Parabolic capacity and equations with measure data

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joint work with F. Petitta and A.Ponce (JEE 2011)

Consider the evolution problem

$$\begin{cases} u_t - \Delta_p u + h(u) = \mu & \text{in } Q := (0, T) \times \Omega, \\ u(0) = u_0 & \text{in } \Omega, \\ u = 0 & \text{on } (0, T) \times \partial\Omega, \end{cases}$$

where Ω is a bounded susbet in $\mathbb{R}^N$, $u_0 \in L^1(\Omega)$. Main features:

- $h(s)$ satisfies the absorption condition:

$$h(s)s \geq 0 \qquad \text{for } |s| \text{ large}$$

- μ is a bounded Radon measure on Q (space-time measure)

Rmk: The p-Laplacian can be replaced by any divergence form operator $A(u) = -\mathrm{div}(a(t, x, \nabla u))$ acting similarly on $L^p(0, T; W_0^{1,p}(\Omega))$.

Known facts:

- The case $\mu \in L^1(Q)$ is well known since the works (among others) by P. Bénilan, L. Boccardo, T. Gallouët, M. Pierre & friends, sons, nephews.....

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- You cannot solve for any measure μ.
  Ex: if $h(s) = |s|^m s$ [Baras-Pierre].

  The obstruction comes from the regularizing of absorption terms (Vs singular data μ). Basic estimate:

$$\|h(u)\|_{L^1} \le \|\mu\|_{\mathcal{M}(\Omega)}$$

  But the compactness of $h(u)$ may be missing.
  (cfr. [Brezis-Friedman] for the case of initial data....)

The elliptic case

$$-\Delta_p u + h(u) = \mu \qquad \text{in } \Omega$$

- If μ does not charge the $W_0^{1,p}$- capacity, then $\exists$ a solution [Boccardo-Gallouët-Orsina]

  Moreover, if h is nondecreasing, uniqueness holds for entropy or renormalized solutions.

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- You can solve generically, i.e. for every $h(u)$, only if μ does not charge sets of null $W_0^{1,p}$- capacity [Brezis-Marcus-Ponce, with $p = 2$].

  Related question (somehow limiting case): the obstacle problem. You can solve the obstacle problem

$$-\Delta u + \beta(u) \ni \mu$$

  if and only if μ does not charge the $W_0^{1,2}$- capacity [Brezis-Ponce, Dall'Aglio-Leone].

Elliptic case: why it is OK when $\mu << p$-capacity?

Answer [Boccardo-Gallouët-Orsina]:

$$\mu << cap_{W_0^{1,p}(\Omega)} \quad \Longleftrightarrow \quad \mu \in L^1 + W^{-1,p'}(\Omega)$$

Then:

$$\mu = f + \mathrm{div}(F) \longrightarrow \text{ approximate by } \mu_n = f_n + \mathrm{div}(F_n),$$

$$f_n \to f \quad \text{in } L^1(\Omega), \qquad F_n \to F \quad \text{in } L^{p'}(\Omega)$$

You can localize from the equation

$$\int_{\{|u_n|>k+1\}} |h(u_n)|dx \le \int_{\{|u_n|>k\}} |f_n|dx + C_0 \int_{\{|u_n|>k\}} |F_n|^{p'} dx$$

Strong convergence of f_n, $F_n \Rightarrow$ equi–integrability of $h(u_n)$

$$\longrightarrow \quad \text{compactness of } h(u_n)$$

The evolution case

In the evolution case, the natural capacity is associated to the parabolic p-Laplace operator $u_t - \Delta_p u$.

Def. of parabolic p-capacity:
([Pierre] for $p = 2$, [Droniou-P.-Prignet] for $p \neq 2$)

$$\mathrm{cap}_W(U) = \inf \left\{ \|u\|_W : u \in W,\ u \geq \chi_U \text{ a.e. in } Q \right\},$$

if U is open, where

$$W = \left\{ u \in L^p(0,T;V) : u_t \in L^{p'}(0,T;V') \right\},\ V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$$

The norm in W is

$$\|u\|_W = \|u\|_{L^p(0,T;V)} + \|u_t\|_{L^{p'}(0,T;V')}$$

Extension to to arbitrary Borel subsets $B \subset Q$ is done as usual.

In [Droniou-P.-Prignet]:

(i) we give a representation for (space-time) measures which do not charge sets of null parabolic capacity:

$$\mu << \mathrm{cap}_W \quad \Rightarrow \quad \mu = f + \chi + g_t \quad \text{in } \mathcal{D}'(Q)$$

where

$$f \in L^1(Q)\,, \quad \chi \in L^{p'}(0,T;W^{-1,p'}(\Omega))\,, \quad g \in L^p(0,T;V)\,.$$

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(ii) we introduce a renormalized formulation for those measures data, based on shift of the time component g_t.

Roughly speaking (according to [DPP] formulation):

$$u \text{ is a ren. sol. of } u_t - \Delta_p u = \mu = f + \chi + g_t \text{ if}$$

$$v := u - g \text{ solves } v_t - \Delta_p(v + g) = f + \chi.$$

Later developments: [Droniou-Prignet] for entropy formulation, [Petitta] for general measures.

- Advantages: one can reproduce many of the elliptic arguments on v.
- Disadvantages: in this approach the renormalization (truncation principle) takes place on $u - g$ rather than on u.

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  Pb: This cannot be extended to lower order terms ! Replacing $h(u)$ into $h(v + g)$ affetcs the absorption structure.
- If one can construct a decomposition with $g \in L^\infty(Q)$, then everything is OK. Is it possible? No.
  Ex [PPP]: There exists measures $\mu << \text{cap}_W$ for which no decomposition can be found with bounded g.

Summary (motivation of our work):

- A natural notion of capacity exists related to parabolic operator.
- A decomposition result for measures which do not charge such capacity exists.
- The renormalized formulation developed in [DPP] relies on such decomposition. It provides well-posedness for the pure p-Laplace operator. It relies on truncation on $u - g$ rather than on u.
- However, this formulation is not suitable when dealing with lower order terms. New strategies may be required.

Rmk: a similar (significant) example is the parabolic obstacle problem, new strategy was needed [Andreianov-Sbihi-Wittbold '08].

Back to

$$u_t - \Delta_p u + h(u) = \mu \quad + \; IBC$$

We change our strategy to deal with measures $<<$ cap_W.

Given μ, approximate with smooth μ_n.
We wish to deduce compactness from the only estimate

$$\int\int_{\{|u_n|>k\}} |h(u_n)|dxdt \leq \int\int_{\{|u_n|>k\}} |\mu_n|dxdt$$

To this purpose :

- We prove that the level sets $\{|u_n| > k\}$ have uniform small capacity
- We take μ_n so that it is equi-diffuse.

Def. [Brezis-Ponce]: μ_n is equi-diffuse if $\forall \varepsilon > 0 \; \exists \; \eta > 0$:

$$\mathrm{cap}_p(E) < \eta \quad \Longrightarrow \quad |\mu_n|(E) < \varepsilon \quad \forall n$$

Typical example: convolution ! $\mu_n = \mu \star \rho_n$.

The key point (and new ingredient) is the estimate of capacity of level sets.

Theorem

Let $u \in W$ be a (regular) solution of

$$\begin{cases} u_t - \Delta_p u = \mu & \text{in } Q, \\ u = u_0 & \text{on } \{0\} \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega. \end{cases}$$

Then,

$$\mathrm{cap}_p(\{|u| > k\}) \leq C \max\left\{ \frac{1}{k^{\frac{1}{p}}}, \frac{1}{k^{\frac{1}{p'}}} \right\} \quad \forall k \geq 1, \qquad (1)$$

where $C = C\left(\|\mu\|_{\mathcal{M}(Q)}, \|u_0\|_{L^1(\Omega)} \right)$.

Thanks to the capacity estimate:

- we prove compactness of $h(u_n)$, hence, passing to the limit, existence of solutions.
- we are led to a new definition of renormalized solution.

Indeed, truncating the equation with respect to u_n we observe:

$$(T_k(u_n))_t - \Delta_p(T_k(u_n)) = \mu_n^k ,$$

and we estimate the error

$$|\mu_n^k - \mu| \quad \lesssim \quad |\mu_n|\chi_{\{|u_n|>k\}}$$

Then:

uniform capacity estimate + equi-diffuse property of μ_n

$$\Rightarrow \quad |\mu_n^k - \mu| \text{ is small as } k \to \infty \text{ (uniformly in } n)$$

Renormalized solution - new version

We suggest to use the following definition of renormalized solution for such measure data:

Definition

$u \in L^1(Q)$ is a renormalized solution if $T_k(u) \in L^p(0, T; W_0^{1,p}(\Omega))$ for every $k > 0$ and if there exist measures $\lambda_k \in \mathcal{M}(Q)$:

$$(T_k(u))_t - \Delta_p(T_k(u)) + h(T_k(u)) = \mu + \lambda_k$$

and

$$\lim_{k\to\infty} \|\lambda_k\|_{\mathcal{M}(Q)} = 0$$

- Very close to the def. used for conservation laws [Bénilan-Carrillo-Wittbold] and to some versions used for elliptic equations [Dal Maso-Malusa], [Dal Maso-Murat-Orsina-Prignet].

In this formulation, we prove the L^1-contraction principle.

Theorem

If u_1, u_2 are renormalized solutions with data (u_{01}, μ_1), (u_{02}, μ_2),

$$\int_\Omega (u_1 - u_2)^+(t)\, dx + \int_0^t \int_\Omega (h(u_1) - h(u_2))\mathrm{sign}^+(u_1 - u_2) dx d\tau$$
$$\leq \|(u_{01} - u_{02})^+\|_{L^1(\Omega)} + \|(\mu_1 - \mu_2)^+\|_{\mathcal{M}(Q)}$$

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Corollary

If the measure μ does not charge sets of null parabolic capacity, the absorption problem

$$\begin{cases} u_t - \Delta_p u + h(u) = \mu & \text{in } Q := (0, T) \times \Omega, \\ u(0) = u_0 \,, \quad u_{|(0,T)\times\partial\Omega} = 0, & \end{cases}$$

admits solution for every h. If h is nondecreasing, the renormalized solution is unique.

We also prove several properties of the new renormalized formulation:

- When $h = 0$, this definition implies the definition of renormalized solution given in [DPP].
- We prove estimates & stability for renormalized solutions
- We prove the energy asymptotics

$$\frac{1}{\delta}\int_{\{m<|u|<m+\delta\}} a(t,x,\nabla u)\nabla u\,dxdt \leq \int_{\{|u_0|>m\}} |u_0|\,dx + |\mu|\,(E_m)\,,$$

for some set E_m such that $\mathrm{cap}_p(E_m) \leq C \max\left\{m^{-\frac{1}{p}}, m^{-\frac{1}{p'}}\right\}$, where $C = C(\|\mu\|_{\mathcal{M}(Q)}, \|u_0\|_{L^1(\Omega)})$.

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In particular, we recover the usual estimate for L^1-data:

$$\lim_{m\to\infty} \int_{\{m<|u|<m+1\}} a(t,x,\nabla u)\nabla u \, dxdt = 0$$

Final comments

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- We have investigated one more definition of renormalized solution, which seems to be more suitable for nonlinear parabolic equations with lower order terms and measure data.
- In this framework, we have extended to parabolic equations the existence and uniqueness results known in the elliptic case with general absorption $h(u)$.
- Still many results of the elliptic theory with measure data are missing in the evolution case. Main difficulties:

  - u (as well as $T_k(u)$) may not have a cap-quasi continuous representative.

  - the decomposition of measures with respect to parabolic capacity does not match well with the localization required by truncations of u.

We hope to have developed some technical tools to handle similar problems, at least in some situations.

Thanks for the attention !