The Hex game and its mathematical side

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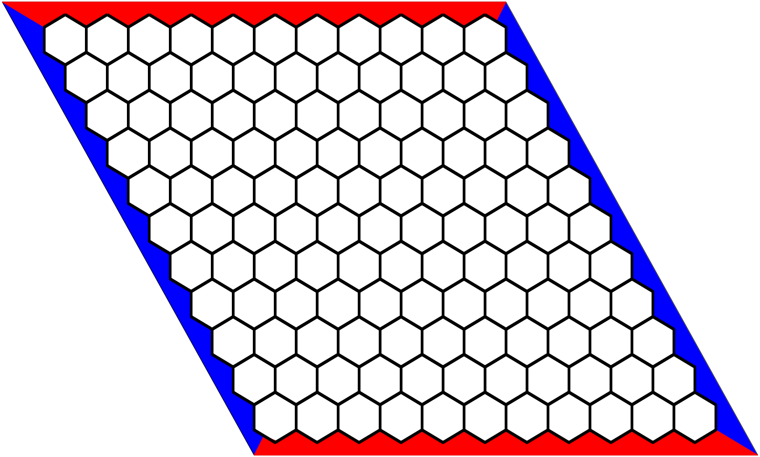

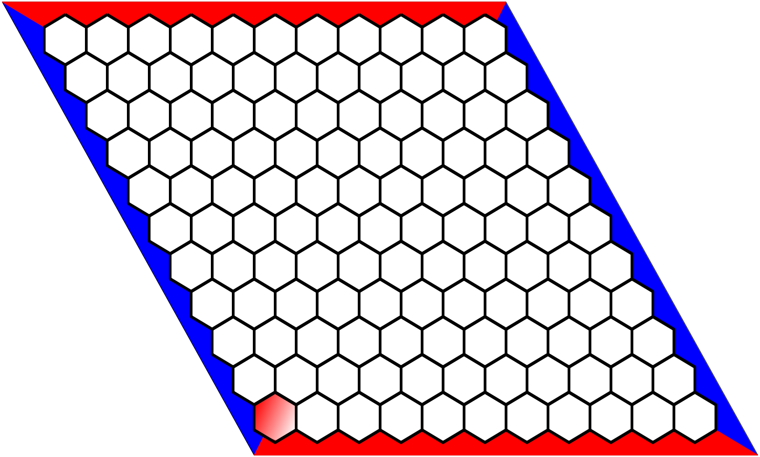

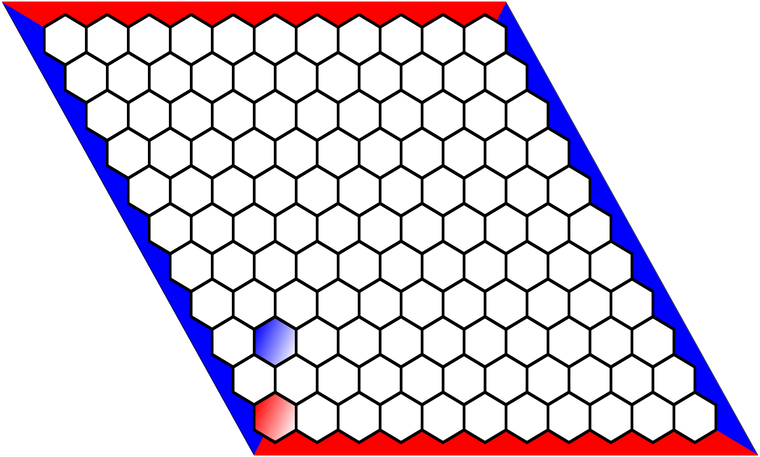

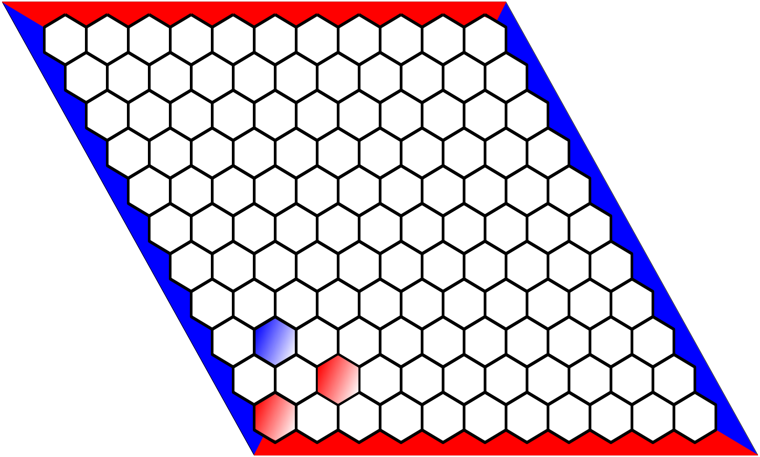

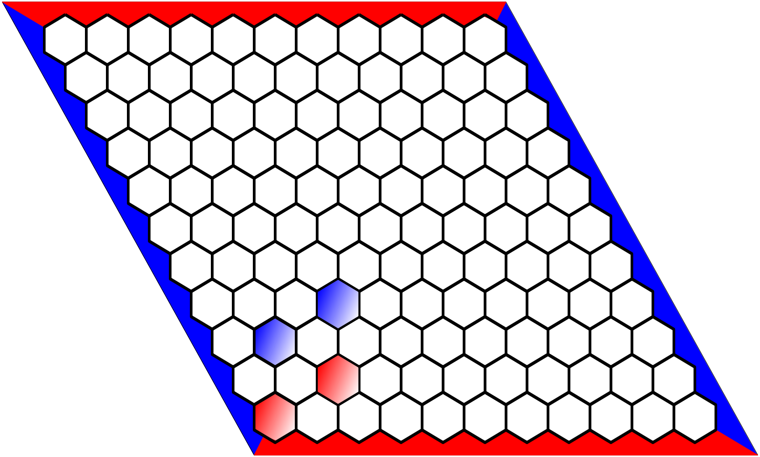

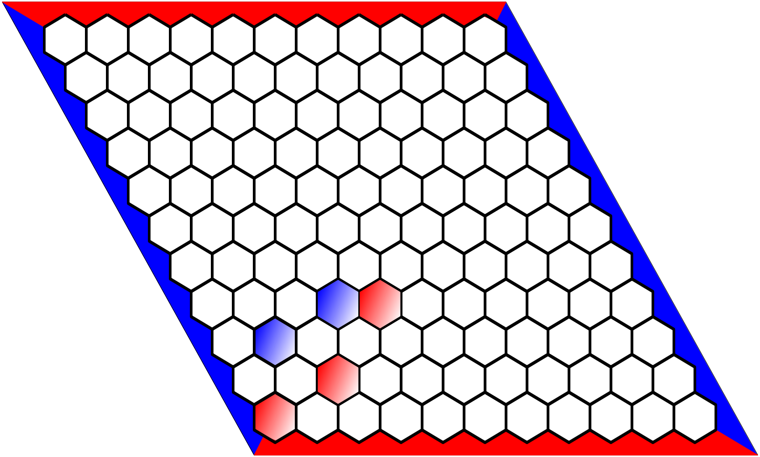

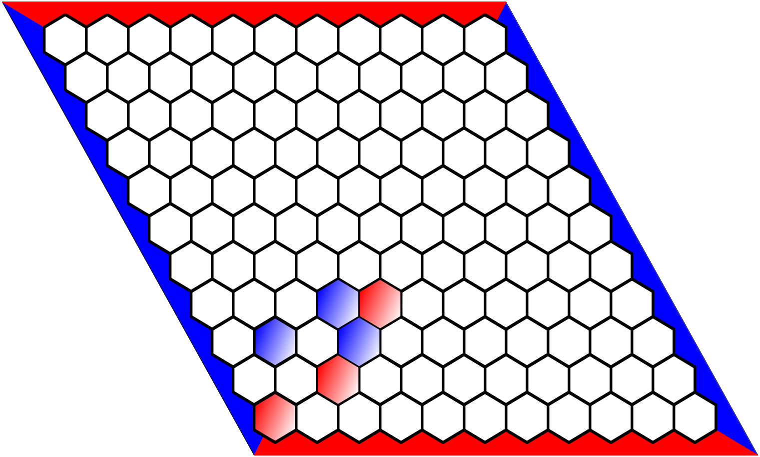

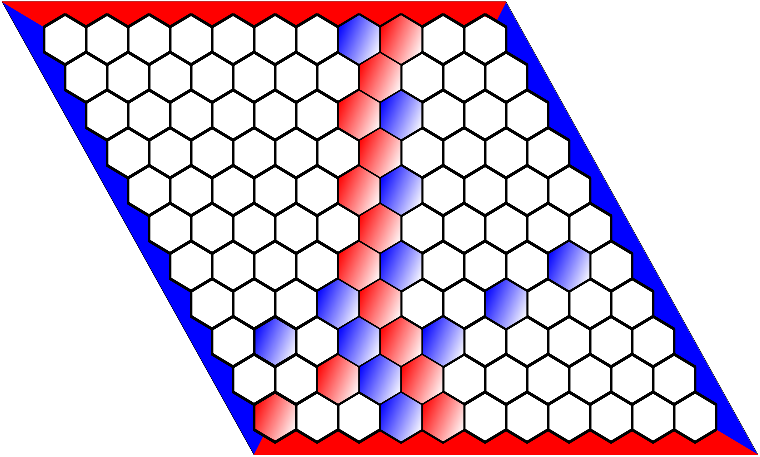

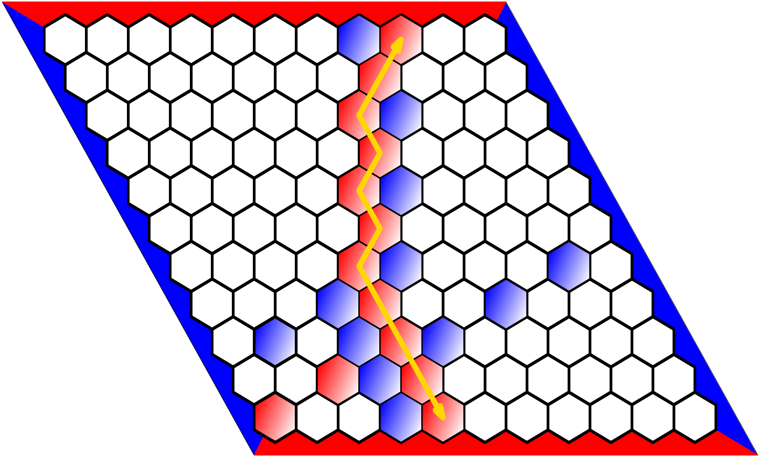

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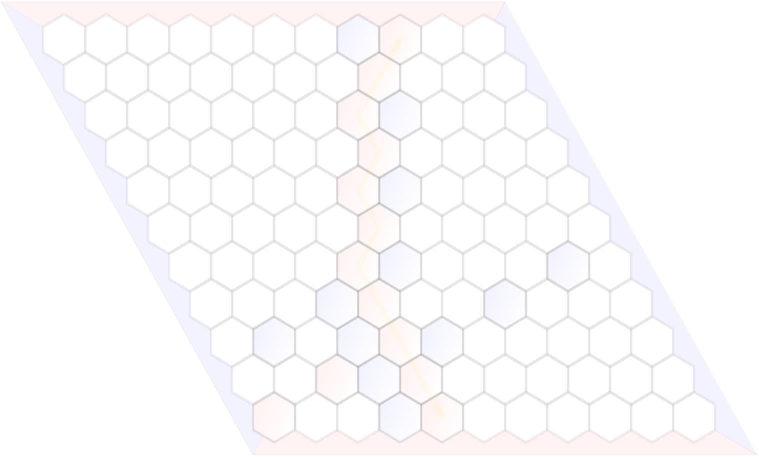
- in 1952 the game is marketed as HEX

How to play to win?



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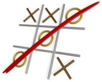

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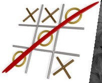

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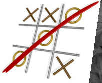

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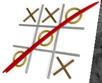

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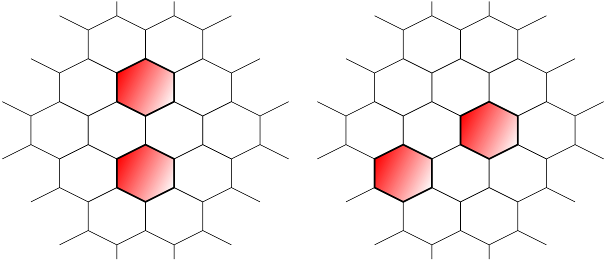
- For the boards of size 10×10 and larger, no one knows the winning strategy.

So how to play?

A hint

Try to “build bridges” :



..and prevent your adversary from building them.

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- A strategy for the Red player is a function $S : \Omega_R \to \Omega_B$ which respects the rules of HEX, that is $S(\omega) \in \mathrm{succ}(\omega)$ for all $\omega \in \Omega$.

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- A strategy for the Blue player is a function $S : \Omega_B \to \Omega_R$ which respects the rules of HEX, that is $S(\omega) \in \mathrm{succ}(\omega)$ for all $\omega \in \Omega$.

We split the terminal configurations acoording to their winner in 3 disjoint subsets :

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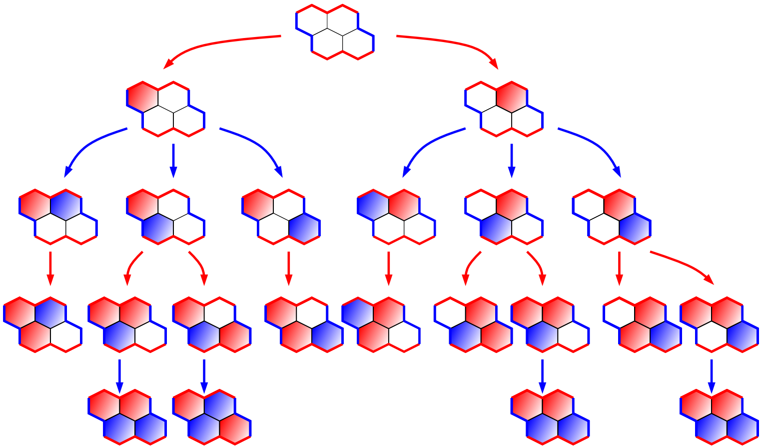
Winning strategy for Red

A strategy S of the Red player is *winning* if for every complete play $(\omega_i)_{i=0}^m \subset \Omega$ which satisfies

$$\omega_{2i+1} = S(\omega_{2i}) \text{ for all } i < m/2$$

we have necessarily $\omega_m \in R$.

Warm-up : the 2×2 case



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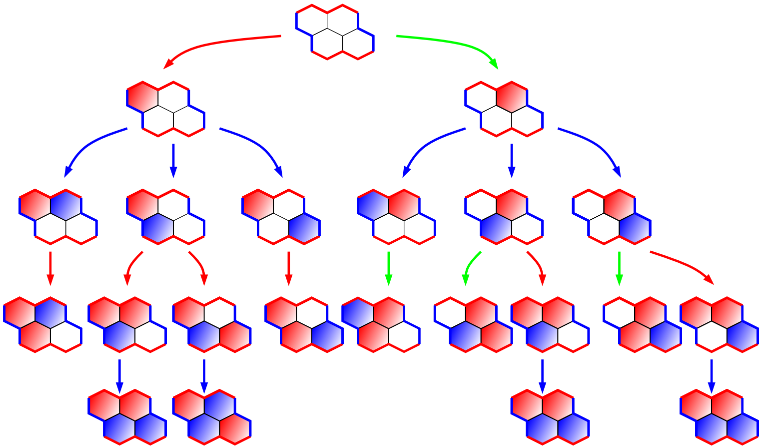

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Exercise

- Find a winning strategy for Red on the 3×3 board.
- What about if we forbid Red to play the central tile in the first turn?

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- Both players win – we get a contradiction.

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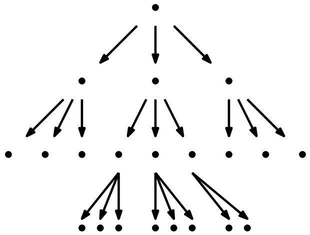

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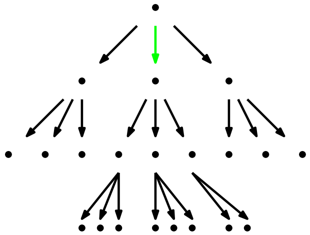

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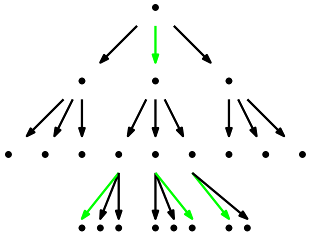

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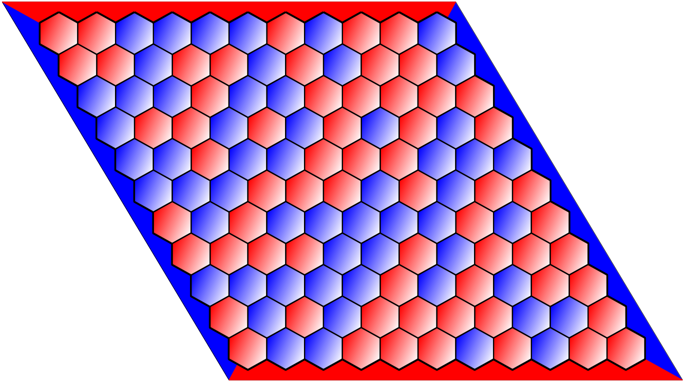
- This is the famous "Theorem of Hex"

Theorem of Hex

Theorem (J. Nash, 1952)

Let $n \in \mathbb{N}$. Let us suppose that every tile of the $n \times n$ board is painted either by red or by blue. Then there exists either a red path which conects the red sides or a blue path which connects the blue sides.



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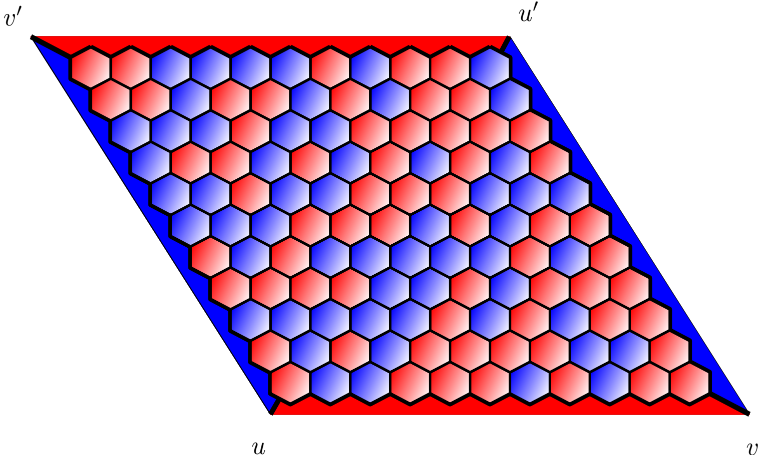

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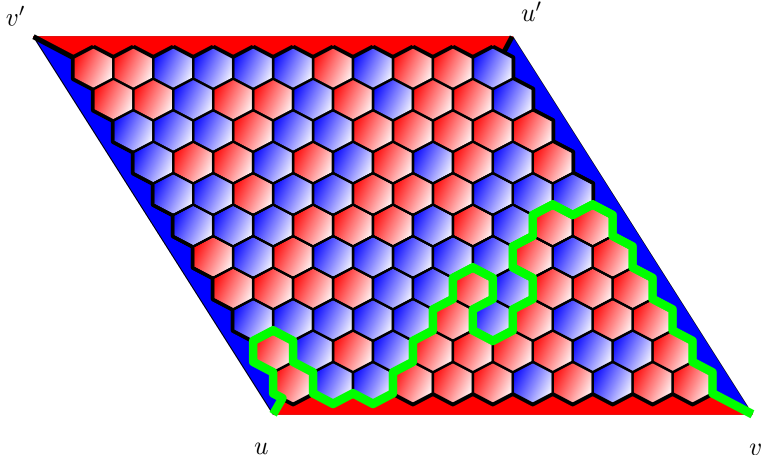

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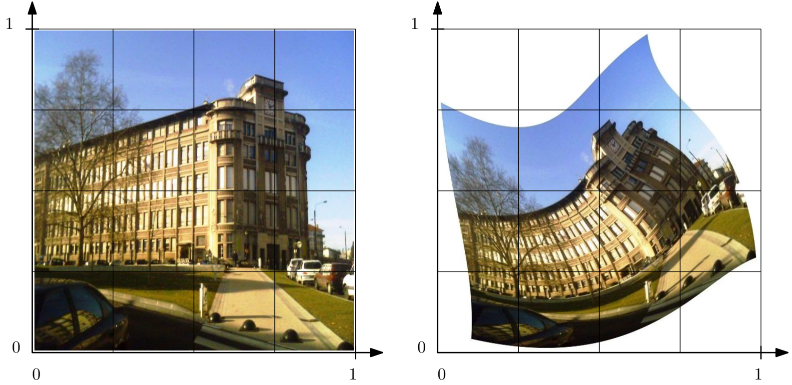
- The theorem of Hex implies a fundamental theorem in *topology* :

Theorem (Brouwer's fixed point theorem,1909)

Let $f : [0,1]^2 \to [0,1]^2$ *be a continuous function. Then there exists* $x \in [0,1]^2$ *such that* $f(x) = x$.

At least one point did not move



Exercise : find it!

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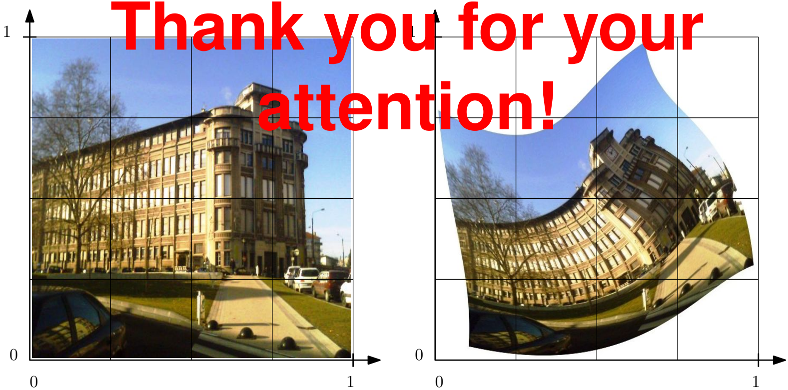

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