H^*(A', B') \to H^*(A, B)$ of abelian groups. We also use notation:

$$\tilde{H}^n(A') = \begin{cases} 0 & \text{for } n = 0, \\ H^n(A') & \text{for } n > 0. \end{cases}$$

For more details on algebraic topology we refer to [47].

Definition 5.2 *For $f : (A, B) \to (A', B')$ the cup-length,* $\mathrm{cupl}(f)$ *is defined as follows; when $f^* : H^*(A', B') \to H^*(A, B)$ is not a trivial map,* $\mathrm{cupl}(f)$ *is defined as the maximal integer $k \geq 0$ such that there exist elements α_1, $\cdots$, $\alpha_k \in \tilde{H}^*(A')$ and $\beta \in H^*(A', B')$ with*

$$\begin{aligned} f^*(\alpha_1 \smile \cdots \smile \alpha_k \smile \beta) &= f^*(\alpha_1) \smile \cdots \smile f^*(\alpha_k) \smile f^*(\beta) \\ &\neq 0 \text{ in } H^*(A, B). \end{aligned}$$

When $f^* = 0 : H^*(A', B') \to H^*(A, B)$, we define $\mathrm{cupl}(f) = -1$.

We note that $\mathrm{cupl}(f) = 0$ if $f^* \neq 0 : H^*(A', B') \to H^*(A, B)$ and $\tilde{H}^*(A') = 0$.

As fundamental properties of $\mathrm{cat}(f)$ and $\mathrm{cupl}(f)$, we have

Proposition 5.3 *(i) For* $f : (A, B) \to (A', B')$, $\mathrm{cat}(f) \geq \mathrm{cupl}(f) + 1$.

(ii) For $f : (A, B) \to (A', B')$, $f' : (A', B') \to (A'', B'')$,

$$\mathrm{cupl}(f' \circ f) \leq \min\{\mathrm{cupl}(f'), \mathrm{cupl}(f)\}.$$

(iii) If $f, g : (A, B) \to (A', B')$ *are homotopic, then* $\mathrm{cupl}(f) = \mathrm{cupl}(g)$.

The proof of these statements can be found in [7, Lemma 2.7], [7, Lemma 2.6 (a)] and [7, Lemma 2.6(b)] respectively. Finally we recall

Definition 5.4 *For a set* (A, B)*, we define the relative category* $\mathrm{cat}(A, B)$ *and the relative cup-length* $\mathrm{cupl}(A, B)$ *by*

$$\mathrm{cat}(A, B) = \mathrm{cat}(id_{(A,B)} : (A, B) \to (A, B)),$$
$$\mathrm{cupl}(A, B) = \mathrm{cupl}(id_{(A,B)} : (A, B) \to (A, B)).$$

We also set

$$\mathrm{cat}(A) = \mathrm{cat}(A, \emptyset), \quad \mathrm{cupl}(A) = \mathrm{cupl}(A, \emptyset).$$

The following lemma which is due to Bartsch [4] is one of the keys of our proof and we make use of the continuity property of Alexander-Spanier cohomology.

Lemma 5.5 *Let* $K \subset \mathbb{R}^N$ *be a compact set. For a d-neighborhood* $K_d = \{x \in \mathbb{R}^N; \mathrm{dist}\,(x, K) \leq d\}$ *and* $I = [0, 1]$, $\partial I = \{0, 1\}$*, we consider the inclusion*

$$j : (I \times K, \partial I \times K) \to (I \times K_d, \partial I \times K_d)$$

defined by $j(s, x) = (s, x)$*. Then for* $d > 0$ *small,*

$$\mathrm{cupl}(j) \geq \mathrm{cupl}(K).$$

Proof. Let $k = \mathrm{cupl}(K)$ and let $\alpha_1, \cdots, \alpha_k \in H^*(K)$ $(* \geq 1)$ be such that $\alpha_1 \cup \cdots \cup \alpha_k \neq 0$. By the continuity property of Alexander-Spanier cohomology (see [47, Theorem 6.6.2]), for $d > 0$ small there exists $\alpha_1^d, \cdots, \alpha_k^d \in H^*(K_d)$ such that $i_d^*(\alpha_i^d) = \alpha_i$ for $i = 1, 2, \cdots, k$, where $i_d : K \to K_d$ is the inclusion.

By the Künneth formula, the cross products give us the following isomorphisms:

$$\begin{aligned}
\times : & \; H^0(I) \otimes H^n(K) \simeq H^n(I \times K), \\
\times : & \; H^1(I, \partial I) \otimes H^n(K) \simeq H^{n+1}(I \times K, \partial I \times K), \\
\times : & \; H^0(I) \otimes H^n(K_d) \simeq H^n(I \times K_d), \\
\times : & \; H^1(I, \partial I) \otimes H^n(K_d) \simeq H^{n+1}(I \times K_d, \partial I \times K_d).
\end{aligned}$$

Let $\tau \in H^1(I, \partial I) \simeq \mathbb{F}$ be a non-trivial element and for any set A we denote by $1_A \in H^0(A)$ the unit of the cohomology ring $H^*(A)$. Set $\tilde{\beta} = \tau \times 1_{K_d} \in H^1(I \times K_d, \partial I \times K_d)$ and $\tilde{\alpha}_i = 1_I \times \alpha_i^d \in H^*(I \times K_d)$, then we have

$$j^*(\tilde{\beta}) = \tau \times 1_K, \quad j^*(\tilde{\alpha}_i) = 1_I \times \alpha_i.$$

Thus

$$\begin{aligned}
j^*(\tilde{\beta} \smile \tilde{\alpha}_1^d \smile \cdots \smile \alpha_k^d) &= j^*(\tilde{\beta}) \smile j^*(\tilde{\alpha}_1^d) \smile \cdots \smile j^*(\alpha_k^d) \\
&= (\tau \times 1_K) \cup (1_I \times \alpha_1) \smile \cdots \smile (1_I \times \alpha_k) \\
&= \tau \times (\alpha_1 \smile \cdots \smile \alpha_k) \neq 0.
\end{aligned}$$

Thus we have $\mathrm{cupl}(j) \geq \mathrm{cupl}(K)$. ∎

Now we have all the ingredients to give the

Proof of Theorem 1.1. We observe that for $\varepsilon > 0$ small

$$\#\{u \in \mathcal{X}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}} \setminus \mathcal{X}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}};\ J'_\varepsilon(u) = 0\} \geq \mathrm{cat}(\mathcal{X}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}, \mathcal{X}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}). \quad (5.1)$$

Using Proposition 4.1, (5.1) can be proved in a standard way (c.f. Theorem 4.2 of [37]).

By (i) of Proposition 5.3, we have

$$\mathrm{cat}(\mathcal{X}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}, \mathcal{X}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}) \geq \mathrm{cupl}(\mathcal{X}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}, \mathcal{X}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}) + 1. \quad (5.2)$$

Since $\Psi_\varepsilon \circ \Phi_\varepsilon = \Psi_\varepsilon \circ id_{(\mathcal{X}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}, \mathcal{X}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}})} \circ \Phi_\varepsilon$, it follows from (ii) of Proposition 5.3 that

$$\begin{aligned}
\mathrm{cupl}(\Psi_\varepsilon \circ \Phi_\varepsilon) &\leq \mathrm{cupl}(id_{(\mathcal{X}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}, \mathcal{X}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}})}) \\
&= \mathrm{cupl}(\mathcal{X}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}, \mathcal{X}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}). \quad (5.3)
\end{aligned}$$

Now recall that, by our choice of ν_0, we have $\Omega([0,\nu_1]) \subset K_d$. Thus, letting

$$\begin{aligned}\tau: \quad & ([1-s_0, 1+s_0] \times \Omega([0,\nu_1]), ([1-s_0,1+s_0] \setminus \{1\}) \times \Omega([0,\nu_1])) \\ & \to ([1-s_0,1+s_0] \times K_d, ([1-s_0,1+s_0] \setminus \{1\}) \times K_d)\end{aligned}$$

be the inclusion we see, using Proposition 4.2, that $\tau \circ \Psi_\varepsilon \circ \Phi_\varepsilon$ is homotopic to the inclusion $j: (I \times K, \partial I \times K) \to (I \times K_d, \partial I \times K_d)$. Thus on one hand, by (ii) of Proposition 5.3,

$$\mathrm{cupl}(\Psi_\varepsilon \circ \Phi_\varepsilon) \geq \mathrm{cupl}(\tau \circ \Psi_\varepsilon \circ \Phi_\varepsilon). \tag{5.4}$$

On the other hand, by (iii) of Proposition 5.3,

$$\mathrm{cupl}(\tau \circ \Psi_\varepsilon \circ \Phi_\varepsilon) = \mathrm{cupl}(j). \tag{5.5}$$

At this point using (5.2)–(5.5) and Lemma 5.5, we deduce that

$$\mathrm{cat}(\mathcal{X}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}}, \mathcal{X}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}) \geq \mathrm{cupl}(K) + 1.$$

Thus by (5.1), J_ε has at least $\mathrm{cupl}(K) + 1$ critical points in $\mathcal{X}_{\varepsilon,\delta}^{E(m_0)+\hat{\delta}} \setminus \mathcal{X}_{\varepsilon,\delta}^{E(m_0)-\hat{\delta}}$. Recalling Proposition 3.2, this completes the proof of the Theorem. ∎

Proof of Remark 1.3. From the proof of Proposition 3.1 we know that for any $\nu_0 > 0$ small enough the critical points u_ε^i, $i = 1, \ldots, \mathrm{cupl}(K)+1$ satisfy

$$||u_\varepsilon^i(x) - U^i(x - x_\varepsilon^i)||_{H^1} \to 0$$

where $\varepsilon x_\varepsilon^i = \varepsilon\Upsilon(u_\varepsilon^i) + o(1) \to x_0^i \in ([0,\nu_0])$ and $U^i \in \widehat{S}$. Thus $w_\varepsilon^i(x) = u_\varepsilon^i(x + x_\varepsilon^i)$ converges to $U^i \in \widehat{S}$. Now observing that these results holds for any $\nu_0 > 0$ and any $\ell_0 > E(m_0 + \nu_0)$ we deduce, considering sequences $\nu_0^n \to 0$, $\ell_0^n \to E(m_0)$ and making a diagonal process, that it is possible to assume that each w_ε^i converges to a least energy solution of

$$-\Delta U + m_0 U = f(U), \quad U > 0, \quad U \in H^1(\mathbb{R}^N).$$

Clearly also

$$u_\varepsilon^i(x) \leq Cexp(-c|x - x_\varepsilon^i|), \quad \text{for some } c, C > 0$$

and this ends the proof. ∎

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